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A neural network identification technique for a foil-air bearing under variable speed conditions and its application to unbalance response analysis

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Abstract

This paper proposes and studies the non-parametric system identification of a foil-air bearing (FAB). This research is motivated by two advantages: (a) it removes computational limitations by replacing the air film and foil structure equations by a displacement/force relationship; (b) it can capture complications that cannot be easily modelled, if the identification is based on empirical data. A Recurrent Neural Network (RNN) is trained to identify the full numerical model of a FAB over a wide range of speeds. The variable-speed RNN-FAB model is then successfully validated against benchmark results in two ways: (i) by subjecting it to different input data sets; (ii) by using it in the harmonic balance (HB) solution process for the unbalance response of a rotor-bearing system. In either case, the results from the identified variable-speed RNN maintain very good correlation with the benchmark over a wide range of speeds, in contrast to an earlier identified constant-speed RNN, demonstrating the great potential of this method in the absence of self-excitation effects.

Keywords: foil-air bearings; nonlinear vibration; system identification; neural networks; harmonic balance

Nomenclature

$(\)'$	differentiation with respect to τ
$\overline{(\)}$	mean value in Fourier series of $(\)$ (eqs. (19, 20))
$(\)_{\cos, \sin}^{(q)}$	sin, cos terms in Fourier series of $(\)$ (eqs. (19, 20))
$\widetilde{(\)}$	network-normalisation of $(\)$ (Figure 2)
A	amplitude of external excitation chirp signal (per bearing) (eqs. (12a,b))
$\mathbf{b}^{(j)}$	vector of bias values for RNN layer no. j (eqs. (16))
c	undeformed radial clearance of FAB
$\mathbf{f}$	nonlinear vector function eq. (22)
$\mathbf{g}^{(j)}(\)$	vector operator comprising neuron transfer functions (eqs. (16))
F_u, F_v	radial, tangential FAB forces (Figure 1)
F_x, F_y	Cartesian FAB forces (Figure 1)
$F_{\text{ext},x}, F_{\text{ext},y}$	external excitation (per bearing) for training (eqs. (12a,b))
$\check{F}_u, \check{F}_v$	RNN outputs for F_u, F_v
i, j	$i = 1 \dots N_z, j = 1 \dots N_\theta$ (FD grid counters)
j	$j = 1 \dots J$ (counter for RNN layers)
$\tilde{h}, \tilde{\tilde{h}}$	air film gap divided by c in $(\zeta, \theta), (\zeta, \phi)$ resp. (Figure 1)
$k_b, \tilde{k}_b$	dimensional, non-dimensional foil stiffness
L	axial length of FAB
m_r	half-mass of rotor
n_1, n_2, n_3	number of delays in respective RNN channels (Figures 2, 4)
N_z, N_θ	number of points of FD grid along ζ, θ directions
$p(\zeta, \theta)$	absolute air pressure at (ζ, θ) for FAB
$p_a, \tilde{p}$	atmospheric pressure, p/p_a resp.

q, Q	harmonic no., maximum harmonic no. resp. (eqs. (19, 20))
r_u	radius of mass unbalance
R	undeformed radius of FAB
$S^{(j)}$	number of neurons in j th layer (Figure 4)
t	time in seconds
u, v	radial, tangential axes (Figure 1)
$\mathbf{v}$	vector of unknowns in HB equation (23)
$\dot{\mathbf{v}}, \dot{\mathbf{v}}_0$	iterate of $\mathbf{v}$, initial approximation of $\mathbf{v}$
$\tilde{w}, \tilde{\tilde{w}}$	foil deflection divided by c in (ζ, θ) , (ζ, ϕ) resp.
$\tilde{w}_{i,j}$	$= \tilde{w}(\zeta_i, \theta_j, \tau)$
$\tilde{\mathbf{w}}$	vector of values of $\tilde{w}$ at discrete ζ, θ values
W	static load in $-y$ direction
$\mathbf{W}^{(j)}$	matrix of weights for RNN layer no. j (eqs. (16))
x, y	Cartesian displacements at FAB (Figure 1)
z	axial displacement from bearing mid-section
$\varepsilon_{x,y}$	$= x/c, y/c$
ε	$= \sqrt{\varepsilon_x^2 + \varepsilon_y^2}$
δ	network error (Figure 2)
Δ	time resolution for RNN (Figure 2)
$\Delta\zeta, \Delta\theta$	spacing of FD grid in ζ, θ directions
ζ	$= z/R$
ζ_i	discrete value of $\zeta, i = 1 \dots N_z$
θ	angular local bearing coordinate (Figure 1)
θ_j	discrete value of $\theta, j = 1 \dots N_\theta$
ϕ	angular coordinate measured from u axis

γ	attitude angle (rad)
Λ_{ref}	bearing number at reference speed, defined under eq. (2)
Λ	bearing number at actual speed
μ	air viscosity
η	damping loss factor of foil structure
$\psi(\zeta, \theta, \tau)$	$= \tilde{p}\tilde{h}$
$\psi_{i,j}$	$= \psi(\zeta_i, \theta_j, \tau)$
Ψ	vector of values of ψ at discrete ζ, θ values
$\hat{\psi}(\zeta, \phi, \tau)$	ψ in (ζ, ϕ) frame (Figure 1)
τ	non-dimensional time ($= \Omega_{\text{ref}}t/2$)
τ_k	$= (k - 1)\Delta, k = 1, 2, \dots$
$\tau_{\text{init}}, \tau_{\text{fin}}$	initial, final values for τ in frequency sweep
χ	$= \gamma' - \tilde{\Omega}$
Ω	rotational speed (rad/s)
Ω_{ref}	reference rotational speed (rad/s)
$\tilde{\Omega}$	$= \Omega/\Omega_{\text{ref}}$
ω	general circular frequency (rad/s)
ω_{ext}	circular frequency of chirp signal (eqs. (12))
$\tilde{\omega}, \tilde{\omega}_{\text{ext}}$	$= \omega/\Omega_{\text{ref}}, \omega_{\text{ext}}/\Omega_{\text{ref}}$

1. INTRODUCTION

A FAB (Figure 1) supports the shaft by means of a cushion of air bounded by a flexible foil structure. The introduction of the foil structure resolves the problems associated with the very tight radial clearance required by a plain air bearing. With a FAB, the hydrodynamic air film pressure generated as the shaft turns pushes the foil boundary away, allowing the shaft to become completely airborne. The dynamics of FAB turbomachinery are governed by the interaction between the turbomachine, air films and the foil structures and exhibit nonlinear effects [1].

Due to compressibility, the Reynolds Equation (RE) governing the air film is a *state* equation since it includes time as an independent variable [1, 2]. The use of Finite Difference (FD)/Finite Element (FE)/Control Volume methods [3-9] to discretize the RE over the air film, creates a grid of $N_z \times N_\theta$ points representing the pressure field, turning the RE into a set of $N_z \times N_\theta$ first order ordinary differential equations (ODEs) with time as the independent variable (state equations). As observed in [1, 2], due to the computational burden so introduced, the simultaneous solution of the state equations of the air film, foil and rotor has typically been avoided. In works such as [3-8] the air-film ODEs are uncoupled from the foil and rotor ODEs and approximated as algebraic equations; these latter equations were solved iteratively for the *current* pressure distribution using the rotor state variables at the *previous* time step [1, 2].

Recent research by the co-author [1, 2] has shown that an efficient simultaneous solution is possible in both the time and frequency domains. Two alternative transformations, respectively based on FD [1, 2] and Galerkin Reduction [2] were presented that preserved the state equations of the air films. These were solved simultaneously with the other state equations in the time domain using a readily available implicit integrator, wherein the necessary Jacobian matrix was efficiently calculated either through a vectorised formulation [2] or through symbolic computation [1]. The basic strategy for the simultaneous time-domain solution introduced by the co-author in [2] has been successfully adopted by independent researchers [10] (without computation-accelerating features like Galerkin, vectorisation, or symbolic computing) and has also given good correlation with experimental results [10, 1].

