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SMOOTH DENSITIES OF THE LAWS OF PERTURBED DIFFUSION PROCESSES

LIHU XU, WEN YUE, AND TUSHENG ZHANG

ABSTRACT. Under some regularity conditions on $b$, $\sigma$ and $\alpha$, we prove that the solution of the following perturbed stochastic differential equation

$$X_t = x + \int_0^t b(X_s) \, ds + \int_0^t \sigma(X_s) \, dB_s + \alpha \sup_{0 \leq s \leq t} X_s, \quad \alpha < 1$$

admits smooth densities for all $0 < t \leq t_0$, where $t_0 > 0$ is some finite number.

Keywords: Perturbed diffusion processes, Malliavin differentiability, Smooth density.

Mathematics Subject Classification (2000): 60H07.

1. INTRODUCTION

There have been a considerable body of literatures devoted to the study of perturbed stochastic differential equations (SDEs), see [1]-[7], [9], [11], [12]. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space with filtration $\{\mathcal{F}_t\}_{t \geq 0}$, let $\{B_t\}_{t \geq 0}$ be a one-dimensional standard $\mathcal{F}_t$-Brownian Motion. Suppose that $\sigma(x), b(x)$ are Lipschitz continuous functions on $\mathbb{R}$. It was proved in [5] that the following perturbed stochastic differential equation:

$$(1.1) \quad X_t = x + \int_0^t b(X_s) \, ds + \int_0^t \sigma(X_s) \, dB_s + \alpha \sup_{0 \leq s \leq t} X_s, \quad \forall \alpha < 1,$$

admits a unique solution. If $|\sigma(x)| > 0$, it was shown in [12] that the law of $X_t$ is absolutely continuous with respect to Lebesgue measure, i.e. the law of $X_t$ admits a density for $t > 0$.

There seem no results on the smoothness of the density of the law of a perturbed diffusion process. This paper aims to partly fill in this gap. The smoothness of densities is a popular topic in stochastic analysis and has been intensively studied for several decades. We refer readers to [8], [10] and references therein. Our approach to proving the smoothness of densities is by Malliavin calculus, so let us first recall some well known results on Malliavin calculus [8] to be used in this paper.

Let $\Omega = C_0(\mathbb{R}_+)$ be the space of continuous functions on $\mathbb{R}_+$ which are zero at zero. Denote by $\mathcal{F}$ the Borel $\sigma$-field on $\Omega$ and $\mathbb{P}$ the Wiener measure, then the canonical
coordinate process \( \{\omega_t, t \in \mathbb{R}_+\} \) on \( \Omega \) is a Brownian motion \( B_t \). Define \( \mathcal{F}_t^0 = \sigma(B_s, s \leq t) \) and denote by \( \mathcal{F}_t \) the completion of \( \mathcal{F}_t^0 \) with respect to the \( \mathbb{P} \)-null sets of \( \mathcal{F} \).

Let \( H := L^2(\mathbb{R}_+, \mathcal{B}, \mu) \) where \((\mathbb{R}_+, \mathcal{B})\) is a measurable space with \( \mathcal{B} \) being the Borel \( \sigma \)-field of \( \mathbb{R}_+ \) and \( \mu \) being the Lebesgue measure on \( \mathbb{R}_+ \). We denote the norm of \( H \) by \( \| \|_H \). For any \( h \in H \), \( W(h) \) is defined by
\[
(1.2) \quad W(h) = \int_0^\infty h(t)dB_t.
\]

Note that \( \{W(h), h \in H\} \) is a Gaussian Process on \( H \).

We denote by \( C^\infty_p(\mathbb{R}^n) \) the set of all infinitely differentiable functions \( f : \mathbb{R}^n \to \mathbb{R} \) such that \( f \) and all of its partial derivatives have polynomial growth. Let \( \mathcal{S} \) be the set of smooth random variables defined by
\[
\mathcal{S} = \{ F = f(W(h_1), ..., W(h_n)); h_1, ..., h_n \in H, n \geq 1, f \in C^\infty_p(\mathbb{R}^n) \}.
\]

