ON THE GELFAND-KIRILLOV CONJECTURE FOR THE W-ALGEBRAS ATTACHED TO THE MINIMAL NILPOTENT ORBITS.

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Abstract. Consider the W-algebra $H$ attached to the minimal nilpotent orbit in a simple Lie algebra $g$ over an algebraically closed field of characteristic $0$. We show that if an analogue of the Gelfand-Kirillov conjecture holds for such a W-algebra, then it holds for the universal enveloping algebra $U(g)$. This, together with a result of A. Premet, implies that the analogue of the Gelfand-Kirillov conjecture fails for some W-algebras attached to the minimal nilpotent orbit in Lie algebras of types $B_n$ ($n \geq 3$), $D_n$ ($n \geq 4$), $E_6$, $E_7$, $E_8$, and $F_4$.

Key words: W-algebra, Gelfand-Kirillov conjecture, noncommutative localization.

1. Introduction

Classical works, see for example [Cn3, Theorems 5.1, 5.4], show that any right Noetherian ring has the quotient field which is a noncommutative skew field. In this framework it is natural to ask whether or not such a skew field is isomorphic to a quotient field of a suitable Weyl algebra over a commutative field. A more precise version of this question is known as the Gelfand-Kirillov conjecture: 'Whether or not the quotient field of the universal enveloping algebra of any algebraic Lie algebra is isomorphic to some Weyl skew field?'. In this paper we study a similar question for some W-algebras.

The solution of the original Gelfand-Kirillov conjecture for Lie algebras of type $A$ and some other cases was settled by I. Gelfand and A. Kirillov themselves [GK1, GK2] and is positive. A version of this problem for the W-algebras attached to type $A$ Lie algebras was considered in [FMO], where the authors provide a positive solution of the corresponding problem. We refer the reader to [Ch] for a more extensive discussion on the Gelfand-Kirillov conjecture.

In his paper [Pr2], A. Premet shows that the Gelfand-Kirillov conjecture fails for $U(g)$ if $g$ is simple and $g$ is not of type $A_\ell$, $C_\ell$, or $G_2$. Another result of the same author [Pr1] shows that for a simple Lie algebra $g$ we have that $U(g)$ is "almost equal" to the tensor product of some W-algebra with a suitable Weyl algebra.

The goal of this paper is to modify the result of [Pr1], i.e. to show that the quotient field of $U(g)$ is isomorphic to the quotient field of the tensor product of the same W-algebra with a suitable Weyl algebra. This, together with results of [Pr2], implies that the Gelfand-Kirillov conjecture fails for some W-algebras. It worth mentioning that such W-algebras are deeply studied in [Pr1] and explicit generators and relations are known for them.

From now on the base field for all objects is an algebraically closed field $F$ of characteristic 0.

2. W-algebras

A W-algebra $U(g,e')$ is a finitely generated algebra attached to a semisimple Lie algebra $g$ and an $sl_2$-triple $(e', h', f')$ inside $g$ (see for example [Pr1]). The isomorphism class of such an algebra $U(g,e')$ depends only on the conjugacy class of $e'$. We are particularly interested in W-algebras attached to an element $e'$ from the minimal nonzero nilpotent orbit in $g$. We take an explicit presentation by generators and relations for such a W-algebra [Pr1, Theorem 1.1] and modify the notation of [Pr1] a little. Namely, to a simple Lie algebra $g$ we attach a reductive Lie algebra $g_{e'}(0)$ with a $g_{e'}(0)$-module $g(1)$. Then the algebra $H$ (this is a notation of [Pr1] for such W-algebras) would be generated by $g_{e'}(0), g(1)$ and an additional element $C$ subject to the following relations:

(i) $xy - yx = [x, y]$ for all $x, y \in g_{e'}(0)$, where $[x, y]$ is the Lie bracket of $g_{e'}(0)$;
(ii) $xy - yx = x \cdot y$ for all $x \in g_{e'}(0), y \in g(1)$, where $x \cdot y$ is the action operator of the element $x \in g_{e'}(0)$ applied to $y \in g(1)$;
(iii) $C$ is central in $H$;
(iv) $xy - yx = \frac{1}{2}(x, y)(C - \Theta_{Cas} - c_0) + F(x, y)$ where $(x, y)$ denote the skew-symmetric $g_{e'}(0)$-invariant bilinear form on $g(1)$, $\Theta_{Cas}$ is the Casimir element of $U(g_{e'}(0))$, $c_0$ is a constant depending on $g$, $F(x, y)$ is a skew symmetric function on $g(1)$ with values in $U(g_{e'}(0))$, see [Pr1, Theorem 1.1].