In contrast to time domain integration, a frequency domain method like Harmonic Balance (HB) is restricted to a periodic steady-state response [2]. However, it is much faster since it calculates it directly, in contrast to the time domain integration, which has to integrate step-by-step past the transient stage to arrive to the steady-state response. In [2], the HB method was developed for the simultaneous solution of the FAB-rotor system equations and its efficacy was successfully demonstrated. The unknowns of the HB problem comprise the Fourier (harmonic) coefficients of the state variables and (in the case of the self-excitation problem) the fundamental frequency. For the FD grid size and number of harmonics used in [2], the number of unknowns was in excess of 7000 but the solution could be achieved in a fraction of a second on a standard platform using a suitable predictor-corrector scheme [11, 12]. However, it was noted in [2] that the use of a finer FD grid with the same number of harmonics was not possible on the platform used since the memory could not accommodate the sheer size of the resulting Jacobian.

The main contribution of the present paper is the nonlinear identification of a foil-air bearing (FAB). The identification of a system involves the extraction of its input/output relationship and, in the case of the FAB, this necessitates a technique that accounts for system memory. By applying the identification technique to the full numerical model of the FAB, the air-film and foil structure equations are effectively replaced by an input (journal displacement/velocity) - output (bearing force) relationship. The use of the identified model in lieu of the full numerical FAB model within the HB solution procedure for a rotor-bearing problem will therefore avoid the computational problem with HB highlighted in the previous paragraph. More significantly, the proof of principle of the identification approach could potentially pave the way for capturing complications that cannot easily be modelled, by basing the identification on experimental, rather than numerical, training data.

The preferred identification strategy for capturing nonlinearities and other complexities in bearings is non-parametric (“black-box”) since it makes no assumption on the nature of the underlying relationship between input and output, unlike parametric identification [13]. Chebyshev polynomial interpolation [14-16] and

neural networks (NNs) [13, 17] fall in this category. Identification by Chebyshev polynomials has been shown in [14-16] to faithfully replicate the full numerical model of a squeeze-film bearing, thereby allowing its substitution by such polynomials in a rotordynamic calculation, bringing computational benefits. NNs have been shown in [13] to be able to capture features that are not easy to model (like geometric irregularities in squeeze-film bearings) when the identification is based on experimental training data. It should be noted however, that, in contrast to parametric identification, one cannot attach any physical meaning to the constants of the relationship extracted in non-parametric identification (coefficients, in case of Chebyshev, and weights/biases, in the case of NNs).

The proposed non-parametric identification of FABs would be fundamentally different from that of oil bearings (like squeeze-film damper bearings, performed in [14-16]). The latter systems are “memoryless” since the instantaneous bearing force is a function of the *instantaneous* journal displacement/velocity inputs – this should be evident from the incompressible Reynolds Equation (RE) [14]. Hence, identification of oil bearings could be performed using either Chebyshev polynomials [14-16] or *static* NNs [17]. In contrast, FABs have memory since they are governed by the state equations of the air film and the foil, respectively introduced by the compressible RE and the foil structure damping. This means that the output (force) of the system to be identified depends on the *time history* of the journal inputs and the output. Hence, Chebyshev interpolation is not suitable and the neural network (NN) needs to be dynamic. A dynamic NN (Recurrent Neural Network, RNN) has been used successfully by the co-author to identify a system with memory (a magnetorheological damper, where the force output is governed by state equations that are subjected to two input time histories: displacement and voltage) [13].

The co-author’s previous work on HB [2] was focused on self-excited vibration. The application of the identification work in the present paper is the solution of the unbalance response by HB in the absence of self-excitation and sub-synchronous activity. In this respect, this application is similar to the work of Li and Flowers [18], which is the only other HB analysis of FAB rotor systems known to the authors. However, the work of Li and Flowers completely disregarded the air film, their identified bearing model being a simple

polynomial force-deflection relation based on the foil structure only. Moreover, in [18], for the purpose of the HB analysis, the bearing forces in the x, y directions were decoupled and the shaft orbits were assumed to be circular and centred within the bearing housing. No such assumptions are made in the present paper. It is noted that identified polynomial force-deflection bearing models based on the foil only were also used more recently in [18, 19] and applied to time domain integration analysis. While it is recognised that such neglect of the air film can be justified for certain conditions (particularly high rotor speeds) [19], it is clearly not universally applicable, even at high rotor speeds, as recognised in [20].

The preliminary version of this paper's research was presented by the authors in a conference paper [21]. In that paper, a pair of RNNs (for the radial and tangential bearing forces respectively) was trained at a *single* rotational speed (i.e. for a *fixed bearing number*) from the full numerical (FD) model of the FAB. These RNNs were then validated over a narrow range of rotational speeds and a reasonable degree of correlation with the benchmark solution was achieved. Nonetheless, significant discrepancies were observed, even over the narrow speed range considered, and these were attributed to the variation of the bearing number with speed. In order to apply the RNNs over a wide range of speeds, it was therefore suggested in the conclusions of [21] to divide the speed range into short bandwidths and train separate pairs of networks for each. This is a rather clumsy solution and the authors have since developed a technique that is capable of identifying a *single* pair of networks that is effective over a wide range of speeds. It is these variable-speed RNNs that are presented in the present paper and the basis for their identification is established by studying the theoretical model of the FAB (section 2). The identification itself is done in section 3 and the methodology for applying the trained RNNs to a rotor-bearing problem is presented in section 4. In section 5, the effectiveness of the variable-speed RNNs relative to the constant-speed RNNs of [21] is demonstrated over a much wider speed range than that used in [21]. Apart from this significant improvement, the present journal paper contains additional details that were not included in the conference publication [21] (e.g. regarding the network architecture, the FD discretization procedure for the generation of data generation used in training and validation, the effect of the static load).

2. BASIS FOR NON-PARAMETRIC IDENTIFICATION

Despite non-parametric identification being “black-box”, it is still necessary to make an informed choice of the controlling inputs for the specified outputs of the system to be identified (i.e. the FAB). This is particularly important in view of the fundamental difference between the present identification task and that previously performed on oil bearings, as discussed above, and to devise an identification strategy that is effective over a range of speeds. The basis for the identification can be established by studying the theoretical model of the FAB. In the following theoretical presentation, the non-dimensional time τ is defined as

$$\tau = \Omega_{\text{ref}} t / 2 \quad (1)$$

where t is the dimensional time and Ω_{ref} (rad/s) is an arbitrarily chosen fixed reference rotational speed. This is in contrast with the authors’ preliminary study [21], where the actual rotational speed Ω was used instead of Ω_{ref} . This change served to devise an identification strategy that was effective over a wide range of speeds. It is noted that the introduction of the 2 in the definition of τ is non-essential and merely follows previous practice by the co-author [1, 2, 21] to present the Reynolds Equation (RE) in a neat format.