Let \( F \in \mathcal{S} \), define its Malliavin derivative \( D_tF \) by
\[
(1.3) \quad D_tF = \sum_{i=1}^n \partial_i f(W(h_1), ..., W(h_n))h_i(t),
\]
and its norm by
\[
\| |F| |_{1,2} = [\mathbb{E}(|F|^2) + \mathbb{E}(|D_tF|^2_H)]^{\frac{1}{2}},
\]
where \( |DF|^2_H = \int_0^\infty |D_tF|^2\mu(dt) \). Denote by \( \mathbb{D}^{1,2} \) the completion of \( \mathcal{S} \) under the norm \( \| |.| |_{1,2} \). We further define the norm
\[
\| |F| |_{m,2} = \left[ \mathbb{E}(|F|^2) + \sum_{k=1}^m \mathbb{E}(\|D^k F\|_{H^\otimes k}^2) \right]^{\frac{1}{2}}.
\]

Similarly, \( \mathbb{D}^{m,2} \) denotes the completion of \( \mathcal{S} \) under the norm \( \| |.| |_{m,2} \).

We shall use the following two propositions:

**Proposition 1.1** (Proposition 1.2.3 of [8]). Let \( \phi : \mathbb{R}^d \to \mathbb{R} \) be a continuously differentiable function with bounded partial derivatives. Suppose that \( F = (F^1, \ldots, F^d) \) is a random vector whose components belong to the space \( \mathbb{D}^{1,2} \). Then \( \phi(F) \in \mathbb{D}^{1,2} \), and
\[
D(\phi(F)) = \sum_{i=1}^d \partial_i \phi(F)DF^i.
\]

**Proposition 1.2** (Proposition 2.1.5 of [8]). If \( F \in \mathbb{D}^{\infty,2} \) with \( \mathbb{D}^{\infty,2} = \bigcap_{m \geq 1} \mathbb{D}^{m,2} \) and \( \|DF\|_{H^\otimes 1} \in L^p(\Omega) \), then the density of \( F \) belongs to the space \( C^\infty(\mathbb{R}) \) of infinitely continuously differentiable functions.

Throughout this paper, for a bounded measurable function \( f \), we shall denote
\[
\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|.
\]
2. Main Results

Throughout this paper, we need to assume $\alpha < 1$ to guarantee that Eq. (1.1) has a unique solution [5]. Furthermore, it is shown in [12] that

**Theorem 2.1.** ([12, Theorem 3.1]) Let $(X_t)_{t \geq 0}$ be the unique solution to Eq. (1.1). Then $X_t \in \mathbb{D}^{1,2}$ for all $t > 0$.

**Theorem 2.2.** ([12, Theorem 3.2]) Assume that $\sigma$ and $b$ are both Lipschitz continuous, and $|\sigma(x)| > 0$ for all $x \in \mathbb{R}$. Then, for $t > 0$, the law of $X_t$ is absolutely continuous with respect to Lebesgue measure.

In this paper, we shall prove the following results about the smoothness of densities:

**Theorem 2.3.** Assume that $b$ is bounded smooth with $\|b^\prime\|_\infty < \infty$ and that $\sigma(x) \equiv \sigma$. If $\alpha < 1$, $t_0 > 0$ and $b$ satisfy

$$\theta(t_0, \alpha, b) < 1/2,$$

with $\theta(t_0, \alpha, b) := \sqrt{2\|b^\prime\|^2_\infty t_0^2 + 8\alpha^2 + b^\prime(t_0)^2 + 4\alpha^2}$, then the law of $X_t$ in (1.1) admits a smooth density for all $t \in (0, t_0]$.

**Theorem 2.4.** Assume that $b$ is bounded smooth with $\|b^\prime\|_\infty < \infty$ and that $\sigma$ is bounded smooth with $\|\sigma^\prime\|_\infty < \infty$, $\|\sigma''\|_\infty < \infty$ and $\inf_{x \in \mathbb{R}} |\sigma(x)| > 0$. Let

$$F(y) = \int_x^y \frac{1}{\sigma(u)} du, \quad y \in (-\infty, \infty)$$

and $\tilde{b}(x) = \frac{b(F^{-1}(x))}{\sigma(F^{-1}(x))} - \frac{1}{2} \sigma'(F^{-1}(x))$, then $\tilde{b}$ is bounded smooth with $\|\tilde{b}^\prime\|_\infty < \infty$. If $\alpha < 1$, $t_0 > 0$ and $\tilde{b}$ satisfy

$$\theta(t_0, \alpha, \tilde{b}) < 1/2$$

with $\theta(t_0, \alpha, \tilde{b}) := \sqrt{2\|\tilde{b}^\prime\|^2_\infty t_0^2 + 8\alpha^2 + \|\tilde{b}^\prime(t_0)^2 + 4\alpha^2}$, then the law of $X_t$ in (1.1) admits a smooth density for all $t \in (0, t_0]$.

