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Bringing Order to Chaos: Hybrid Modelling of a Discontinuous Chaotic System

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Abstract—We propose a computational-oriented perspective within the study of discontinuous chaotic systems, and provide insights into the modelling, control and simulation of chaotic systems with switching dynamics. In particular, the Lorenz system in its piecewise-linear version is studied. This system is reinterpreted within the hybrid-automaton framework, and what is referred to as the *Lorenz hybrid automaton* is established. Furthermore, a discontinuous control which eliminates the chaotic behaviour and steers the trajectories to a desired equilibrium is proposed. An integral characteristic of the modelling framework is that the controlled system, exhibiting three discontinuity surfaces, is reduced to the composition of several *Lorenz hybrid automata*. The approach proposed here is especially useful in order to specify the transitions between the different system operation modes, which becomes a crucial problem due to the existence of multiple switching surfaces.

I. MOTIVATION

What is the best model to describe variable structure systems with multiple switching elements and exhibiting several types of complex behaviours? How to specify all their transitions in a deterministic way to facilitate the simulation? Every mathematical model is an approximation of the real world, and is full of limitations. However, some models are better than others at describing the evolution of certain physical and engineering systems. The hybrid systems framework, in particular, the computational hybrid-automaton one is very adequate for modelling and controlling complex systems [1], [2], [3].

This paper is devoted to the specification and the computational abstraction of a particular discontinuous system with chaotic behaviour: the piecewise-linear (PWL) Lorenz system. The computational framework used is that of hybrid automata [2], [3], [4], [5], [6].

There has been great success in analysing and controlling the Lorenz system and what is called the generalised Lorenz family; there is a vast literature about this topic, for example [7], [8], [9]. However, there are still open problems concerning the modelling and simulation, especially, when these systems are discontinuous [10], [11], [12]. Generally, the discontinuous or PWL version of the Lorenz system presents two discontinuity surfaces, and the problem of non-uniqueness and even non-existence of solutions has to be

faced when the trajectories either cross or slide on the discontinuity surfaces. Furthermore, due to the coexistence of several discontinuity surfaces, the problem of specification of the transitions between the system operation modes is critical, and special care has to be taken in the numerical integration [13], [14], [15]. The use of a hybrid automaton to describe the system is presented here as a solution to this problem.

This paper is inspired by Navarro's previous results related to the modelling and discrete abstraction of discontinuous dynamical systems (DDSs) under the hybrid-automaton framework [4], [5], [6].

The *Lorenz hybrid automaton* with 6 discrete locations is given. To prove the efficacy of the hybrid-automaton framework, a second hybrid automaton with 18 locations is proposed. This second automaton corresponds to the controlled PWL Lorenz system with three discontinuity surfaces, which is obtained by considering a sliding-mode-based control. The control strategy is based on inserting a discontinuity surface on which the system presents the desired behaviour and forces the system trajectories to slide on this surface. The *controlled Lorenz hybrid automaton*, by following the computer science divide-and-conquer principle, is obtained by means of the composition of three *Lorenz hybrid automata*. Although the discontinuous Lorenz system can be effectively stabilised with smooth or even linear controllers, we propose a discontinuous-type controller in order to include within the system a third discontinuity surface, and show the potential of the framework to model and analyse discontinuous systems with multiple switching elements.

The analysis and control of discontinuous chaotic systems is not new, and has been solved in different ways [16], [17], [18], [19]. The novel contribution of this paper is the reinterpretation of these systems as hybrid automata, which leads to an alternative modelling and simulation framework. Whilst a particular example is used, the proposed approach is applicable to a broader class of systems and provides new insights into the modelling, analysis and control of discontinuous chaotic systems.

II. THE DISCONTINUOUS CHAOTIC SYSTEM

The starting point of our analysis is the smooth Lorenz system with the following form [20]:

$$\begin{aligned}\dot{x}_1 &= a(x_2 - x_1), \\ \dot{x}_2 &= x_1(\rho - x_3) - x_2, \\ \dot{x}_3 &= x_2x_1 - bx_3.\end{aligned}\tag{1}$$

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Fig. 1. Lorenz chaotic attractor.