For a FAB of radius R and length L (Figure 1), letting $p(\zeta, \theta, \tau)$ denote the air film pressure (absolute), where $\zeta = \frac{z}{R}$, and denoting $\partial(\quad)/\partial\tau$ as $(\quad)'$, the pressure function is governed by the isothermal RE:

$$\psi' = \frac{1}{\Lambda_{\text{ref}}} \left\{ \frac{\partial}{\partial\theta} \left[\psi \left(\tilde{h} \frac{\partial\psi}{\partial\theta} - \psi \frac{\partial\tilde{h}}{\partial\theta} \right) \right] + \frac{\partial}{\partial\zeta} \left[\psi \left(\tilde{h} \frac{\partial\psi}{\partial\zeta} - \psi \frac{\partial\tilde{h}}{\partial\zeta} \right) \right] \right\} - \tilde{\Omega} \frac{\partial\psi}{\partial\theta} \quad (2)$$

In eq. (2), $\Lambda_{\text{ref}} = \frac{6\mu\Omega_{\text{ref}}}{p_a} \left(\frac{R}{c} \right)^2$ is the bearing number at the reference rotational speed where μ is the air viscosity, c is the radial clearance with no foil deflection and p_a the atmospheric pressure. $\tilde{\Omega} = \Omega/\Omega_{\text{ref}}$ is the non-dimensional rotational speed. The actual bearing number in eq. (2) (i.e. that at the actual rotational

speed Ω) is therefore $\Lambda = \tilde{\Omega}\Lambda_{\text{ref}}$. The dependent variable in eq. (2) is $\psi = \tilde{p}\tilde{h}$ where $\tilde{p} = \frac{p}{p_a}$ and $\tilde{h}$ the non-dimensional air-film gap at a position (ζ, θ) :

$$\tilde{h} = 1 - \varepsilon_x \cos \theta - \varepsilon_y \sin \theta + \tilde{w} \quad (2a)$$

In equation (2a): $\varepsilon_{x,y} = x/c, y/c$ are the non-dimensional Cartesian displacements of J relative to B (Figure 1) (B may be dynamic); $\tilde{w} = \frac{w}{c}$ is the non-dimensional foil deflection in the radial direction at a position (ζ, θ) . This latter variable would be governed by the foil-structure model. If, for illustrative purposes, a simple localized spring-damper model is used, as in [6, 7]:

$$\frac{d\tilde{w}}{d\tau} = \frac{2\tilde{\omega}}{\eta} \left\{ \frac{1}{\tilde{k}_b} \left(\frac{\psi}{\tilde{h}} - 1 \right) - \tilde{w} \right\} \quad (3)$$

where: $\tilde{k}_b = k_b c / p_a$ is the non-dimensional form of k_b , the stiffness per unit area of the foil structure (N/m^3) and the equivalent viscous damping is $k_b \eta / \omega$ (Ns/m) where η is the hysteretic damping loss factor, ω is the frequency of the vibration and $\tilde{\omega} = \omega / \Omega_{\text{ref}}$.

The Cartesian bearing forces are determined by the integration of the air film pressure distribution in FAB:

$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = -p_a R^2 \int_{\zeta=0}^{L/R} \int_{\theta=0}^{2\pi} \left(\frac{\psi}{\tilde{h}} - 1 \right) \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} d\theta d\zeta \quad (4)$$

Let ψ and $\tilde{w}$ be discretised over the spatial domain:

$$\boldsymbol{\Psi}(\tau) = [\cdots \quad \psi_{i,j}(\tau) \quad \cdots]^T, \quad \tilde{\mathbf{w}}(\tau) = [\cdots \quad \tilde{w}_{i,j}(\tau) \quad \cdots]^T \quad (5a,b)$$

where $\psi_{i,j}(\tau) = \psi(\zeta_i, \theta_j, \tau)$, $\tilde{w}_{i,j}(\tau) = \tilde{w}(\zeta_i, \theta_j, \tau)$, $i = 1, \dots, N_z$, $\theta = \theta_j$, $j = 1, \dots, N_\theta$; if FD (for example) is used to approximate the partial derivatives with respect to (ζ, θ) , then eqs. (2, 3) result in a system of first order time-based ODEs (state equations) with $\boldsymbol{\psi}$, $\tilde{\mathbf{w}}$ as the state variables [1, 2] (see Appendix A). Hence, eqs. (2, 3) can be solved for prescribed *input time histories* $\varepsilon_x(\tau)$, $\varepsilon_y(\tau)$ (eq. (2a)) and $\tilde{\Omega}(\tau)$ (eq. (2)), and given initial conditions of the state variables $\boldsymbol{\psi}$, $\tilde{\mathbf{w}}$, to determine the *outputs* F_x , F_y at time τ . The *memory* of the system is the result of the dependence of the output on the time histories of the inputs and the initial conditions of the state variables.

Notice that $\tilde{\omega}$ in eq. (3) need not be regarded as a controlling input since it is reflected in the time histories of $\varepsilon_x(\tau)$, $\varepsilon_y(\tau)$. In any case, η is only defined for harmonic vibration. Hence, if $\varepsilon_x(\tau)$, $\varepsilon_y(\tau)$ are not harmonic functions, eq. (3) would be need to be approximated (for the determination of the arbitrary vibration response in the time domain) by basing the equivalent viscous damping coefficient on the fundamental frequency of the time histories of $\varepsilon_x(\tau)$, $\varepsilon_y(\tau)$, assumed to be periodic. Also, frequency will not appear in the foil model if an alternative damping model (that does not use the hysteretic loss factor e.g. Coulomb [3]) is used.

From the above, the number of controlling inputs can be reduced from three to two by omitting the non-dimensional rotational speed $\tilde{\Omega}$ under the following conditions:

- a) Rotational speed is kept constant – this effectively means that the air film equation (2) is approximated using a constant bearing number $\Lambda = \tilde{\Omega}\Lambda_{\text{ref}}$, as done in [21].
- b) Rotational speed can be variable, *provided that it is reflected in the time histories of $\varepsilon_x(\tau)$, $\varepsilon_y(\tau)$.*

Condition (b) prevails when the $\varepsilon_x(\tau)$, $\varepsilon_y(\tau)$ are driven by rotational unbalance without any self-excitation effects (i.e. their fundamental frequency is Ω). It is noted that in [21], an approach based on condition (a) was taken even though $\varepsilon_x(\tau)$, $\varepsilon_y(\tau)$ were driven by rotational unbalance without any self-excitation effects.

In the present paper, an approach based on condition (b) is taken. As can be seen from the following analysis, it is possible to redefine the two remaining inputs to the system so as to relax, to a certain extent, the proviso on condition (b), making the system to be identified (i.e. the FAB) more robust to variation in the rotational speed. To achieve this, it is necessary to transform to the radial, tangential (u, v) coordinate system, which rotates according to the attitude angle γ and has a spatial domain based on (ζ, ϕ) rather than (ζ, θ) (see Figure 1):

$$\psi(\zeta, \theta, \tau) \equiv \hat{\psi}(\zeta, \phi, \tau), \tilde{w}(\zeta, \theta, \tau) \equiv \tilde{w}(\zeta, \phi, \tau), \tilde{h}(\zeta, \theta, \tau) \equiv \tilde{h}(\zeta, \phi, \tau), \text{ where } \phi = \theta - \pi/2 - \gamma$$

Eqs. (2, 2a, 3) then become:

$$\hat{\psi}' = \frac{1}{\Lambda_{\text{ref}}} \left\{ \frac{\partial}{\partial \phi} \left[\hat{\psi} \left(\tilde{h} \frac{\partial \hat{\psi}}{\partial \phi} - \hat{\psi} \frac{\partial \tilde{h}}{\partial \phi} \right) \right] + \frac{\partial}{\partial \zeta} \left[\hat{\psi} \left(\tilde{h} \frac{\partial \hat{\psi}}{\partial \zeta} - \hat{\psi} \frac{\partial \tilde{h}}{\partial \zeta} \right) \right] \right\} + \chi \frac{\partial \hat{\psi}}{\partial \phi} \quad (6)$$

where

$$\tilde{h} = 1 + \varepsilon \cos \phi + \tilde{w} \quad (6a)$$

$$\chi = \gamma' - \tilde{\Omega} \quad (6b)$$

$$\tilde{w}' = \frac{2\tilde{\omega}}{\eta} \left\{ \frac{1}{\tilde{k}_b} \left(\frac{\hat{\psi}}{\tilde{h}} - 1 \right) - \tilde{w} \right\} + \gamma' \frac{\partial \tilde{w}}{\partial \phi} \quad (7)$$

where:

$$\varepsilon = \sqrt{\varepsilon_x^2 + \varepsilon_y^2} \quad , \quad \gamma' = (\varepsilon_x \varepsilon_y' - \varepsilon_y \varepsilon_x') / \varepsilon^2 \quad (8a,b)$$

The system outputs are now the radial and tangential components of the bearing forces (F_u, F_v) , which are given by:

$$\begin{bmatrix} F_u \\ F_v \end{bmatrix} = -p_a R^2 \int_{\zeta=0}^{L/R} \int_{\theta=0}^{2\pi} \left(\frac{\hat{\psi}}{\tilde{h}} - 1 \right) \begin{bmatrix} \cos \phi \\ \sin \phi \end{bmatrix} d\phi d\zeta \quad (9)$$

$\hat{\psi}$ and $\tilde{\omega}$ can be similarly discretised over the spatial domain (ζ, ϕ) , and the application of FD will transform eqs. (6, 7) into state equations.