**Proofs of Theorems 2.3 and 2.4:** The main idea is to use Proposition 1.2 to prove the two theorems. To verify the conditions in Proposition 1.2, it suffices to prove that $X_t \in \mathbb{D}^{m,2}$ for all $m \geq 1$ and $\|DFX_t\|_H \geq c > 0$ a.s. for some constant $c > 0$.

Theorem 2.3 immediately follows from Lemmas 3.1 and 3.4 below.

Now we prove Theorem 2.4. Recall $Y_t = \int_{X_0}^{X_t} \frac{1}{\sigma(u)} du$ in Lemma 3.5 below, by the condition of $\sigma$, $F$ is a continuous and strictly increasing function with bounded derivative and thus

$$\|DY_t\|_H = \|DF(X_t)\|_H \leq \frac{1}{\inf_{x \in \mathbb{R}} |\sigma(x)|} \|DX_t\|_H.$$

Hence, by Lemmas 3.1 and 3.5 below, under the same condition as in Theorem 2.4 we have

$$\|DX_t\|_H \geq \inf_{x \in \mathbb{R}} |\sigma(x)| \cdot \|DY_t\|_H \geq \inf_{x \in \mathbb{R}} |\sigma(x)| \cdot \frac{[1 - 2\theta(t_0, \alpha, \tilde{b})]t}{2(1 + 2\|\tilde{b}^\prime\|_\infty t_0^2 + 2\alpha^2)} \quad t \in [0, t_0].$$
Hence, $X_t$ admits a smooth density for all $t \in (0, t_0]$.

3. AUXILIARY LEMMAS

It is well known that $\|DX_t\|_H$ has the following representation [12] for all $t > 0$:

$$\|DX_t\|_H = \left(\int_0^t |D_rX_t|^2 dr\right)^{\frac{1}{2}}$$

with $D_rX_t$ satisfying

$$D_rX_t = \sigma(X_r) + \int_r^t D_r b(X_s) ds + \int_r^t D_r \sigma(X_s) dB_s + \alpha D_r \left(\sup_{0 \leq s \leq t} X_s\right).$$

We shall often use the following fact ([12], [8])

$$D_rX_t = 0 \text{ if } r > t,$$

$$\left\|D\left(\sup_{0 \leq s \leq t} X_s\right)\right\|_H \leq \sup_{0 \leq s \leq t} \|DX_s\|_H,$$

where

$$\left\|D\left(\sup_{0 \leq s \leq t} X_s\right)\right\|_H^2 = \int_0^t \left|D_r \left(\sup_{0 \leq s \leq t} X_s\right)\right|^2 dr, \quad \|DX_t\|_H^2 = \int_0^t |D_rX_t|^2 dr.$$

3.1. $X_t$ is an element in $\mathbb{D}^{m,2}$ for all $t > 0$ and $m \geq 1$.

**Lemma 3.1.** Let $X_t$ be the solution of the perturbed stochastic differential equation (1.1), and suppose that the coefficients $b$ and $\sigma$ are smooth with bounded derivatives of all orders. Then $X_t$ belongs to $\mathbb{D}^{m,2}$ for all $t > 0$ and all $m \geq 1$.

**Proof.** We shall use Picard iteration to prove the lemma. Letting $X^0_t = x_0$ for all $t > 0$, define $X^{n+1}_t$ be the unique, adapted solution to the following equation:

$$X^{n+1}_t = x_0 + \int_0^t \sigma(X^n_s) dB_s + \int_0^t b(X^n_s) ds + \alpha \max_{0 \leq s \leq t} (X^{n+1}_s),$$

which obviously implies

$$\max_{0 \leq s \leq t} (X^{n+1}_s) = x_0 + \max_{0 \leq s \leq t} \left(\int_0^t \sigma(X^n_s) dB_s + \int_0^t b(X^n_s) ds + \alpha \max_{0 \leq s \leq t} (X^{n+1}_s)\right).$$