Below we write explicitly Lie algebras $g_{e'}(0)$ and $g_{e'}(0)$-modules $g(1)$ for all simple Lie algebras $g$:

Table 1: $g_{e'}(0)$-module $g(1)$
Involutions of $W$-algebras

The Weyl algebra and its field which is both the left and the right quotient field and is universal in the appropriate sense. (elements of the form $ab$ for example [Cn2, Chapter 7 and p. 486]). There is a notion of the left (elements of the form $a^{-1}b$) and the right (elements of the form $ab^{-1}$) skew fields, and they do coincide if both exist. This is the case if the algebra in question has no zero divisors and has finite Gelfand-Kirillov dimension. All algebras in this paper (and thus all $W$-algebras) have no zero divisors and have finite Gelfand-Kirillov dimension, and thus they have the quotient field which is both the left and the right quotient field and is universal in the appropriate sense.

We now are ready to formulate a precise version of our main result.

**Theorem 2.1.** The quotient fields of $U(g)$ and $H \otimes W_d$ are isomorphic, where $W_d := F[z_1, ..., z_d, \partial_1, ..., \partial_d]$ is a Weyl algebra and $d := \frac{1}{2} \dim g(1) + 1$.

The proof of Theorem 2.1 goes as follows. First we recall in Section 3 a result of A. Premet which states that $U(g)$ is “almost birationally isomorphic” to the tensor product of $H$ with a Weyl algebra. The precise statement of this fact goes through some involutions of a Weyl algebra and $H$. We provide another presentation of these involutions in Section 4 and then complete the proof in Section 5.

Further, we will write $A \cong B$ if the quotient fields of the two associative algebras $A$ and $B$ are isomorphic.

### 3. INVOLUTIONS OF $W$-ALGEBRAS

The algebra $H$ has an involution $\sigma$ (see [Pr1, 2.2]) whose action on generators is given by the formulas

1. $\sigma(x) = x$ if $x \in g(0)$ or $x = C$;
2. $\sigma(x) = -x$ if $x \in g(1)$.

The following lemma is the first step in the proof of Theorem 2.1.

**Lemma 3.1.** We have that

\[ U(g) \cong (W_d \otimes H)\tau^{\otimes \sigma}, \]

where $W_d$ is the Weyl algebra over $F$ generated by $z_1, ..., z_d, \partial_1, ..., \partial_d$ satisfying the relations

\[ [\partial_i, z_j] = \delta_{ij} \quad (1 \leq i, j \leq d), \]

and $\tau$ is the involution on $W_d$ such that

\[ \tau(z_i) = -z_i, \tau(\partial_i) = -\partial_i \quad (1 \leq i \leq d). \]

**Proof.** It follows from the result of [Pr1, 1.5] that

\[ U(g) \cong (A_c \otimes H)\tau^{\otimes \sigma} \]

where $A_c \cong W_{d-1} \otimes (F[h] \ast \langle \Delta \rangle)$. Observe that $F[h] \ast \langle \Delta \rangle \cong F[z_1, \partial_1]$ via maps

\[ h \leftrightarrow z_1\partial_1, \quad \Delta \leftrightarrow z_1. \]

We identify $W_{d-1}$ with $F[z_2, \partial_2, ..., z_d, \partial_d]$ and the quotient field of $F[h] \ast \langle \Delta \rangle$ with the quotient field of $F[z_1, \partial_1]$. Then we have

\[ \tau(z_i) = -z_i, \quad \tau(\partial_i) = -\partial_i. \]

The next lemma is a useful refinement of the previous one.