In the limiting case $\tilde{k}_b \rightarrow \infty$ (i.e. a plain air bearing without a flexible foil boundary), eq. (7) disappears and $\tilde{\omega} \rightarrow 0$ in eq. (6a). In this scenario, from eq. (6), the controlling inputs can be regarded to be the time histories $\varepsilon(\tau)$ and $\chi(\tau)$ (defined by eq. (6b)) *regardless* of the rotational speed Ω and without the need for the proviso in condition (b). In the case where $\tilde{k}_b$ is finite (i.e. FAB), $\varepsilon(\tau)$ and $\chi(\tau)$ are not enough to solve the system since one would also need to know $\tilde{\Omega}(\tau)$ to determine $\gamma'(\tau)$ (from eq. (6b)) to use in eq. (7). Hence, the proviso in condition (b) is still needed to maintain the number of inputs at two. However, it is clear that redefining the inputs as $\varepsilon(\tau)$ and $\chi(\tau)$ will make the system to be identified generally less dependent on variations in Ω .

It is noted that, in the authors' preliminary work, although the (u, v) coordinate system was used, τ was defined as $\tau = \Omega t/2$ rather than $\tau = \Omega_{\text{ref}} t/2$ i.e. in eq. (1), $\Omega = \Omega_{\text{ref}} (\Rightarrow \Lambda_{\text{ref}} = \Lambda \text{ and } \tilde{\Omega} = 1)$. Hence, in [21], the controlling inputs of the identified system were $\varepsilon(\tau)$, $\gamma'(\tau)$ for a *fixed* rotational speed Ω (condition (a)), limiting its applicability.

3. NEURAL NETWORK IDENTIFICATION

Based on the study above, the proposed non-parametric model will be a Recurrent Neural network (RNN), which is an implementation of the Nonlinear Autoregressive Exogenous (NARX) identification scheme used in time-series modelling [22]. The schematic diagram of the network for F_u , and the scheme used to train it, is illustrated in Figure 2 (a similar separate networks is used for F_v). The RNN's *external* inputs will be the signals $\varepsilon(\tau_k)$ and $\chi(\tau_k) = \gamma'(\tau_k) - \tilde{\Omega}(\tau_k)$, where τ_k denotes discrete times $\tau = (k - 1)\Delta$ where $k = 1, 2, \dots$ and Δ is the sampling time resolution. This input is normalized for network use to a range of $[-1, 1]$ [22] (where $\widetilde{(\quad)}$ denotes the network-normalised version of the variable $(\quad)$). The network's output

estimate for $F_u(\tau_k)$ is denoted by $\tilde{F}_u(\tau_k)$, and its network-normalised form $\hat{\tilde{F}}_u(\tau_k)$ is internally fed back to the RNN. The system memory is therefore accounted for by the fact that the net has internal feedback and is *dynamic* (i.e. contains multiple delays of the signals in the Tapped Delay Lines (TDLs)). The delayed versions of $\tilde{\varepsilon}(\tau_k)$, $\tilde{\chi}(\tau_k)$, $\hat{\tilde{F}}_u(\tau_k)$ constitute the internal input to the network, where TDL no.1 is a multi-channel line carrying signals $\tilde{\varepsilon}(\tau_{k-1})$, $\tilde{\varepsilon}(\tau_{k-2})$, ..., $\tilde{\varepsilon}(\tau_{k-n_1})$ in the respective n_1 channels, TDL no. 2 carries signals $\tilde{\chi}(\tau_{k-1})$, $\tilde{\chi}(\tau_{k-2})$, ..., $\tilde{\chi}(\tau_{k-n_2})$ in the respective n_2 channels,.....etc.

Figure 2 also illustrates the identification (training) procedure – the parameters of a given assumed network architecture are optimized to minimise the difference (error) δ between the true (target) output $\hat{\tilde{F}}_u(\tau_k)$ and the network output $\tilde{F}_u(\tau_k)$ when the RNN is presented with experimentally-obtained input/output training data. This is discussed further in the following two subsections.

3.1 Training Data Generation

With reference to Figure 2, the identification of the RNN requires training data sets comprising the time history of the true output for given time histories of the inputs. These data sets can be generated in two possible ways, depending on whether the FAB being identified is a numerical model or actual hardware:

(i) Bearing in isolation – *prescribing* $\varepsilon_x(\tau)$, $\varepsilon_y(\tau)$ and $\Omega(\tau)$ and determining the resulting FAB force signals

$$F_x(\tau), F_y(\tau);$$

(ii) Bearing as part of a rotor system (“test rig”) – applying excitation forces to the rotating system over a range of rotational speeds and determining the resulting journal state variable time histories $\varepsilon_x(\tau)$, $\varepsilon_y(\tau)$ and the corresponding FAB force signals $F_x(\tau)$, $F_y(\tau)$.

In either case, the input signals to the identified model, $\varepsilon(\tau)$, $\chi(\tau)$, can be determined from eqs. (8a), (8b), (6b) and the outputs $F_u(\tau)$, $F_v(\tau)$ determined using the transformation:

$$\begin{bmatrix} F_u \\ F_v \end{bmatrix} = \begin{bmatrix} -\sin \gamma & \cos \gamma \\ -\cos \gamma & -\sin \gamma \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix} \quad (10)$$

where $\gamma(\tau)$ is determined by $\varepsilon_x(\tau)$, $\varepsilon_y(\tau)$.

Method (i) can only be used for the identification of a numerical model (by solving equations (2), (3) for the prescribed inputs). Method (ii) can be used for both numerical and experimental identification. It is this latter method that is used in this paper for the identification of the numerical model of a FAB. The rotor system used for both training data generation and validation of the identification is shown in Figure 3.

The equations of the rotor system in Figure 3 are given by:

$$\varepsilon_x'' = 4 \frac{F_x + F_{\text{ext},x}}{m_r c \Omega_{\text{ref}}^2} \quad (11a)$$

$$\varepsilon_y'' = 4 \frac{F_y + F_{\text{ext},y} - W}{m_r c \Omega_{\text{ref}}^2} \quad (11b)$$

where m_r is the mass of one symmetric half of the rotor, and W is the static load per bearing in the $-y$ direction.

In this procedure the external dynamic forces $F_{\text{ext},x}$, $F_{\text{ext},y}$ (per bearing) were not unbalance forces but orthogonally-phased chirp signals with *constant* amplitude A and steadily increasing frequency $\omega_{\text{ext}}(\tau)$:

$$F_{\text{ext},x} = A \sin \left\{ 2\tilde{\omega}_{\text{ext}}(\tau_{\text{init}})\tau + \left(\frac{d\tilde{\omega}_{\text{ext}}}{d\tau} \right) \tau^2 \right\} \quad (12a)$$

$$F_{\text{ext},y} = A \cos \left\{ 2\tilde{\omega}_{\text{ext}}(\tau_{\text{init}})\tau + \left(\frac{d\tilde{\omega}_{\text{ext}}}{d\tau} \right) \tau^2 \right\} \quad (12b)$$

where $\tilde{\omega}_{\text{ext}} = \omega_{\text{ext}}/\Omega_{\text{ref}}$ and:

$$\frac{d\tilde{\omega}_{\text{ext}}}{d\tau} = \frac{\tilde{\omega}_{\text{ext}}(\tau_{\text{fin}}) - \tilde{\omega}_{\text{ext}}(\tau_{\text{init}})}{\tau_{\text{fin}} - \tau_{\text{init}}} \quad (13)$$

where τ_{init} , τ_{fin} are the non-dimensional start and end times of the frequency sweep. The rotational speed Ω was set to be equal to the external excitation frequency:

$$\Omega(\tau) = \omega_{\text{ext}}(\tau) \quad (14)$$

i.e. the rotational speed *was variable over the duration of the chirp signal* (unlike the preliminary work in [21]) covering the operational range of applicability of the RNNs to be trained.