Therefore,

$$\max_{0 \leq s \leq t} (X^{n+1}_s) = \frac{x_0}{1 - \alpha} + \frac{1}{1 - \alpha} \max_{0 \leq s \leq t} \left(\int_0^t \sigma(X^n_s) dB_s + \int_0^t b(X^n_s) ds\right),$$
this and (3.4) further gives
\[ X_t^{n+1} = x_0 + \int_0^t \sigma(X_u^n)dB_u + \int_0^t b(X_u^n)du + \frac{\alpha}{1 - \alpha} \max_{0 \leq s \leq t} \left( \int_0^s \sigma(X_u^n)dB_u + \int_0^s b(X_u^n)du \right). \]

By the above representation of \( X_t^{n+1} \) and a standard method [5], for every \( t > 0 \) we have

\[ \lim_{n \to \infty} X_t^n = X_t \quad \text{in} \quad L^2(\Omega). \]

Let \( m \geq 1 \), it is standard to check that \( X_t^n \in \mathbb{D}^{m,2} \) for every \( t > 0 \) and \( n \geq 1 \) [12, Theorem 3.1]. By a similar argument as in [12, Theorem 3.1], we have

\[ \sup_{n \geq 1} \mathbb{E} \left[ \| D^k X_t^n \|_{H^{\otimes k}}^2 \right] < \infty, \quad k = 1, \ldots, m. \]

Next we prove \( X_t \in \mathbb{D}^{m,2} \) by the argument of [8, Lemma 1.2.3]. Indeed, by (3.6), there exists some subsequence \( D^k X_t^{n_j} \) weakly converges to some \( \alpha_k \) in \( L^2(\Omega, H^{\otimes k}) \) for \( k = 1, \ldots, m \). By (3.5) and the remark immediately below [8, Proposition 1.2.2], the projections of \( D^k X_t^{n_j} \) on any Wiener chaos converge in the weak topology of \( L^2(\Omega) \), as \( n_j \) tends to infinity, to those of \( \alpha_k \) for \( k = 1, \ldots, m \). Hence, \( X_t \in \mathbb{D}^{m,2} \) and \( D^k X_t = \alpha_k \) for \( k = 1, \ldots, m \). Moreover, for any weakly convergent subsequence the limit must be equal to \( \alpha_1, \ldots, \alpha_m \) by the same argument as above, and this implies the weak convergence of the whole sequence. \( \square \)

3.2. Additive noise case. If \( \sigma(x) \equiv \sigma \), then Eq. (3.1) reads as

\[ D_r X_t = \sigma + \int_0^t D_r b(X_s)ds + \alpha D_r \left( \sup_{0 \leq s \leq t} X_s \right). \]

Lemma 3.2. Let \( t > 0 \) be arbitrary and \( b \) be bounded smooth with \( \| b' \|_{\infty} < \infty \). For all \( 0 < t_1 < t_2 \leq t \), we have

\[ \| DX_{t_2} \|_H^2 - \| DX_{t_1} \|_H^2 \leq 2 \left[ \sqrt{2}\| b' \|_{\infty}^2 (t_2 - t_1)^2 + \| b' \|_{\infty}^2 (t_2 - t_1)^2 + 4\alpha^2 \right] \sup_{0 \leq s \leq t} \| DX_s \|_H^2. \]

Proof. It is easy to see that

\[ \| DX_{t_2} \|_H^2 - \| DX_{t_1} \|_H^2 = \int_0^{t_2} (D_r X_{t_2})^2 dr - \int_0^{t_1} (D_r X_{t_1})^2 dr \leq I_1 + I_2, \]

where

\[ I_1 := \int_{t_1}^{t_2} (D_r X_{t_2})^2 dr, \quad I_2 := \int_0^{t_1} \left| (D_r X_{t_2})^2 - (D_r X_{t_1})^2 \right| dr. \]
We claim that

\[
I_1 = \int_{t_1}^{t_2} (D_x X_{t_2} - D_x X_{t_1})^2 \, dr \leq \int_0^{t_2} (D_x X_{t_2} - D_x X_{t_1})^2 \, dr,
\]
by (3.8) we have

\[
I_1 \leq 2 \left[ \|b'\|^2_\infty (t_2 - t_1)^2 + 4\alpha^2 \right] \sup_{0 \leq s \leq t} \|DX_s\|^2_H.
\]