The system state vector is defined as $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$, and $a = 10$, $b = \frac{8}{3}$ and $\rho = 28$. It is well-known that this system for these parameters exhibits a chaotic attractor, which is shown in Fig. 1

The chaotic behaviour of the Lorenz system can be reproduced by a discontinuous version of (1), which is referred to as *piecewise-linear (PWL) Lorenz system* and has the form: [10], [11], [12]:

$$\begin{aligned}\dot{x}_1 &= a(x_2 - x_1), \\ \dot{x}_2 &= \text{sign}(x_1)(\rho - x_3) - qx_2, \\ \dot{x}_3 &= \text{sign}(x_2)x_1 - bx_3,\end{aligned}\quad (2)$$

with $a = 1.2$, $b = 0.15$, $\rho = 7.0$, $q = 0.1$. For these parameters, the PWL Lorenz system exhibits chaotic behaviour.

Let define two scalar functions $s_1 = x_1$ and $s_2 = x_2$. System (2) has two discontinuity surfaces:

$$S_1 := \{x \in \mathbb{R}^3 : s_1 = x_1 = 0\}, \quad S_2 := \{x \in \mathbb{R}^3 : s_2 = x_2 = 0\}.$$

S_1 and S_2 can intersect. Now, the equivalent dynamics on S_1 and S_2 will be derived. Utkin's equivalent control method [21] will be used for this purpose, but additional considerations have to be taken into account due to the fact that on either S_1 or S_2 , a second-order sliding mode appears [22], [23], [24]. By derivating twice s_1 and by considering $s_1 = \dot{s}_1 = \ddot{s}_1 = 0$, the relations $\text{sign}(x_1) = qx_2/(\rho - x_3)$ and $x_2 = 0$ are obtained. Notice that $\text{sign}(x_1)$ plays the role of the equivalent control. Consequently, it is obtained that $\text{sign}(x_1) = 0$. Following the same procedure for s_2 , that is, by derivating twice s_2 and by considering $s_2 = \dot{s}_2 = \ddot{s}_2 = 0$, the relations $\text{sign}(x_2) = bx_3/x_1$ and $x_3 = \rho$ are obtained. To sum up, we have established that the equivalent dynamics on S_1 and S_2 have the form:

$$\dot{x} = f_{s_1}(x) = \begin{pmatrix} 0 \\ 0 \\ -bx_3 \end{pmatrix}, \quad \dot{x} = f_{s_2}(x) = \begin{pmatrix} -ax_1 \\ 0 \\ 0 \end{pmatrix}. \quad (3)$$

We again remark that for the case $x_1 = x_2 = 0$, the dynamics $\dot{x} = f_{s_1}(x)$ will be considered.

We will specify and simulate system (2) by means of a hybrid automaton in order to reproduce the Lorenz chaotic attractor for the PWL Lorenz system. In addition, a discontinuous control is proposed in order to eliminate this chaotic behaviour and drive the system trajectories to the origin.

III. THE PWL LORENZ SYSTEM AS A HYBRID AUTOMATON

System (2) can be represented as a hybrid automaton, which will be referred to as the *Lorenz hybrid automaton* (H_L). The proposal of H_L is inspired by the *DDS hybrid automaton* - that is, a special hybrid automaton for describing discontinuous dynamical systems (DDSs) — proposed by Navarro-López in [4], [5], [6]. The following general hybrid automaton with inputs and outputs, which was also proposed in [4], [5], [6], will be used as the basic model.

Definition 1. A hybrid automaton with inputs and outputs is a collection

$$H = (Q, E, \mathcal{X}, \Sigma, \mathcal{U}, O, \mathcal{Y}, \text{Dom}, \mathcal{F}, \text{Init}, G, R, h, r)$$

where:

- $Q = \{q_1, q_2, \dots, q_N\}$ is a finite set of discrete locations.
- $\mathcal{X} \subseteq \mathbb{R}^n$, $\mathcal{U} \subseteq \mathbb{R}^m$ and $\mathcal{Y} \subseteq \mathbb{R}^m$ are the state, input and output continuous spaces, respectively. The state of H is $(q, x) \in Q \times \mathcal{X}$.
- $E \subseteq Q \times Q$ is a finite set of edges called transitions.
- $\Sigma = \{\sigma_1, \sigma_2, \dots, \sigma_M\}$ is a finite set of symbols labelling the transitions and representing the discrete input events.
- $O = \{o_1, o_2, \dots, o_K\}$ is a finite set of symbols representing the discrete output events.
- $\text{Dom} : Q \rightarrow 2^{\mathcal{X} \times \mathcal{U}}$ is the location domain. Dom assigns a set of continuous states and inputs to each discrete state $q_i \in Q$, thus, $\text{Dom}(q_i) \subset \mathcal{X} \times \mathcal{U}$. As long as the system is within location q_i , the continuous state x must satisfy $x \in \text{Dom}(q_i)$.
- $\mathcal{F} = \{f_{q_i}(x, u) : q_i \in Q\}$ is the collection of vector fields describing the continuous dynamics such that $f_{q_i} : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{X}$. Each $f_{q_i}(x, \cdot)$ is assumed to be Lipschitz continuous on the location domain for q_i in order to ensure that the solution within $\text{Dom}(q_i)$ exists and is unique.
- $\text{Init} \subseteq Q \times \mathcal{X}$ is a set of initial states.
- $G : E \rightarrow 2^{\mathcal{X}}$ is a guard set. Function G assigns to each transition $e = (q_i, q_j) \in E$ a set of continuous states ($G(e) \subset \mathcal{X}$). Each guard enables the change of discrete location.
- $R : E \times \mathcal{X} \times \mathcal{U} \rightarrow 2^{\mathcal{X}}$ is a reset map for the continuous states for each transition. It is assumed that $\forall e \in E$, $G(e) \neq \emptyset$ and $\forall x \in G(e)$, $R(e, x, u) \neq \emptyset$.
- $h : Q \times \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{Y}$ is the continuous output mapping, there is one for each location.
- $r : Q \times \mathcal{X} \times \Sigma \times \mathcal{U} \rightarrow O$ is the discrete output map, there is one for each location.

In order to define the transitions of H_L , two properties are considered:

- x_1 and x_2 cannot change sign without crossing the discontinuity surface. That is, x_1 (or x_2) cannot change from $x_1 > 0$ to $x_1 < 0$ directly, it has to visit first the discrete location in which $x_1 = 0$.
- Due to numerical problems for detecting the zero crossing, a zero band is considered, and instead of having $x_1 = 0$, $x_1 > 0$ and $x_1 < 0$, $|x_1| \leq \delta$, $x_1 > \delta$ and $x_1 < -\delta$

are considered, respectively, with $\delta = 10^{-6}$. The same consideration is for x_2 .

Definition 2. The *Lorenz hybrid automaton* with 6 discrete states (H_L) describing the dynamics of system (2) is a particular case of H with,

- $Q = \{q_1, q_2, q_3, q_4, q_5, q_6\}$, $\mathcal{X} \subseteq \mathbb{R}^3$.
- $E = \{(q_1, q_3), (q_1, q_4), (q_2, q_3), (q_2, q_4), (q_3, q_1), (q_3, q_2), (q_3, q_4), (q_3, q_5), (q_3, q_6), (q_4, q_1), (q_4, q_2), (q_4, q_3), (q_4, q_5), (q_4, q_6), (q_5, q_3), (q_5, q_4), (q_6, q_3), (q_6, q_4)\}$.
- $\Sigma = \{a, b, c, d, e, f, g, h, i\}$. $\Sigma = O$. One edge label is assigned to each type of guard.
- $Dom(q_1) = \{\mathbf{x} \in \mathbb{R}^3 : x_1 > \delta, x_2 > \delta\}$, $Dom(q_2) = \{\mathbf{x} \in \mathbb{R}^3 : x_1 > \delta, x_2 < -\delta\}$, $Dom(q_3) = \{\mathbf{x} \in \mathbb{R}^3 : |x_1| > \delta, |x_2| \leq \delta\}$, $Dom(q_4) = \{\mathbf{x} \in \mathbb{R}^3 : |x_1| \leq \delta\}$, $Dom(q_5) = \{\mathbf{x} \in \mathbb{R}^3 : x_1 < -\delta, x_2 > \delta\}$, $Dom(q_6) = \{\mathbf{x} \in \mathbb{R}^3 : x_1 < -\delta, x_2 < -\delta\}$.
- Dynamics for each discrete location:

$$\begin{aligned} \mathbf{f}_{q_1} &= \begin{pmatrix} a(x_2 - x_1) \\ \rho - x_3 - qx_2 \\ x_1 - bx_3 \end{pmatrix}, \mathbf{f}_{q_2} = \begin{pmatrix} a(x_2 - x_1) \\ \rho - x_3 - qx_2 \\ -x_1 - bx_3 \end{pmatrix}, \\ \mathbf{f}_{q_3} &= \begin{pmatrix} -ax_1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{f}_{q_4} = \begin{pmatrix} 0 \\ 0 \\ -bx_3 \end{pmatrix}, \\ \mathbf{f}_{q_5} &= \begin{pmatrix} a(x_2 - x_1) \\ x_3 - qx_2 - \rho \\ x_1 - bx_3 \end{pmatrix}, \mathbf{f}_{q_6} = \begin{pmatrix} a(x_2 - x_1) \\ x_3 - qx_2 - \rho \\ -x_1 - bx_3 \end{pmatrix}. \end{aligned}$$

The case $x_1 = x_2 = 0$ is considered in location q_4 .