For prescribed $F_{\text{ext},x}$, $F_{\text{ext},y}$, the input/output training data sets $\{\varepsilon(\tau), \chi(\tau)\} / \{F_u(\tau), F_v(\tau)\}$ were generated by solving equations (2), (3) and (11a,b) for $\varepsilon_x(\tau)$, $\varepsilon_y(\tau)$ and $F_x(\tau)$, $F_y(\tau)$. FD discretisation was used to approximate the partial derivatives with respect to (ζ, θ) ; the resulting state equations were then solved by numerical integration in the time domain using an implicit integrator [1, 2]. It is noted that, in this work, unlike [1, 2], the foil deflection in eq. (3) was allowed to vary in the ζ direction, rather than basing it on an average pressure in the ζ direction. The FD discretisation procedure to obtain the state equations, which was not presented in [1, 2], is therefore presented in Appendix A of this paper.

Since the journal displacement time history was not prescribed (being the result of the interaction of the rotor eqs. (11a,b), with the FAB eqs. (2) and (3)), $\tilde{\omega}$ in eq. (3) was set to the non-dimensional excitation frequency $\tilde{\omega}_{\text{ext}}$ (i.e. a slowly varying equivalent viscous damping coefficient $k_b\eta/\omega_{\text{ext}}$ was assumed).

The above procedure was applied to the bearing with parameters as in Table 1 (used in other works e.g. [2, 6, 7]). The half-mass m_r of the rotor system (eqs. (11a,b)) was 3.061 kg. The static load was set to $W = 1.5m_rg$ since this ensured stable operation of the FAB (i.e. no self-excitation effects) over a much wider operational speed range than $W = m_rg$ (which was used in the authors' preliminary work [21] and had resulted in a much more limited testing regime for the RNN). The frequency sweep (eq. (13)) was performed over an interval of 0 to 63 seconds and covered frequencies ω_{ext} in the range 166.67 Hz to 400 Hz. From eq. (14), the rotational speed Ω was simultaneously varied from 10,000 rpm (166.67 rev/s) to

24,000 rpm (400 rev/s). The amplitude A of the excitation was set at 1.5 N. The constant value of Ω_{ref} was arbitrarily set to 17,000 rpm (its value is immaterial).

3.2 RNN Architecture and Training

The generic architecture of the RNN enclosed within the dashed box in Figure 2 is shown in Figure 4. With reference to Figure 4, if $S^{(j)}$ is the number of neurons in the j^{th} layer, and J the total number of layers, then $S^{(J)} = 1$ since the output layer (layer no. J) has the 1×1 signal $\tilde{F}_u(t_k)$ as its output. Let $\mathbf{p}_k$ be the $(n_1 + n_2 + n_3) \times 1$ column matrix comprising the signal inputs to layer no. 1:

$$\mathbf{p}_k = \left[\tilde{\varepsilon}(\tau_{k-1}) \quad \cdots \quad \tilde{\varepsilon}(\tau_{k-n_1}) \quad \tilde{\chi}(\tau_{k-1}) \quad \cdots \quad \tilde{\chi}(\tau_{k-n_2}) \quad \tilde{F}_u(\tau_{k-1}) \quad \cdots \quad \tilde{F}_u(\tau_{k-n_3}) \right]^T \quad (15)$$

where n_1, n_2, n_3 are the numbers of delays in the respective TDLs. If $\mathbf{a}_k^{(j)}$ is the $S^{(j)} \times 1$ vector comprising the signal outputs of the j^{th} layer, then [13]:

$$\mathbf{a}_k^{(1)} = \mathbf{g}^{(1)}(\mathbf{W}^{(1)}\mathbf{p}_k + \mathbf{b}^{(1)}) \quad (16a)$$

$$\mathbf{a}_k^{(j)} = \mathbf{g}^{(j)}(\mathbf{W}^{(j)}\mathbf{a}_k^{(j-1)} + \mathbf{b}^{(j)}), \quad j = 2, \dots, J \quad (16b)$$

where $\mathbf{W}^{(j)}$ and $\mathbf{b}^{(j)}$ are respectively the matrix of weights and vector of biases of the j^{th} layer and $\mathbf{g}^{(j)}(\cdot)$ is a vector operator comprising the transfer functions of the neurons of the j^{th} layer. The network output $\mathbf{a}_k^{(J)} = \tilde{F}_u(t_k)$. It is also noted that the network is always initialised such that, in $\mathbf{p}_k$ (eq. (15)):

$$\tilde{\varepsilon}(\tau_{k-n}), \tilde{\chi}(\tau_{k-n}), \tilde{F}_u(\tau_{k-n}) = 0, \quad \tau_{k-n} < \tau_1 \quad (17a,b,c)$$

where n is an arbitrary positive integer.

With reference to Figures 2 and 4, for the radial force (F_u), the RNN had four layers with 5, 4, 2, 1 neurons respectively and $n_1 = 4$ and $n_2 = 4$ delays for the external inputs and $n_3 = 4$ delays for the feedback. For the tangential force (F_v), the RNN had three layers with 5, 4, 1 neurons respectively and $n_1 = 3$ and

$n_2 = 3$ delays for the external inputs and $n_3 = 3$ delays for the feedback. In both cases, tangent-sigmoid transfer functions [22] were used for the neurons, except the one in the final (output) layer, which was a linear transfer function.

The neural networks were implemented in a general purpose technical computing software that has a neural network facility [22]. With reference to the training procedure in Figure 2, the Levenberg-Marquardt optimization method [22] was used to optimize the weights and biases of the network, the goal of the optimization being the convergence of the mean square value of the error δ (MSE) to a value below a set tolerance specification. It is noted that the above described network architectures were chosen after various options were considered – the criterion for choosing one option over another being the rapidity of convergence of the MSE towards the set tolerance.

4. RNN-FAB APPLICATION TO HB ANALYSIS

As part of the validation process, the RNN-identified FAB model is used to predict the rotational unbalance response of the system in Figure 3 using the Harmonic Balance (HB) analysis (i.e. the RNN model in Figure 2 replaces eqs. (2) and (3)). Adapting the rotor equations (11a,b) for this problem gives

$$\varepsilon_x'' = 4 \frac{F_x + m_r r_u \Omega^2 \sin(2\tilde{\Omega}\tau)}{m_r c \Omega_{\text{ref}}^2} \quad (18a)$$

$$\varepsilon_y'' = 4 \frac{F_y - m_r r_u \Omega^2 \cos(2\tilde{\Omega}\tau) - W}{m_r c \Omega_{\text{ref}}^2} \quad (18b)$$

where the external dynamic forces $F_{\text{ext},x}$, $F_{\text{ext},y}$ in eqs. (11a,b) have been replaced by unbalance forces, r_u being the unbalance eccentricity and the external static load is $W = 1.5m_r g$, as before. Assuming periodic vibration with fundamental frequency equal to that of the unbalance excitation (Ω), the response can be represented by a Fourier series:

$$\varepsilon_x = \bar{\varepsilon}_x + \sum_{q=1}^Q \varepsilon_{x\cos}^{(q)} \cos 2\tilde{\Omega}q\tau + \varepsilon_{x\sin}^{(q)} \sin 2\tilde{\Omega}q\tau \quad (19a)$$

$$\varepsilon_y = \bar{\varepsilon}_y + \sum_{q=1}^Q \varepsilon_{y\cos}^{(q)} \cos 2\tilde{\Omega}q\tau + \varepsilon_{y\sin}^{(q)} \sin 2\tilde{\Omega}q\tau \quad (19b)$$