**Lemma 3.2.** We have that

\[ U(g) \cong W_{d-1} \otimes (F[z_d, \partial_d] \otimes H)^{\tau'}^{\otimes \sigma}, \]

where $F[z_d, \partial_d]$ is the Weyl algebra in one variable with involution $\tau'$ determined by

\[ \tau'(z_d) = -z_d, \quad \tau'(\partial_d) = -\partial_d. \]
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Proof. We consider the birational automorphisms of $W_d$ determined by the pair of maps:

$$z_i \leftrightarrow \frac{z_i}{z_d} \quad (i < d), \quad z_d \leftrightarrow z_d$$

$$\partial_i \leftrightarrow z_d \partial_i \quad (i < d), \quad \partial_d \leftrightarrow \partial_d + \frac{1}{z_d}(z_1 \partial_1 + \ldots + z_{d-1} \partial_{d-1}).$$

One can check that the involution $\tau$ on $W_d$ is equivalent via this automorphism to the involution $\tau'$ on $W_{d-1} \otimes F[z_d, \partial_d]$ such that $\tau'$ preserves the first factor pointwise and the action of $\tau'$ on $[F[z_d, \partial_d]$ is given by the formula

$$\tau'(z_d) = -z_d, \quad \tau' (\partial_d) = -\partial_d.$$

Thus we have that

$$U(\mathfrak{g}) \approx (W_d \otimes H)^{\tau' \otimes \sigma} = W_{d-1} \otimes (F[z_d, \partial_d] \otimes H)^{\tau' \otimes \sigma}.$$

We would like to mention that a philosophically similar decomposition of $U(\mathfrak{g})$ into a tensor product of a W-algebra with a Weyl algebra modulo some completion exists for any W-algebra, see [Lo, Theorem 1.2.1]. It would be interesting if the present paper could be at all elaborated with regard to a more general picture. Some steps in this direction are done in [Tik, Theorem 3.1]

4. INVOLUTIONS OF $H$ AND $\mathfrak{sl}_2$-TRIPLES.

In this section we relate $\sigma : H \to H$ with some $\mathfrak{sl}_2$-triple in $\mathfrak{g}_C(0)$. Namely we prove the following lemma.

Lemma 4.1. Let $H$ and $\sigma$ be as in Sections 2, 3 and assume that $\mathfrak{g}$ is not of type $A$. Then there exists an $\mathfrak{sl}_2$-triple $\{e, h, f\}$ such that

1. the adjoint action of $\{e, h, f\}$ on $H$ is locally finite and thus it can be integrated to the action of an algebraic group $SL_2(F)$,

2. the action of $\sigma$ on $H$ coincides with the action of the only nonunit element of the centre of $SL_2(F)$ on $H$.

Proof. We claim that a suitable $\mathfrak{sl}_2$-triple is an $\mathfrak{sl}_2$-triple with a regular nilpotent $e$. Thus we pick an $\mathfrak{sl}(2)$-triple

$$\{e, h, f\} \subset \mathfrak{g}_C(0)$$

with a regular nilpotent $e$. It is clear that the adjoint action of such an $\mathfrak{sl}_2$-triple preserves both spaces $\mathfrak{g}_C(0)$ and $\mathfrak{g}(1)$, and commutes with $C$. Thus $\{e, h, f\}$ acts locally finitely on $H$.

The fact that $e$ is regular nilpotent implies that $\mathfrak{g}_C(0)$ is isomorphic to a direct sum of simple odd-dimensional $\{e, h, f\}$-modules. Thanks to Table 1, $\mathfrak{g}(1)$ is a simple $\mathfrak{g}_C(0)$-module, and therefore $\mathfrak{g}(1)$ has the unique simple $\{e, h, f\}$-submodule of maximal dimension $V$. We have that

- if $V$ is odd dimensional then all other simple $\{e, h, f\}$-submodules of $\mathfrak{g}_C(1)$ are odd dimensional,

- if $V$ is even dimensional then all other simple $\{e, h, f\}$-submodules of $\mathfrak{g}_C(1)$ are even dimensional.