Further observe

\[
I_2 \leq \left[ \int_0^{t_1} (D_x X_{t_2} - D_x X_{t_1})^2 \, dr \right]^{1/2} \left[ \int_0^{t_1} |D_x X_{t_2} + D_x X_{t_1}|^2 \, dr \right]^{1/2}
\]
\[
\leq \sqrt{2} \left[ \int_0^{t_1} (D_x X_{t_2} - D_x X_{t_1})^2 \, dr \right]^{1/2} \left[ \int_0^{t_1} |D_x X_{t_2}|^2 + |D_x X_{t_1}|^2 \, dr \right]^{1/2}
\]
\[
\leq \sqrt{2} \left[ \int_0^{t_2} (D_x X_{t_2} - D_x X_{t_1})^2 \, dr \right]^{1/2} \left[ \int_0^{t_2} |D_x X_{t_2}|^2 \, dr + \int_0^{t_1} |D_x X_{t_1}|^2 \, dr \right]^{1/2}
\]
\[
\leq 2 \left[ \int_0^{t_1} (D_x X_{t_2} - D_x X_{t_1})^2 \, dr \right]^{1/2} \sup_{0 \leq s \leq t} \|DX_s\|_H
\]
\[
\leq 2 \left[ \int_0^{t_2} (D_x X_{t_2} - D_x X_{t_1})^2 \, dr \right]^{1/2} \sup_{0 \leq s \leq t} \|DX_s\|_H,
\]

this inequality and (3.8) gives

\[
I_2 \leq 2 \sqrt{2} \left[ \|b'\|^2_\infty (t_2 - t_1)^2 + 4\alpha^2 \right] \sup_{0 \leq s \leq t} \|DX_s\|^2_H.
\]

Combining the estimates of \(I_1\) and \(I_2\), we immediately get the desired inequality in the lemma.

It remains to prove (3.8). By (3.7), we have

\[
(D_x X_{t_2} - D_x X_{t_1})^2 \leq 2 \left[ \int_{t_1}^{t_2} D_x b(X_s) \, ds \right]^2 + 2\alpha^2 \left| D_x \left( \sup_{0 \leq s \leq t_1} X_s \right) - D_x \left( \sup_{0 \leq s \leq t_2} X_s \right) \right|^2
\]
\[
\leq 2 \left[ \int_{t_1}^{t_2} D_x b(X_s) \, ds \right]^2 + 4\alpha^2 \left| D_x \left( \sup_{0 \leq s \leq t_1} X_s \right) \right|^2 + 4\alpha^2 \left| D_x \left( \sup_{0 \leq s \leq t_2} X_s \right) \right|^2.
\]
By Hölder inequality, (3.2) and Proposition 1.1, we have
\[ \int_0^{t_2} \left( \int_{t_1}^{t_2} D_r b(X_s) ds \right)^2 dr \leq \|b'\|_\infty^2 \int_0^{t_2} (t_2 - t_1) \int_{t_1}^{t_2} |D_r X_s|^2 ds dr \]
\[ = \|b'\|_\infty^2 (t_2 - t_1) \int_{t_1}^{t_2} \int_0^s |D_r X_s|^2 dr ds \]
\[ \leq \|b'\|_\infty^2 (t_2 - t_1)^2 \sup_{0 \leq s \leq t} \|D X_s\|_H^2. \]
Moreover, by (3.3) and (3.2) we have
\[ \int_0^{t_2} \left( \sup_{0 \leq s \leq t_2} X_s \right)^2 dr \leq \sup_{0 \leq s \leq t} \|D X_s\|_H^2 \leq \sup_{0 \leq s \leq t} \|D X_s\|_H^2, \]
\[ \int_0^{t_2} \left( \sup_{0 \leq s \leq t_1} X_s \right)^2 dr = \int_0^{t_1} \left( \sup_{0 \leq s \leq t} X_s \right)^2 dr \leq \sup_{0 \leq s \leq t} \|D X_s\|_H^2. \]
Collecting the above four inequalities, we immediately get the desired (3.8). □