Likewise, the bearing forces can be represented by a Fourier series:

$$F_x = \bar{F}_x + \sum_{q=1}^Q \left(F_{x\cos}^{(q)} \cos 2\tilde{\Omega}q\tau + F_{x\sin}^{(q)} \sin 2\tilde{\Omega}q\tau \right) \quad (20a)$$

$$F_y = \bar{F}_y + \sum_{q=1}^Q \left(F_{y\cos}^{(q)} \cos 2\tilde{\Omega}q\tau + F_{y\sin}^{(q)} \sin 2\tilde{\Omega}q\tau \right) \quad (20b)$$

and the Fourier coefficients in eqs. (20a,b) will be nonlinear functions of the Fourier coefficients of the journal displacements in (18a,b). Neglecting non-synchronous harmonics, the HB equations are given by:

$$\bar{F}_x = 0 \quad (21a)$$

$$\bar{F}_y - W = 0 \quad (21b)$$

$$-\varepsilon_{x\cos}^{(1)} - \frac{1}{m_r c \Omega^2} F_{x\cos}^{(1)} = 0 \quad (21c)$$

$$-\varepsilon_{y\cos}^{(1)} - \frac{1}{m_r c \Omega^2} F_{y\cos}^{(1)} + \frac{r_u}{c} = 0 \quad (21d)$$

$$-\varepsilon_{x\sin}^{(1)} - \frac{1}{m_r c \Omega^2} F_{x\sin}^{(1)} - \frac{r_u}{c} = 0 \quad (21e)$$

$$-\varepsilon_{y\sin}^{(1)} - \frac{1}{m_r c \Omega^2} F_{y\sin}^{(1)} = 0 \quad (21f)$$

These nonlinear algebraic equations can be expressed as

$$\mathbf{f}(\mathbf{v}) = \mathbf{0} \quad (22)$$

where $\mathbf{f}$ is a nonlinear vector function of the vector $\mathbf{v}$:

$$\mathbf{v} = \left[\bar{\varepsilon}_x \quad \bar{\varepsilon}_y \quad \varepsilon_{x\cos}^{(1)} \quad \varepsilon_{y\cos}^{(1)} \quad \varepsilon_{x\sin}^{(1)} \quad \varepsilon_{y\sin}^{(1)} \right]^T \quad (23)$$

The system of equations (22) can then be solved for each individual rotational speed Ω over a range speeds to yield a set of solutions defining a “speed response curve”. As in [2], the continuation technique used to

advance the solution procedure along the curve used a predictor-corrector approach where the initial approximation $\dot{\mathbf{v}}_0$ (or “predictor”) for the solution $\mathbf{v}$ at a point on the curve was obtained from the previous point. Equation (15) was then solved by the damped Newton-Raphson method (the “corrector”) [12]. The initial approximation for the first point on the curve (only) was provided by the Fourier coefficients of a time-domain solution of eqs. (2), (3) and (18a,b) (the “benchmark” solution, see following section). Figure 5 illustrates the flow chart for the corrector step at a given solution point Ω of the speed response curve.

5. RESULTS AND DISCUSSION

The validation process for the RNN identification considered the unbalance response of the system in Figure 3 over a range of rotational speeds 10,000 rpm-24,000 rpm. The benchmark (reference) solution was obtained by solving equations (2), (3) and (18a,b) at a various discrete speeds Ω over this range, as described in Section 3.1 for the training data generation. The benchmark solution revealed self-excitation (linear instability of the static equilibrium configuration) beyond around 24,296 rpm, which is why the validation speeds were limited to 24,000 rpm. Notice that the stability range in the present study is much wider than in [21], which was 10,000 rpm to 15,000 rpm for the same foil-air bearing (FAB) but with a static load of W of $m_r g$ rather than the present value of $1.5m_r g$.

The validation was done in two stages:

- (i) Using the benchmark response as input to the RNN and comparing its output with the benchmark output;
- (ii) Comparing the orbit response computed by the RNN-FAB HB rotor-bearing analysis (section 4) with the benchmark.

In the following validation test results, unless otherwise stated, the unbalance per bearing ($m_r r_u$ per bearing in eqs. (18a,b)) was set to 10 g-mm. This means that, for the validation tests, the external dynamic excitation force amplitudes ranged from 10.97 N (@10,000 rpm i.e. 167 Hz) to 63.17 N (@24,000 rpm i.e. 400 Hz). In sharp contrast, the dynamic external excitation used to generate the training data was a chirp

signal of constant amplitude of *only* 1.5 N with frequencies ranging from 166.67 Hz to 400 Hz (see section 3.1). Hence, although the rotor-bearing system used for the validation was the same as that used to generate the training data, the validation data sets presented in the following sub-sections were independent from the training data since the conditions at the FAB were clearly different.

For each of the validation studies (i), (ii) above, results from two kinds of RNNs are shown:

- The *variable-speed RNNs* (for F_u , F_v) introduced in this paper, with inputs $\varepsilon(\tau)$, $\chi(\tau)$ (as shown in Figure 2); these were trained as described in section 3.1, wherein the rotational speed was in synch with the external chirp excitation (eq. (14)).
- The *constant-speed RNNs* (for F_u , F_v) introduced in the preliminary paper [21], with inputs $\varepsilon(\tau)$, $\gamma'(\tau)$; these were trained as described in section 3.1 with a chirp excitation signal ranging from 166.67 Hz to 400 Hz, but with the rotational speed kept fixed at a value corresponding to the middle of the frequency sweep (17,000 rpm) i.e. eq. (14) *does not apply* ($\Omega(\tau) \neq \omega_{\text{ext}}(\tau)$) and the network corresponds to an FAB with fixed bearing number $\Lambda = \tilde{\Omega}\Lambda_{\text{ref}}$ in eq. (2).

The details of the optimised architecture of the variable speed RNNs were described in section 3.2. The optimised architectures of the constant speed RNNs were the same except that the numbers of neurons in the respective four layers were 5-4-1-1 for and 3-3-3-1 for F_u , F_v respectively.

5.1 Validation of the identified RNN using new sets of input/output data

In these tests, the benchmark solutions for $\varepsilon_x, \varepsilon_y$, $\varepsilon_x', \varepsilon_y'$ were transformed to ε and χ (or γ' in the case of the constant-speed RNNs) and the latter were then fed to the RNNs for the radial and tangential bearing forces. Their outputs were then compared with the “true” benchmark forces (for the same inputs).

Generally, as can be seen in Figures 6-9, the identified RNNs could predict the hydrodynamic forces reasonably well. With regard to the predictions for the radial force F_u , the superiority of the variable-speed RNN over the constant-speed RNN is only slightly evident when looking at the time histories (Figure 6).

The distinction between the predictions from the two RNNs becomes more apparent when the mean and first three harmonics (amplitude and phase) of the predicted force time histories are extracted and compared with the benchmark values over the full range of speeds (see Figure 7). Both variable-speed RNN and constant-speed RNN give good agreement with the benchmark for the first harmonic (Figure 7(b)). The variable-speed RNN predictions clearly give better agreement with the benchmark with regard to the mean (Figure 7(a)) and the second and third harmonics (Figures 7(c,d)).

On the other hand, with regard to the predictions for the tangential force F_v , the superiority of the variable-speed RNN over the constant-speed RNN is clearly evident from the time histories (Figure 8). This superiority becomes even more apparent when the mean and first three harmonics (amplitude and phase) of the predicted force time histories are extracted and compared with the benchmark values over the full range of speeds (see Figure 9). As can be seen, the variable-speed RNN predictions for the tangential force give far superior agreement with the benchmark, compared to those from the constant-speed RNN, with regard to the first harmonic (Figure 9(b)), as well as the mean (Figure 9(a)) and higher harmonics (Figures 9(c,d)). In the preliminary paper [21], which used the constant-speed RNN, inferior predictions for the tangential force F_v were also observed over the much narrow speed range considered in [21]. The most likely reason for this is that the effect of the constant speed assumption is the fixing of the bearing number, which also factors the viscosity in its definition. For this reason, the fixing of the bearing number has more effect on the tangential force F_v than the radial force F_u (the latter being more of a reaction to fluid squeezing than fluid friction). The results in Figure 10 show that the variable-speed RNN predictions maintain good agreement with the benchmark predictions when the unbalance is doubled to 20 g-mm.