Statement (2) is equivalent to the second case. Thus we left to check that $V$ is even dimensional.

Space $\mathfrak{g}(1)$ carries a symplectic form $(u, v) \ (u, v \in \mathfrak{g}(1))$, see Section 2. This form is $\mathfrak{g}_C(0)$-invariant and thus it is $\{e, h, f\}$-invariant. The restriction of $(u, v)$ to $V$ must be nondegenerate, and therefore $V$ is even dimensional. □

Now we will use an $\mathfrak{sl}_2$-triple in $H$ to “almost” decompose $H$ as a tensor product of two algebras. We do this in several steps and the first one is as follows.

4.1. Some localizations of $H$. The goal of this subsection is to define an extension of $H$ by an element $e^{-\frac{1}{2}}$ together with a natural involution $e^{-\frac{1}{2}} \to -e^{-\frac{1}{2}}$ in a proper way (the result will be called $H[e^{-\frac{1}{2}}]$).

It is clear from the construction that the skew field of $H[e^{-\frac{1}{2}}]$ has left dimension 2 over the skew field of $H$. In [Cn1] P. Cohn used skew polynomials to classify such extensions of left degree 2. Our approach to the definition of $H[e^{-\frac{1}{2}}]$ is quite similar. The difference is that we rely on the properties of the locally nilpotent derivations of algebras in characteristic 0, while [Cn1] works in a much more general setting but with skew fields.

Let $H$ be an associative algebra and $e \in H$ be such that the operator

$$\text{ad} \ e : H \to H \quad (x \to [e, x] := ex - xe)$$

is an $\mathfrak{sl}_2$-triple in $H$ to “almost” decompose $H$ as a tensor product of two algebras. We do this in several steps and the first one is as follows.
acts locally nilpotently on $H$. Then we set $(\widehat{H, \text{ade}}, \frac{1}{2})$ to be the algebra which is, as a vector space, a direct sum of $((\mathbb{Z} \cup \{\frac{1}{2}\}) \times \mathbb{Z}, \frac{1}{2})$-copies of $H$, with multiplication

$$(xt^n) \cdot (yt^m) = \sum_{i=0}^{\infty} \left( \binom{n}{i} \right) x(ad e)^i(y)t^{m+n-i}$$

for $x, y \in H, m, n \in (\mathbb{Z}, \frac{1}{2})$, where we denote by $xt^n$ the element $x \in H$ in the $n$th copy of $H$. We wish to note that even if formally the above sum is infinite, it is essentially finite as ad $e$ is a locally nilpotent operator. One can explicitly check that the algebra $(\widehat{H, \text{ade}}, \frac{1}{2})$ is associative.

One can think about $(\widehat{H, \text{ade}}, \frac{1}{2})$ as about a power series extension of $H/a$ power series extension of a localization of $H$.

As the next step, we determine a two-sided ideal of $(\widehat{H, \text{ade}}, \frac{1}{2})$, namely the ideal generated by $(e - t)$, but it is quite useful to write it more explicitly. The following lemma is straightforward.

**Lemma 4.2.** a) The vector space $I$ spanned by

$$xt^{n+m} - xe^n t^m, \text{ for all } n \in \mathbb{Z}, m \in (\mathbb{Z}, \frac{1}{2})$$

is a two-sided ideal of $(\widehat{H, \text{ade}}, \frac{1}{2})$.

b) $I$ is generated as a two-sided ideal by $(e - t)$.

We denote by $H[e^{-\frac{1}{2}}]$ the quotient $(\widehat{H, \text{ade}}, \frac{1}{2})/I$ and by $ev$ the respective map $ev : (\widehat{H, \text{ade}}, \frac{1}{2}) \to H[e^{-\frac{1}{2}}]$. The reason for this notation is that $H[e^{-\frac{1}{2}}]$ is generated by

$$ev(H), ev(t^\frac{1}{2}), ev(t^{-\frac{1}{2}})$$

and $ev(t) = ev(e)$. The following lemma is implied by Lemma 4.2.