**Lemma 3.3.** Let \( b \) be bounded smooth with \( \|b'\|_\infty < \infty \), we have
\[ \sup_{0 \leq s \leq t} \|D X_s\|_H^2 \geq \frac{\sigma^2 t}{2(1 + 2\|b'\|_\infty^2 t^2 + 2\alpha^2)}, \quad t > 0. \]

**Proof.** By (3.7) and using \((a + b)^2 \geq \frac{1}{2}a^2 - b^2\), we have
\[ (D_r X_t)^2 \geq \frac{1}{2} \sigma^2 - \left[ \int_r^t D_r b(X_s) ds + \alpha D_r \left( \sup_{0 \leq s \leq t} X_s \right) \right]^2 \]
\[ \geq \frac{1}{2} \sigma^2 - 2 \left( \int_r^t D_r b(X_s) ds \right)^2 - 2\alpha^2 \left[ D_r \left( \sup_{0 \leq s \leq t} X_s \right) \right]^2. \]
Further observe
\[ \int_0^t \left( \int_r^t D_r b(X_s) ds \right)^2 dr \leq \int_0^t (t - r) \int_r^t |D_r b(X_s)|^2 ds dr \]
\[ \leq \int_0^t (t - r) \|b'\|_\infty^2 \int_r^t |D_r X_s|^2 ds dr \]
\[ \leq t \|b'\|_\infty^2 \int_0^t \int_r^t |D_r X_s|^2 ds dr \]
\[ = t \|b'\|_\infty^2 \int_0^t \|D X_s\|_H^2 ds \]
\[ \leq t^2 \|b'\|_\infty^2 \sup_{0 \leq s \leq t} \|D X_s\|_H^2, \]
where the second inequality is by Proposition 1.1. Hence,
\[ \| DX_t \|_H^2 \geq \frac{\sigma^2 t}{2} - 2\| b' \|_\infty^2 t^2 \sup_{0 \leq s \leq t} \| DX_s \|_H^2 - 2\| b' \|_\infty^2 t^2 \| DX_s \|_H^2, \]
where the last inequality is by (3.3).

This clearly implies
\[ \sup_{0 \leq s \leq t} \| DX_s \|_H^2 \geq \frac{\sigma^2 t}{2} - 2\| b' \|_\infty^2 t^2 \| DX_s \|_H^2, \]
which immediately yields the desired bound.

Lemma 3.4. Let \( b \) is bounded smooth with \( \| b' \|_\infty < \infty \) and \( \sigma(x) \equiv \sigma \) with \( \sigma \neq 0 \). If \( \alpha < 1, t_0 > 0 \) and \( b \) satisfy
\[ \theta(t_0, \alpha, b) < 1/2 \]
with \( \theta(r, \alpha, b) := \sqrt{2\| b' \|_\infty^2 r^2 + 8\alpha^2} + \| b' \|_\infty^2 r^2 + 4\alpha^2 \) for \( r > 0 \), then
\[ (3.12) \quad \| DX_t \|_H^2 \geq \frac{[1 - 2\theta(t, \alpha, b)]\sigma^2 t}{2(1 + 2\| b' \|_\infty^2 t^2 + 2\alpha^2)}, \quad t \in [0, t_0]. \]

Proof. Let \( t \in [0, t_0] \). For all \( 0 \leq t_1 \leq t_2 \leq t \), by Lemma 3.2, we have
\[ \| DX_{t_2} \|_H^2 - \| DX_{t_1} \|_H^2 \leq 2\theta(t_2 - t_1, \alpha, b) \sup_{0 \leq s \leq t} \| DX_s \|_H^2. \]

Hence, for all \( s \in [0, t] \),
\[ \| DX_s \|_H^2 \leq \| DX_t \|_H^2 - \| DX_t \|_H^2 + \| DX_s \|_H^2 \leq 2\theta(t - s, \alpha, b) \sup_{0 \leq s \leq t} \| DX_s \|_H^2, \]
and consequently
\[ \sup_{0 \leq s \leq t} \| DX_s \|_H^2 \leq 2\theta(t, \alpha, b) \sup_{0 \leq s \leq t} \| DX_s \|_H^2 + \| DX_t \|_H^2. \]

The above inequality and (3.10) further give
\[ \| DX_t \|_H^2 \geq [1 - 2\theta(t, \alpha, b)] \sup_{0 \leq s \leq t} \| DX_s \|_H^2 \geq [1 - 2\theta(t_0, \alpha, b)] \| DX_t \|_H^2. \]