Figures 6-10 clearly show that the variable-speed RNN-FAB is reliably reproducing the original (benchmark) model output including the trends. This is most evident by comparing Figures 8 and 10 for the tangential force and noting the clear departure of both RNN and benchmark force signals from an approximate sinusoidal waveform as the unbalance is doubled from 10 g-mm (Figure 8) to 20 g-mm (Figure 10).

Despite the evident superiority of the variable-speed RNNs over the constant-speed RNNs, a close inspection of Figures 6-10 reveals some visible discrepancies between the variable-speed RNNs and the benchmark predictions. These can be considered inevitable, especially considering that the amplitude of the unbalance force is much higher than the amplitude of the excitation force used for training the RNNs (e.g. for 10 g-mm unbalance, the amplitude of the unbalance force ranges from around 7 to 40 times the amplitude of the excitation force signal used in training). However, the discrepancy in the tangential force predictions at 10,000 rpm for 10 g-mm unbalance (Figure 8) cannot be explained at the time of writing of this paper, especially considering that the discrepancy at the same speed is drastically reduced as the unbalance is doubled to 20 g-mm (compare Figure 10 at 10,000 rpm with Figure 8 at the same speed).

In Figures 6, 8 and 10 it is noted that the RNN prediction has an initial short-lived transient (in the form of a spike) – this is due to the necessity of initializing the TDLs in Figure 2 by setting the initial conditions defined in eqs. (17a,b,c). This spike means that it is difficult to use the RNN-FAB model in a transient (time domain integration) solution of a rotor-bearing system. However, it causes no problem in a frequency domain (harmonic balance) solution (following section) since this short transient can be removed prior to Fourier analysis in step 6 of the flowchart in Figure 5.

5.2 Validation by application of RNN-FAB to HB analysis

Figure 11 compares the orbit response (ε_y vs ε_x) computed using the RNN-FAB HB analysis (section 4) with the benchmark solution. There is reasonable correlation in terms of orbit size, orientation and general position within the original clearance circle. It is seen that the variable-speed RNN-FAB model generally gives better orbit predictions than the constant-speed RNN-FAB model. Indeed, the orbit prediction from the latter is inferior to that from the variable-speed RNN-FAB model even at the mid-point speed of 17,000 rpm (which was the speed at which the constant-speed RNN was trained). Of course, there are visible discrepancies between the variable-speed RNN-FAB model predictions and the benchmark solution. Such discrepancies inevitably arise from the discrepancies observed in the previous validation exercise (section 5.1). Nonetheless, the RNN predictions follow the trend of the benchmark solution and it is seen that the RNN-identified FAB can be reliably used within the nonlinear solution process for the periodic unbalance response of a rotor-bearing system with fundamental frequency synchronous with the rotational speed.

It is noted, however, that the replacement of the full numerical FAB model (eqs. (5, 6) or (1, 2)) by an identified displacement-force bearing model will remove the facility to calculate the foil deflection (i.e. the expanded clearance at any instant). It is also noted that, for the present system, beyond around 24,296 rpm, the periodic orbit predictions by HB, whether based on the RNN-FAB model or the exact model (like that used in [2]), become unstable and the aperiodic response can only be obtained by time-domain analysis. This instability is related to the instability of the static equilibrium configuration (self-excitation). Transient (time domain) analysis using the RNN-FAB model does not appear to be feasible at this stage due to the initial spikes in the RNN force output (as observed at the end of previous section), which are likely to diverge the trajectory from its final steady-state condition.

5.3 Note on the static load

The above analysis has shown that the performance of the identified RNN-FAB model is very robust to large variations in the amplitude of the external dynamic loading, which was even greater than the static load (45 N) at 10 g-mm unbalance and the top speed of 24,000 rpm. However, it is noted that the same static loading was used for both training and validation. Further investigations showed that an RNN-FAB identified using a given static loading during training generally gave inferior predictions when used within a problem involving a different static load. Hence, the successful transferability of the identified bearing model to a given application depends on the training taking due account of the static loading to be supported by the bearing.

CONCLUSIONS

The identification of compressible fluid bearings is fundamentally different from that of incompressible fluid bearings. A suitably trained recurrent neural network (RNN) has been shown to be capable of identifying the numerical model of a foil-air bearing (comprising the RE and foil structure state equations) through two validation studies. The second of these studies demonstrated the possibility of using the identified model within a nonlinear frequency domain rotordynamic solution process for the vibration response. Separate networks were used for the radial and tangential bearing forces. The limitation identified in the authors' preliminary research, restricting the trained networks to a given bearing number (speed), has been overcome in the present research, wherein a single pair of networks can be trained effectively to cover a full range of speeds.

Although the same system was used for training and validation, the conditions at the FAB were very different as a result of the amplitude of the external dynamic force used in the validation being many times larger than that used in the training. In principle, once properly trained, the identified RNN-FAB model could be used in the rotordynamic analysis of *any* system that hosts it, provided that the training takes due account of the static loading to be supported by the FAB in that application.

The potential demonstrated by the proof of principle in this paper goes beyond computational efficiency advantages since it paves the way for empirical nonlinear identification models.

A significant limitation of the current identification procedure was the restriction to periodic vibration with fundamental frequency synchronous with the excitation. Hence, self-excitation effects, as well as subsynchronous frequencies introduced by high unbalance forces [20], could not be dealt with at this stage.

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APPENDIX A

Since the bearing is open to atmosphere at both ends, symmetry can be exploited and the FD grid need only cover half the axial length of the bearing. With reference to the definitions in eqs. (5a,b), the rectangular grid has $N_z \times N_\theta$ points where $\zeta = \zeta_i, i = 1, \dots, N_z, \theta = \theta_j, j = 1, \dots, N_\theta$. The location $\zeta = \zeta_{N_z}$ corresponds to the middle cross-section of the bearing (i.e. $\zeta_{N_z} = 0.5L/R$). The location $\zeta = \zeta_1$ corresponds to the cross-section that is displaced inwards from the bearing edge by a distance $\Delta\zeta$ i.e. $\zeta_1 = \Delta\zeta$, and the bearing edge, where $\psi = \tilde{h}$ (i.e. $\tilde{p}=1$), is excluded from the FD grid (however, this boundary condition is considered when estimating partial derivatives and integrating).

Using central-difference formulae [12] the partial derivatives in eq. (2) can be approximated as follows.

For ψ and $i = 1, \dots, N_z, j = 1, \dots, N_\theta$:

$$\left. \frac{\partial \psi}{\partial \theta} \right|_{i,j} = \frac{1}{2\Delta\theta} (\psi_{i,j+1} - \psi_{i,j-1}), \left. \frac{\partial^2 \psi}{\partial \theta^2} \right|_{i,j} = \frac{1}{\Delta\theta^2} (\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}) \quad (24a,b)$$

$$\left. \frac{\partial \psi}{\partial \zeta} \right|_{i,j} = \frac{1}{2\Delta\zeta} (\psi_{i+1,j} - \psi_{i-1,j}), \left. \frac{\partial^2 \psi}{\partial \zeta^2} \right|_{i,j} = \frac{1}{\Delta\zeta^2} (\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}) \quad (24c,d)$$

where the following boundary conditions apply:

$$\psi_{i,N_\theta+1} = \psi_{i,1}, \psi_{i,0} = \psi_{i,N_\theta} \quad (25a,b)$$

$$\psi_{N_z+1,j} = \psi_{N_z-1,j}, \psi_{0,j} = \tilde{h}_{0,j} \quad (26a,b)$$

Eqs. (25a,b) are due to periodicity in θ . Eq. (26a) is due to symmetry about the mid-section of the bearing. Eq. (26b) is due to the open-atmosphere boundary condition, where:

$$\tilde{h}_{i,j} = 1 - \varepsilon_x \cos \theta_j - \varepsilon_y \sin \theta_j + \tilde{w}_{i,j} \quad (27)$$

The value of $\tilde{w}_{0,j}$ (to use in $\tilde{h}_{0,j}$) is obtained as follows.