**Lemma 4.3.** a) For any $x \in H[e^{-\frac{1}{2}}]$ there exist $x_1, x_2 \in H, n \in \mathbb{Z}$ such that

$$x = ev(x_1 t^{-n} + x_2 t^{-n-\frac{1}{2}}),$$

b) For any $x_1, x_2, x'_1, x'_2 \in H, n \in \mathbb{Z}$, we have that

$$ev(x_1 t^{-n} + x_2 t^{-n-\frac{1}{2}}) = ev(x'_1 t^{-n} + x'_2 t^{-n-\frac{1}{2}})$$

if and only if for some $m \in \mathbb{Z}_{\geq 0}$ we have that $(x_1 - x'_1)e^m = (x_2 - x'_2)e^m = 0$.

**Corollary 4.4.** If $H$ has no zero-divisors, then $ev|_H : H \to H[e^{-\frac{1}{2}}]$ is injective.

The algebra $H$ defined in Section 2 has a PBW-basis which is a union of $C$ and bases of subspaces $g_{v^e}(0), g(1)$. Therefore $H$ has no zero divisors. Hence, we can identify $H$ with its image in $H[e^{-\frac{1}{2}}]$. We also prefer to avoid whenever possible the notation $ev(t^n)$, using $e^n$ instead. We mention that the linear map

$$\sigma_t(x_1 t^m) \to (-1)^{2m} x_1 t^m$$

is an involution of $(\widehat{H, \text{ade}}, \frac{1}{2})$ which preserves $I$. Thus $\sigma_t$ defines an involution $\sigma_\sigma : H[e^{-\frac{1}{2}}] \to H[e^{-\frac{1}{2}}]$ of $H[e^{-\frac{1}{2}}]$ such that $\sigma_\sigma(e^{-\frac{1}{2}}) = - e^{-\frac{1}{2}}$ and $\sigma_\sigma$ preserves $H$ pointwise.

4.2. **An “almost” decomposition of $H$.** Let $H$ be an associative algebra with no zero-divisors, $\{e, h, f\} \subset H$ be a $\mathfrak{sl}_2$-triple, and let $\sigma : H \to H$ be an involution of $H$ satisfying conditions (1) and (2) of Lemma 4.1. Then $\sigma$ gives rise to the automorphism $\tilde{\sigma}$ of $(\widehat{H, \text{ade}}, \frac{1}{2})$ such that

$$\tilde{\sigma}|_H = \sigma, \quad \tilde{\sigma}(t) = t.$$

This automorphism preserves $I$ and hence defines an automorphism (which we also denote by $\sigma$) of $H[e^{-\frac{1}{2}}]$.

**Lemma 4.5.** The quotient field of $H$ is generated by $H^e, e$ and $h$.

**Proof.** The Casimir $\theta = h^2 + 2(e f + fe)$ of $U(\mathfrak{sl}(2)) = U(\{e, h, f\})$ is contained in $H^e$, and hence

$$f = \frac{\theta - h^2 - 2h}{4} e^{-1}$$

is contained in the envelope of $\{H^e, h, e\}$ in the quotient field of $A$. It follows from the representation theory of $\mathfrak{sl}(2)$ that $H^e$ and $f$ generate $H$, and thus generate the quotient field of $H$. $\square$
We have the natural adjoint action of $h$ on $H^e$. This action induces the nonsimple grading $\{(H^e)_i\}_{i \in \mathbb{Z}_{\geq 0}}$ on $H^e$. We consider the map $\psi : H^e \to H[e^{-\frac{1}{2}}]$ such that $a \to ae^{-\frac{1}{2}}$ for any $a \in (H^e)_i$. Clearly, $\psi$ is a homomorphism of algebras and the image of $\psi$ commutes with both $e$ and $h$. We denote the image of $\psi$ by $(H^e)_\psi$.