Combining the above inequality and Lemma 3.3 immediately gives the desired inequality. \( \square \)
3.3. **Multiplicative noise case.** By the condition of \( \sigma \), we have \( \sup_{x \in \mathbb{R}} \sigma(x) < 0 \) or \( \inf_{x \in \mathbb{R}} \sigma(x) > 0 \). Without loss of generality, we assume that
\[
\inf_{x \in \mathbb{R}} \sigma(x) > 0.
\]
Let us consider the following well known transform
\[
F(X_t) = \int_{X_t}^{X_t} \frac{1}{\sigma(u)} du,
\]
which is easy to see that \( F \) is a strictly increasing function with bounded derivative. Hence,
\[
\sup_{0 \leq s \leq t} F(X_s) = F \left( \sup_{0 \leq s \leq t} X_s \right).
\]
By Itô formula, we have
\[
F(X_t) = \int_0^t \left( \frac{b(X_s)}{\sigma(X_s)} - \frac{1}{2} \sigma'(X_s) \right) ds + B_t + \alpha \int_0^t \frac{1}{\sigma(X_s)} dM_s
\]
where \( M_t = \sup_{0 \leq s \leq t} X_s \). It is easy to see that \( M_t \) is an increasing function of \( t \) and that \( \frac{1}{\sigma(X_s)} \) has a contribution to the related integral only when \( X_s = M_s \). Hence,
\[
F(X_t) = \int_0^t \left( \frac{b(X_s)}{\sigma(X_s)} - \frac{1}{2} \sigma'(X_s) \right) ds + B_t + \alpha \int_0^t \frac{1}{\sigma(M_s)} dM_s.
\]
Since \( M_t \) is a continuous increasing function with respect to \( t \), we have
\[
F(X_t) = \int_0^t \left( \frac{b(X_s)}{\sigma(X_s)} - \frac{1}{2} \sigma'(X_s) \right) ds + B_t + \alpha \sup_{0 \leq s \leq t} F(X_s).
\]
By (3.14),
\[
F(X_t) = \int_0^t \left( \frac{b(X_s)}{\sigma(X_s)} - \frac{1}{2} \sigma'(X_s) \right) ds + B_t + \alpha \sup_{0 \leq s \leq t} F(X_s).
\]
Denote \( Y_t = F(X_t) \), it solves the following perturbed SDE:
\[
Y_t = \int_0^t \hat{b}(Y_s) ds + B_t + \alpha \sup_{0 \leq s \leq t} Y_s
\]
where \( \hat{b}(x) = \frac{b(F^{-1}(x))}{\sigma(F^{-1}(x))} - \frac{1}{2} \sigma'(F^{-1}(x)) \). Applying Lemma 3.4, we get the following lemma about the dynamics \( Y_t \):

**Lemma 3.5.** Assume that \( b \) is bounded smooth and that \( \sigma \) is bounded smooth with \( \| \sigma' \|_{\infty} < \infty \), \( \| \sigma'' \|_{\infty} < \infty \) and \( \inf_{x \geq 0} |\sigma(x)| > 0 \). Then \( b \) is bounded smooth. If \( \alpha < 1 \), \( t_0 > 0 \) and \( b \) satisfy
\[
\theta(t_0, \alpha, \hat{b}) < 1/2
\]
with \( \theta(r, \alpha, \tilde{b}) := \sqrt{2\|\tilde{b}'\|_\infty^2 r^2 + 8\alpha^2 + \|\tilde{b}'\|_\infty^2 r^2 + 4\alpha^2} \) for \( r > 0 \), then

\[
\|DY_t\|_H^2 \geq \left[ 1 - 2\theta(t_0, \alpha, \tilde{b}) \right] t \quad t \in (0, t_0].
\]

**Proof.** It is easy to check that under the conditions in the lemma \( \tilde{b} \) is bounded smooth with \( \|\tilde{b}'\|_\infty < \infty \). Hence, the lemma immediately follows from applying Lemma 3.4 to \( Y_t \).

**REFERENCES**

1. Ph. Carmona, F. Petit, M. Yor, Beta variables as times spent in \([0, \infty)\) by certain perturbed Brownian motions, J. London Math. Soc. 58 (1998), 239-256.