For $\tilde{w}$ and $i = 2, \dots, N_z, j = 1, \dots, N_\theta$:

$$\left. \frac{\partial \tilde{w}}{\partial \theta} \right|_{i,j} = \frac{1}{2\Delta\theta} (\tilde{w}_{i,j+1} - \tilde{w}_{i,j-1}), \left. \frac{\partial^2 \tilde{w}}{\partial \theta^2} \right|_{i,j} = \frac{1}{\Delta\theta^2} (\tilde{w}_{i,j+1} - 2\tilde{w}_{i,j} + \tilde{w}_{i,j-1}) \quad (28a,b)$$

$$\left. \frac{\partial \tilde{w}}{\partial \zeta} \right|_{i,j} = \frac{1}{2\Delta\zeta} (\tilde{w}_{i+1,j} - \tilde{w}_{i-1,j}), \left. \frac{\partial^2 \tilde{w}}{\partial \zeta^2} \right|_{i,j} = \frac{1}{\Delta\zeta^2} (\tilde{w}_{i+1,j} - 2\tilde{w}_{i,j} + \tilde{w}_{i-1,j}) \quad (28c,d)$$

where the following periodicity and symmetry boundary conditions apply:

$$\tilde{w}_{i,N_\theta+1} = \tilde{w}_{i,1}, \tilde{w}_{i,0} = \tilde{w}_{i,N_\theta} \quad (29a,b)$$

$$\tilde{w}_{N_z+1,j} = \tilde{w}_{N_z-1,j} \quad (30)$$

The partial derivatives $\left. \frac{\partial \tilde{w}}{\partial \zeta} \right|_{1,j}, \left. \frac{\partial^2 \tilde{w}}{\partial \zeta^2} \right|_{1,j}$ require the foil deflection at the bearing edge, $\tilde{w}_{0,j}$. According to eq. (3),

since the pressure is atmospheric at the bearing edge, the foil deflection there should be zero. However, this will not be the case in practice, due to the membrane effect. Therefore, instead of imposing a boundary condition of $\tilde{w} = 0$ at the bearing edges, $\tilde{w}_{0,j}$ and the partial derivatives $\left. \frac{\partial \tilde{w}}{\partial \zeta} \right|_{1,j}, \left. \frac{\partial^2 \tilde{w}}{\partial \zeta^2} \right|_{1,j}$ are determined using a

quadratic interpolation using the foil deflections at $i = 1, 2$:

$$\left. \frac{\partial \tilde{w}}{\partial \zeta} \right|_{1,j} = \frac{2}{\Delta\zeta} (\tilde{w}_{2,j} - \tilde{w}_{1,j}) - \left. \frac{\partial \tilde{w}}{\partial \zeta} \right|_{2,j}, \left. \frac{\partial^2 \tilde{w}}{\partial \zeta^2} \right|_{1,j} = \frac{2}{\Delta\zeta} \left\{ \left. \frac{\partial \tilde{w}}{\partial \zeta} \right|_{2,j} - \frac{(\tilde{w}_{2,j} - \tilde{w}_{1,j})}{\Delta\zeta} \right\} \quad (31a,b)$$

$$\tilde{w}_{0,j} = 2 \left. \frac{\partial^2 \tilde{w}}{\partial \zeta^2} \right|_{1,j} (\Delta\zeta)^2 - 2 \left. \frac{\partial \tilde{w}}{\partial \zeta} \right|_{2,j} (\Delta\zeta) + \tilde{w}_{2,j} \quad (31c)$$

Defining $\boldsymbol{\varepsilon} = [\varepsilon_x \ \varepsilon_y]^T$, in addition to the definitions of $\boldsymbol{\psi}$ and $\tilde{\mathbf{w}}$ (eqs. (5a,b), and substituting the above approximations into the air film eqs. (2), (2a) yields the following system of state equations that couples the air film with the foil and rotor subsystems (eqs. (3), (11a,b)):

$$\boldsymbol{\psi}' = \mathbf{g}_{\text{RE}}(\boldsymbol{\psi}, \tilde{\mathbf{w}}, \boldsymbol{\varepsilon}) \quad (32a)$$

$$\tilde{\mathbf{w}}' = \mathbf{g}_{\text{Foil}}(\boldsymbol{\psi}, \tilde{\mathbf{w}}, \boldsymbol{\varepsilon}) \quad (32b)$$

$$\begin{bmatrix} \boldsymbol{\varepsilon}' \\ \boldsymbol{\varepsilon}' \end{bmatrix} = \begin{bmatrix} \boldsymbol{\varepsilon}' \\ \frac{4}{m_r c \Omega_{\text{ref}}^2} \left[F_x(\boldsymbol{\psi}, \tilde{\mathbf{w}}, \boldsymbol{\varepsilon}) + F_{\text{ext},x}(\tau) \right] \\ F_y(\boldsymbol{\psi}, \tilde{\mathbf{w}}, \boldsymbol{\varepsilon}) + F_{\text{ext},x}(\tau) - W \end{bmatrix} \quad (32c)$$

where $\mathbf{g}_{\text{RE}}(\boldsymbol{\psi}, \tilde{\mathbf{w}}, \boldsymbol{\varepsilon})$ is the vector function resulting from the substitution of eqs. (24-31) into the RE (eq. (2)), $\mathbf{g}_{\text{Foil}}(\boldsymbol{\psi}, \tilde{\mathbf{w}}, \boldsymbol{\varepsilon})$ is the vector function arising from the discretization of the foil equation (eq. (3)), and the bearing forces $F_{x,y}$ are computed for given $\boldsymbol{\psi}, \tilde{\mathbf{w}}, \boldsymbol{\varepsilon}$ by applying quadrature formulae [12] to eq. (4). The FD grid used in the computation was 7×72 ($N_z = 7$, $N_\theta = 72$) i.e. the full extent of FAB was covered by a 15×73 mesh. The time history of the state variables $\boldsymbol{\psi}, \tilde{\mathbf{w}}, \boldsymbol{\varepsilon}$ is obtained by applying numerical integration in the time domain using an implicit solver, as done in [1, 2].

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(TDL: tapped delay line; N: network normalisation; N^{-1} : de-normalisation)

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(● *benchmark force*; ■ *force prediction from identified RNN-FAB (variable-speed RNN, defined before section 5.1)*; ◆ *force prediction from identified RNN-FAB (constant-speed RNN, defined before section 5.1)*)

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bearing length (mm)	38.1
bearing diameter (mm)	38.1
clearance (μm)	31.8
foil stiffness k_b (GN/m ³)	4.739
foil damping loss factor η	0.25

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fig1.tiff

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fig2.tiff

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fig8.tiff

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fig9.tiff

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(● benchmark force; ■ force prediction from identified RNN-FAB (variable-speed RNN, defined before section 5.1); ◆ force prediction from identified RNN-FAB (constant-speed RNN, defined before section 5.1))

fig10.tiff

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fig11.tiff

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 (— *black* benchmark time domain solution; - - - *blue* prediction from HB using identified RNN-FAB (variable-speed RNNs, defined before section 5.1); ···· *red* prediction from HB using identified RNN-FAB (constant-speed RNNs, defined before section 5.1))

fig11contd.tiff

Figure 11 (cont'd.): Validation of identified RNN-FAB by application of RNN-FAB to HB analysis at different speeds with unbalance mass of 10 g-mm
 (— *black* benchmark time domain solution; - - - - *blue* prediction from HB using identified RNN-FAB (variable-speed RNNs, defined before section 5.1); ····· *red* prediction from HB using identified RNN-FAB (constant-speed RNNs, defined before section 5.1))