**Lemma 4.6.** The elements $(H^e)_\psi, e^{\frac{1}{2}}, h$ generate the quotient field of $H[e^{-\frac{1}{2}}]$. Moreover

$$H[e^{-\frac{1}{2}}] \approx U(\{e^{\frac{1}{2}}, h\}) \otimes (H^e)_\psi.$$  

**Proof.** The first statement is implied by Lemma 4.5. Thus we now focus on the second statement. It is clear that we have a natural map

$$\gamma : U(\{e^{\frac{1}{2}}, h\}) \otimes (H^e)_\psi \to H[e^{-\frac{1}{2}}]$$

and it is enough to prove that $\gamma$ is injective. Hence it is enough to show that

$$\sum_{i,j \in \mathbb{Z}_{\geq 0}} e^{i/2} h^j H_{ij}, \quad H_{ij} \in (H^e)_\psi,$$

equals zero if and only if $H_{ij} = 0$ for all $i,j$. The action of $ad \, h := [h, \cdot]$ on both $U(\{e^{\frac{1}{2}}, h\}) \otimes (H^e)_\psi$ and $H[e^{-\frac{1}{2}}]$ is semisimple and it is clear that the $i$th eigenspaces will map to the $i$th eigenspace. Thus it is enough to show that

$$\sum_{j \in \mathbb{Z}_{\geq 0}} h^j H_j, \quad H_j \in (H^e)_\psi$$

equals zero if and only if $H_j = 0$ for all $j$. Assume on the contrary that there exist $H_0, ..., H_s$ such that

$$H_0 h^0 + ... + H_s h^s = h^0(H_0) + ... + h^s(H_s) = 0$$

and not all $H_0, ..., H_s$ are equal to 0. Without loss of generality we can assume that

1) $H_0 \neq 0, H_s \neq 0$.

2) $s$ is the smallest possible among all such expressions.

Under these conditions we have that $s > 0$. To proceed, we need the following simple lemma, a proof of which is left to the reader.

**Lemma 4.7.** We have $e^{-1} p(h)e = p(h + 2)$.

According to this lemma, we have

$$0 = e^{-1}(H_0 h^0 + ... + H_0 h^0)e - (H_0 h^0 + ... + H_0 h^0) = 2s H_s h^{s-1} + (...) h^{s-2} + ...$$

It is clear that this new expression is of the same form but of smaller degree in $h$. This is a contradiction.  

**Corollary 4.8.** We have that $H \approx (U(\{e^{\frac{1}{2}}, h\}) \otimes (H^e)_\psi)^{\tau'}$ where

$$\sigma_e(e^{\frac{1}{2}}) = -e^{\frac{1}{2}}, \quad \sigma_e(h) = h, \quad \sigma_e(\psi(a)) = (-1)^s \psi(a)$$

for $a \in (H^e)_i$.

5. **Proof of Theorem 2.1**

If $\mathfrak{g}$ is of type $A$, the statement of Theorem 2.1 follows from the results of [FMO]. Thus we can focus on all other cases and apply the results of Section 4. Lemma 3.2 implies that it is enough to prove the following proposition.

**Proposition 5.1.** If $\mathfrak{g}$ is a simple Lie algebra then

$$(F[z_d, \partial_d] \otimes H)^{\tau' \otimes \sigma} \approx F[z_d, \partial_d] \otimes H,$$

where $\tau'(z_d) = -z_d, \tau'(\partial_d) = -\partial_d$.

According to Section 4, $H[e^{-\frac{1}{2}}]$ carries two involutions: $\sigma$ and $\sigma_e$, which commute with each other. Thus, $\sigma, \sigma_e$ is also an involution of $H[e^{-\frac{1}{2}}]$. We denote this involution by $\sigma_U$. One can easily check that

1) $\sigma_U$ preserves $(H^e)_\psi$ pointwise and preserves $U(\{e^{\frac{1}{2}}, h\})$ as a set,

2) $\sigma$ preserves $U(\{e^{\frac{1}{2}}, h\})$ pointwise and preserves $(H^e)_\psi$ as a set,

3) $\sigma_e = \sigma_U \sigma = \sigma \sigma_U$.