- $G(q_3, q_1) = G(q_4, q_1) = Dom(q_1)$, $G(q_1, q_4) = G(q_5, q_4) = \{\mathbf{x} \in \mathbb{R}^3 : |x_1| \leq \delta, x_2 > \delta\}$, $G(q_3, q_5) = G(q_4, q_5) = Dom(q_5)$, $G(q_1, q_3) = G(q_2, q_3) = \{\mathbf{x} \in \mathbb{R}^3 : |x_2| \leq \delta, x_1 > \delta\}$, $G(q_3, q_4) = \{\mathbf{x} \in \mathbb{R}^3 : |x_1| \leq \delta, |x_2| \leq \delta\}$, $G(q_5, q_3) = G(q_6, q_3) = \{\mathbf{x} \in \mathbb{R}^3 : |x_2| \leq \delta, x_1 < -\delta\}$, $G(q_3, q_2) = G(q_4, q_2) = Dom(q_2)$, $G(q_2, q_4) = G(q_6, q_4) = \{\mathbf{x} \in \mathbb{R}^3 : |x_1| \leq \delta, x_2 < -\delta\}$, $G(q_3, q_6) = G(q_4, q_6) = Dom(q_6)$, $G(q_4, q_3) = Dom(q_3)$.
- $R(q_i, q_j, \mathbf{x}) = \{\mathbf{x}\}$, $\forall i, j \in \{1, \dots, 6\}$, and $i \neq j$.
- $y = h(q_i, \mathbf{x}) = \mathbf{x}^T$, $\forall i \in \{1, \dots, 6\}$.

The graphical representation of the *Lorenz hybrid automaton* is given in Fig. 2. This hybrid automaton was simulated for $q_0 \times \mathbf{x}_0 = q_6 \times (-2.1 \cdot 10^{-4}, -5.20625 \cdot 10^{-3}, 6.895783)^T$ by using the tool Ptolemy II under the environment HyVisual [25], [26]. The resulting chaotic attractor of H_L and the discrete locations visited in order to generate it are shown in Fig. 3. From the figure, as it was expected, it can be appreciated that the hybrid trajectory crosses either S_1 or S_2 without sliding on them.

IV. CONTROLLING THE HYBRID SYSTEM

The control goal is to eliminate the chaotic behaviour and steer the trajectories of the system to the equilibrium $\mathbf{x} = (0, 0, 0)^T$. For this purpose, a control variable u is included in system (2), having now $\dot{x}_1 = a(x_2 - x_1) + u$.

It is well-established that the most appropriate types of controllers in order to stabilise a discontinuous system is a discontinuous or switched-type controller [27]. We will

Fig. 2. Hybrid automaton H_L for system (2).

Fig. 3. Chaotic attractor of the Lorenz hybrid automaton H_L with the visited discrete locations. The chaotic attractor is obtained with a simulation time of 1800secs, and the discrete locations are shown for the first 100secs.

follow this recommendation and propose a discontinuous controller. Indeed, the use of a discontinuous controller — which implies the inclusion of another discontinuity surface in the system — will show how powerful the hybrid-automaton framework is in order to specify the multiple transitions of complex systems. Furthermore, the control scheme proposed can be useful for controlling chaotic discontinuous systems with similar structure to the Lorenz system.

A. Sliding Mode-based Control

The control goal will be achieved by inserting an attractive surface of discontinuity, $s_d = 0$, along which the system trajectories enter a sliding regime and $x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow 0$.

A discontinuous control u is proposed, which is inspired by the control structure proposed in [28].

We define the following scalar function:

$$\begin{aligned} s_d &= (x_1 + x_2 + x_3) + \lambda \int_0^t (x_1(\tau) - x_3(\tau)) d\tau + \\ &+ \lambda \int_0^t (x_2(\tau) - x_3(\tau)) d\tau = \\ &= (x_1 + x_2 + x_3) + \lambda(x_4 + x_5), \quad 0 < \lambda < 1, \\ \dot{x}_4 &= x_1 - x_3, \quad \dot{x}_5 = x_2 - x_3. \end{aligned} \quad (4)$$

For s_d to become zero and make $s_d = 0$ be attractive (that is, to have $s_d \dot{s}_d < 0$), $\dot{s}_d$ is forced to be:

$$\dot{s}_d = -\eta \text{sign}(s_d), \quad (5)$$

with $\eta > 0$, ensuring to have a sliding motion on $s_d = 0$ [28]. From (4) and (5), the following control u is obtained:

$$\begin{aligned} u &= -a(x_2 - x_1) + qx_2 + bx_3 - \lambda(x_1 + x_2 - 2x_3) - \\ &- \text{sign}(x_1)(\rho - x_3) - \text{sign}(x_2)x_1 - \eta \text{sign}(s_d). \end{aligned} \quad (6)$$

By applying (6) to dynamics (2), this system is obtained:

$$\begin{aligned} \dot{x}_1 &= qx_2 + bx_3 - \lambda(x_1 + x_2 - 2x_3) - \\ &- \text{sign}(x_1)(\rho - x_3) - \text{sign}(x_2)x_1 - \eta \text{sign}(s_d), \\ \dot{x}_2 &= \text{sign}(x_1)(\rho - x_3) - qx_2, \\ \dot{x}_3 &= \text{sign}(x_2)x_1 - bx_3. \end{aligned} \quad (7)$$

The dynamics of system (7) on the surface of discontinuity $s_d = 0$ can be obtained by means of Utkin's equivalent control method [21], and have the form:

$$\begin{aligned} \dot{x}_1 &= qx_2 + bx_3 - \lambda(x_1 + x_2 - 2x_3) - \\ &- \text{sign}(x_1)(\rho - x_3) - \text{sign}(x_2)x_1, \\ \dot{x}_2 &= \text{sign}(x_1)(\rho - x_3) - qx_2, \\ \dot{x}_3 &= \text{sign}(x_2)x_1 - bx_3. \end{aligned} \quad (8)$$

The equivalent dynamics on the surfaces S_1 and S_2 are computed by applying Utkin's equivalent control method and following the same procedure as in Section II. The difference now is that the second-order sliding motion appears on S_2 .

B. Hybrid Automaton of the Controlled System

The hybrid automaton which describes the controlled system (7) will be referred to as H_{LC} . It has the following key characteristics:

- H_{LC} is obtained as the composition of three hybrid automata H_L . Each of these three hybrid automata corresponds to: $s_d = 0$ (discrete locations $q_1 - q_6$), $s_d < 0$ (discrete locations $q_7 - q_{12}$) and $s_d > 0$ (discrete locations $q_{13} - q_{18}$).
- The surface $s_d = 0$ is attractive for all $\mathbf{x}$ and all the trajectories reach it in a finite time. Furthermore, once the trajectory reaches $s_d = 0$, it remains there. Consequently, once the system reaches locations $q_1 - q_6$, its future transitions will be restricted to these six locations. This can be considered as a desired *recurrent loop*. In addition, we also consider as unfeasible transitions, the ones going across $s_d = 0$.

- For simulation purposes, a zero band is considered, and $s_d = 0$, $s_d < 0$ and $s_d > 0$ are rewritten as $|s_d| \leq \delta$, $s_d < -\delta$ and $s_d > \delta$, respectively, with $\delta = 10^{-6}$.
- When entering q_4 , x_1 is reset to 0; when entering q_3 , x_2 is reset to 0. When entering the rest of locations, $\mathbf{x}$ is not reset.

The graphical representation of H_{LC} is given in two parts, in Figures 4 and 5.

Fig. 4. Part I of the hybrid automaton for the controlled system. The dotted lines represent the transitions within each group of six locations. The solid-lined transitions represent the transitions to the discontinuity surface $s_d = 0$.