Put

$$W_x := F[x, \partial_x] := F(x, \partial_x)/(x\partial_x - \partial_x x - 1)$$

(polyonomial differential operators in one variable). We denote by $\sigma_x$ the involution of $W_x$ defined by the following formulas

$$\sigma_x(x) = -x, \sigma_x(\partial_x) = -\partial_x.$$
Similarly we define $W_y$ and $\sigma_y$. The quotient fields of $W_y$ and $U(\{\epsilon^{\frac{1}{2}}, h\})$ are isomorphic via the identification $y \mapsto \epsilon^{\frac{1}{2}}$, $y\partial_y \mapsto h$, and the involutions $\sigma_y$ and $\sigma_U$ corresponds to each other under this isomorphism.

We have that $H \approx H[\epsilon^{-1}] \approx (U(\{\epsilon^{\frac{1}{2}}, h\}) \otimes (H^c)^{\psi})^{\sigma \times \sigma_v}$ and thus $H \approx (W_y \otimes (H^c)^{\psi})^{\sigma \times \sigma_v \times \sigma}$. Therefore Proposition 5.1 is implied by the following lemma.

**Lemma 5.2.** We have $(W_x \otimes W_y \otimes (H^c)^{\psi})^{\sigma_v \times \sigma_v \times \sigma_v \times \sigma} \approx (W_x \otimes W_y \otimes (H^c)^{\psi})^{\sigma \times \sigma_v \times \sigma}$, where $\langle \sigma_v \times \sigma, \sigma_y \times \sigma \rangle$ is a group of order $4$ generated by $\sigma_x \times \sigma$ and $\sigma_v \times \sigma$.

To prove Lemma 5.2, we need another set of generators of the quotient field of $W_x \otimes W_y$. Namely, we set

$$z := \frac{y}{x}, \quad \partial_z := x\partial_y, \quad \partial'_z := \partial_x + \frac{y}{x} \partial_y.$$

One can easily check that $z, \partial_z, x, \partial_x$ generates the quotient field of $W_x \otimes W_y$ and that

$$[z, x] = [\partial'_z, \partial_z] = [z, \partial'_x] = [x, \partial_z] = 0, \quad [x, \partial'_x] = [z, \partial'_z] = -1.$$

Put $W'_x := \mathbb{F}[x, \partial'_x], W'_z := \mathbb{F}[z, \partial'_z]$. We have that $\mathbb{F}[x, \partial'_x, z, \partial'_z] \cong W'_x \otimes W'_z$ and that the quotient field of $W_x \otimes W_y$ is identified with the quotient field of $W'_x \otimes W'_z$ under this isomorphism. We have that

$$\sigma_y(z) = -z, \sigma_y(\partial_z) = -\partial_z, \sigma_y(x) = x, \sigma_y(\partial'_z) = \partial'_z$$

and

$$z \sigma_y (z) = z, \sigma_z \sigma_y (\partial_z) = \partial_z, \sigma_z \sigma_y (x) = x - z, \sigma_z \sigma_y (\partial'_z) = -\partial'_z.$$

We set $\sigma'_z := \sigma_z \sigma_y$ and $\sigma_z := \sigma_y$ (this notation is justified by the formula above).

**Proof of Lemma 5.2.** We have that

$$(W_x \otimes W_y \otimes (H^c)^{\psi})^{\langle \sigma_x \times \sigma, \sigma_y \times \sigma \rangle} = (W_x \otimes W_y \otimes (H^c)^{\psi})^{\langle \sigma \times \sigma_v, \sigma_v \times \sigma \rangle} = (W'_x \otimes W'_z \otimes (H^c)^{\psi})^{\langle \sigma_x \times \sigma \rangle} = (W'_x \otimes W'_z \otimes (H^c)^{\psi})^{\sigma \times \sigma}.$$

It only remains to note that $(W'_x)^{\sigma \times \sigma} \cong W_x$ and $(W_y \otimes (H^c)^{\psi})^{\sigma \times \sigma} \cong (W_y \otimes (H^c)^{\psi})^{\sigma \times \sigma} \cong A$. \qed

**Acknowledgements**

I would like to thank A. Premet for many useful discussions on $W$-algebras, which helped me to write this paper. I also thank a referee for dozens of thoughtful suggestions. This work is a part of a project on the Gelfand-Kirillov conjecture for $W$-algebras supported by Leverhulme Trust Grant RPG-2013-293.

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