The main elements of H_{LC} are:

• Discrete locations:

- $\text{Dom}(q_1) = \{\mathbf{x} \in \mathbb{R}^5 : x_1 > \delta, x_2 > \delta, |s_d| \leq \delta\}$.
- $\text{Dom}(q_2) = \{\mathbf{x} \in \mathbb{R}^5 : x_1 > \delta, x_2 < -\delta, |s_d| \leq \delta\}$.
- $\text{Dom}(q_3) = \{\mathbf{x} \in \mathbb{R}^5 : |x_1| > \delta, |x_2| \leq \delta, |s_d| \leq \delta\}$.
- $\text{Dom}(q_4) = \{\mathbf{x} \in \mathbb{R}^5 : |x_1| \leq \delta, |s_d| \leq \delta\}$.
- $\text{Dom}(q_5) = \{\mathbf{x} \in \mathbb{R}^5 : x_1 < -\delta, x_2 > \delta, |s_d| \leq \delta\}$.
- $\text{Dom}(q_6) = \{\mathbf{x} \in \mathbb{R}^5 : x_1 < -\delta, x_2 < -\delta, |s_d| \leq \delta\}$.
- $\text{Dom}(q_7) = \{\mathbf{x} \in \mathbb{R}^5 : x_1 > \delta, x_2 > \delta, s_d < -\delta\}$.
- $\text{Dom}(q_8) = \{\mathbf{x} \in \mathbb{R}^5 : x_1 > \delta, x_2 < -\delta, s_d < -\delta\}$.
- $\text{Dom}(q_9) = \{\mathbf{x} \in \mathbb{R}^5 : |x_1| > \delta, |x_2| \leq \delta, s_d < -\delta\}$.
- $\text{Dom}(q_{10}) = \{\mathbf{x} \in \mathbb{R}^5 : |x_1| \leq \delta, s_d < -\delta\}$.
- $\text{Dom}(q_{11}) = \{\mathbf{x} \in \mathbb{R}^5 : x_1 < -\delta, x_2 > \delta, s_d < -\delta\}$.
- $\text{Dom}(q_{12}) = \{\mathbf{x} \in \mathbb{R}^5 : x_1 < -\delta, x_2 < -\delta, s_d < -\delta\}$.
- $\text{Dom}(q_{13}) = \{\mathbf{x} \in \mathbb{R}^5 : x_1 > \delta, x_2 > \delta, s_d > \delta\}$.
- $\text{Dom}(q_{14}) = \{\mathbf{x} \in \mathbb{R}^5 : x_1 > \delta, x_2 < -\delta, s_d > \delta\}$.
- $\text{Dom}(q_{15}) = \{\mathbf{x} \in \mathbb{R}^5 : |x_1| > \delta, |x_2| \leq \delta, s_d > \delta\}$.
- $\text{Dom}(q_{16}) = \{\mathbf{x} \in \mathbb{R}^5 : |x_1| \leq \delta, s_d > \delta\}$.
- $\text{Dom}(q_{17}) = \{\mathbf{x} \in \mathbb{R}^5 : x_1 < -\delta, x_2 > \delta, s_d > \delta\}$.
- $\text{Dom}(q_{18}) = \{\mathbf{x} \in \mathbb{R}^5 : x_1 < -\delta, x_2 < -\delta, s_d > \delta\}$.

Fig. 5. Part II of the hybrid automaton for the controlled system. The dotted lines represent the transitions within each group of six locations. The solid-lined transitions represent the transitions to the discontinuity surface $s_d = 0$.

- **Dynamics for each location:** for each location $\dot{x}_4 = x_1 - x_3$, $\dot{x}_5 = x_2 - x_3$ is considered, which is part of the controller dynamics, in addition to the system dynamics for x_1 , x_2 and x_3 as follows:

$$\begin{aligned} \mathbf{f}_{q_1} &= \begin{pmatrix} \varphi_1(\mathbf{x}) - x_1 \\ \rho - x_3 - qx_2 \\ x_1 - bx_3 \end{pmatrix}, \mathbf{f}_{q_2} = \begin{pmatrix} \varphi_1(\mathbf{x}) + x_1 \\ \rho - x_3 - qx_2 \\ -x_1 - bx_3 \end{pmatrix} \\ \mathbf{f}_{q_3} &= \begin{pmatrix} -\lambda(x_1 - 2\rho) \\ 0 \\ 0 \end{pmatrix}, \mathbf{f}_{q_4} = \begin{pmatrix} 0 \\ bx_3 - \lambda(x_2 - 2x_3) \\ -bx_3 \end{pmatrix} \\ \mathbf{f}_{q_5} &= \begin{pmatrix} \varphi_2(\mathbf{x}) - x_1 \\ x_3 - qx_2 - \rho \\ x_1 - bx_3 \end{pmatrix}, \mathbf{f}_{q_6} = \begin{pmatrix} \varphi_2(\mathbf{x}) + x_1 \\ x_3 - qx_2 - \rho \\ -x_1 - bx_3 \end{pmatrix} \\ \mathbf{f}_{q_7} &= \begin{pmatrix} \varphi_1(\mathbf{x}) - x_1 + \eta \\ \rho - x_3 - qx_2 \\ x_1 - bx_3 \end{pmatrix}, \mathbf{f}_{q_8} = \begin{pmatrix} \varphi_1(\mathbf{x}) + x_1 + \eta \\ \rho - x_3 - qx_2 \\ -x_1 - bx_3 \end{pmatrix}, \\ \mathbf{f}_{q_9} &= \begin{pmatrix} -\lambda(x_1 - 2\rho) + \eta \\ 0 \\ 0 \end{pmatrix}, \mathbf{f}_{q_{10}} = \begin{pmatrix} 0 \\ bx_3 - \lambda(x_2 - 2x_3) \\ -bx_3 \end{pmatrix} \\ \mathbf{f}_{q_{11}} &= \begin{pmatrix} \varphi_2(\mathbf{x}) - x_1 + \eta \\ x_3 - qx_2 - \rho \\ x_1 - bx_3 \end{pmatrix}, \mathbf{f}_{q_{12}} = \begin{pmatrix} \varphi_2(\mathbf{x}) + x_1 + \eta \\ x_3 - qx_2 - \rho \\ -x_1 - bx_3 \end{pmatrix} \\ \mathbf{f}_{q_{13}} &= \begin{pmatrix} \varphi_1(\mathbf{x}) - x_1 - \eta \\ \rho - x_3 - qx_2 \\ x_1 - bx_3 \end{pmatrix}, \mathbf{f}_{q_{14}} = \begin{pmatrix} \varphi_1(\mathbf{x}) + x_1 - \eta \\ \rho - x_3 - qx_2 \\ -x_1 - bx_3 \end{pmatrix} \end{aligned}$$

$$\mathbf{f}_{q_{15}} = \begin{pmatrix} -\lambda(x_1 - 2\rho) - \eta \\ 0 \\ 0 \end{pmatrix}, \mathbf{f}_{q_{16}} = \begin{pmatrix} 0 \\ bx_3 - \lambda(x_2 - 2x_3) \\ -bx_3 \end{pmatrix}$$

$$\mathbf{f}_{q_{17}} = \begin{pmatrix} \varphi_2(\mathbf{x}) - x_1 - \eta \\ x_3 - qx_2 - \rho \\ x_1 - bx_3 \end{pmatrix}, \mathbf{f}_{q_{18}} = \begin{pmatrix} \varphi_2(\mathbf{x}) + x_1 - \eta \\ x_3 - qx_2 - \rho \\ -x_1 - bx_3 \end{pmatrix}$$

with $\varphi_1(\mathbf{x}) = qx_2 + bx_3 - \lambda(x_1 + x_2 - 2x_3) - (\rho - x_3)$ and $\varphi_2(\mathbf{x}) = qx_2 + bx_3 - \lambda(x_1 + x_2 - 2x_3) + \rho - x_3$.

- **Conditions for a transition (guards).** The 27 symbols associated with the transitions represent the 27 types of guards in the hybrid automaton, that is:

- For $|s_d| \leq \delta$, locations $q_1 - q_6$:

$$\begin{aligned} \mathbf{a} &\Leftrightarrow x_1 > \delta, x_2 > \delta, |s_d| \leq \delta, \\ \mathbf{b} &\Leftrightarrow |x_1| \leq \delta, x_2 > \delta, |s_d| \leq \delta, \\ \mathbf{c} &\Leftrightarrow x_1 < -\delta, x_2 > \delta, |s_d| \leq \delta, \\ \mathbf{d} &\Leftrightarrow x_1 > \delta, |x_2| \leq \delta, |s_d| \leq \delta, \\ \mathbf{e} &\Leftrightarrow |x_1| \leq \delta, |x_2| \leq \delta, |s_d| \leq \delta, \\ \mathbf{f} &\Leftrightarrow x_1 < -\delta, |x_2| \leq \delta, |s_d| \leq \delta, \\ \mathbf{g} &\Leftrightarrow x_1 > \delta, x_2 < -\delta, |s_d| \leq \delta, \\ \mathbf{h} &\Leftrightarrow |x_1| \leq \delta, x_2 < -\delta, |s_d| \leq \delta, \\ \mathbf{i} &\Leftrightarrow x_1 < -\delta, x_2 < -\delta, |s_d| \leq \delta, \end{aligned}$$

- For $s_d < -\delta$, locations $q_7 - q_{12}$:

$$\begin{aligned} \mathbf{j} &\Leftrightarrow x_1 > \delta, x_2 > \delta, s_d < -\delta, \\ \mathbf{k} &\Leftrightarrow |x_1| \leq \delta, x_2 > \delta, s_d < -\delta, \\ \mathbf{l} &\Leftrightarrow x_1 < -\delta, x_2 > \delta, s_d < -\delta, \\ \mathbf{m} &\Leftrightarrow x_1 > \delta, |x_2| \leq \delta, s_d < -\delta, \\ \mathbf{n} &\Leftrightarrow |x_1| \leq \delta, |x_2| \leq \delta, s_d < -\delta, \\ \mathbf{o} &\Leftrightarrow x_1 < 0, |x_2| \leq \delta, s_d < -\delta, \\ \mathbf{p} &\Leftrightarrow x_1 > 0, x_2 < 0, s_d < -\delta, \\ \mathbf{q} &\Leftrightarrow |x_1| \leq \delta, x_2 < 0, s_d < -\delta, \\ \mathbf{r} &\Leftrightarrow x_1 < 0, x_2 < 0, s_d < \delta, \end{aligned}$$

- For $s_d > \delta$, locations $q_{13} - q_{18}$:

$$\begin{aligned} \mathbf{s} &\Leftrightarrow x_1 > \delta, x_2 > \delta, s_d > \delta, \\ \mathbf{t} &\Leftrightarrow |x_1| \leq \delta, x_2 > \delta, s_d > \delta, \\ \mathbf{u} &\Leftrightarrow x_1 < -\delta, x_2 > \delta, s_d > \delta, \\ \mathbf{v} &\Leftrightarrow x_1 > \delta, |x_2| \leq \delta, s_d > \delta, \\ \mathbf{w} &\Leftrightarrow |x_1| \leq \delta, |x_2| \leq \delta, s_d > \delta, \\ \mathbf{x} &\Leftrightarrow x_1 < -\delta, |x_2| \leq \delta, s_d > \delta, \\ \mathbf{y} &\Leftrightarrow x_1 > \delta, x_2 < -\delta, s_d > \delta, \\ \mathbf{z} &\Leftrightarrow |x_1| \leq \delta, x_2 < -\delta, s_d > \delta, \\ \mathbf{ch} &\Leftrightarrow x_1 < -\delta, x_2 < -\delta, s_d > \delta. \end{aligned}$$

The hybrid automaton H_{LC} was simulated for the initial hybrid state $q_0 \times \mathbf{x}_0 = q_{18} \times (-2.1 \cdot 10^{-4}, -5.20625 \cdot 10^{-3}, 6.895783)^T$, and for the controller parameters $\eta = 3$, $\lambda = 0.1$ and $\eta = 0.3$, $\lambda = 0.01$. The results are presented in Figures 6 and 7.

As it was expected, the final discrete location for the controlled system is location q_4 , for which $x_1 = x_2 = x_3 =$

Fig. 6. Response of the controlled Lorenz hybrid automaton H_{LC} with $\eta = 3$ and $\lambda = 0.1$.

Fig. 7. Response of the controlled Lorenz hybrid automaton H_{LC} with $\eta = 0.3$ and $\lambda = 0.01$.

$s_d = 0$ is obtained. Notice that from dynamics $\dot{\mathbf{x}} = f_{q_4}(\mathbf{x})$, the trajectories are ensured to converge to $x_2 = x_3 = x_1 = 0$, because x_1 is always zero entering and remaining within q_4 . Changes in η and λ influence the time of convergence to $s_d = 0$. The bigger η and λ are, the faster s_d approaches zero. However, this implies a higher control effort.

A complete study of the controlled system is necessary. In particular, the analysis of the change of stability properties when some key parameters are changed (e.g., λ), in addition to the analysis of complex behaviours, unexpected transitions and bifurcations due to the interaction of three discontinuity surfaces in the system. This is out of the scope of this paper.

V. CONCLUSIONS

The Lorenz hybrid automaton and the controlled Lorenz hybrid automaton are defined in this paper in order to formulate discontinuous chaotic systems with several discontinuity surfaces within the hybrid-automaton framework. The proposed hybrid automata are especially effective in the specification of the transitions between the different modes of operation of the system, which becomes critical when multiple switching elements are present. The use of a computational framework for specifying the dynamical properties of the system can facilitate the computer-science-based control of chaotic systems.

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