Robust Output Feedback Consensus for Networked Negative-Imaginary Systems

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Abstract—A robust output feedback consensus problem for networked homogeneous Negative-Imaginary (NI) systems is investigated in this technical note. By virtue of NI systems theory, a set of reasonable yet elegant conditions are derived for output consensus under $\Omega_2$ external disturbances as well as NI model uncertainty. As a byproduct, this technical note also reaffirms a previous result in [1] which shows the robustness of networked systems is always worse than that of single agent system. Furthermore, the eventual convergence sets are also characterised for several special NI systems that are commonly studied in the literature. It is shown how the results in this work embed and generalise earlier results for these classes of systems. We show that the natural convergence set boils down to the centroid of the initial pattern when the initial conditions of the controllers are zero. Numerical examples are given to showcase the main results.

Index Terms—Consensus, Cooperative Control, Negative-Imaginary Systems, Robust Control.

I. INTRODUCTION

NI systems are, broadly speaking, systems with a negative imaginary frequency response. This class of systems has received extensive attention in recent years [2], [3] since it was introduced in [4] and found its most successful application in the area of nano-positioning control [5] where co-located force actuation and position measurement are typical [6]. NI systems theory has also been widely applied to the control of flexible structures with highly-resonant dynamics, which is typically a challenging task to tackle via classical methods. Robust stability analysis of interconnected systems with mixed NI and small-gain properties has also been studied in [7].

The area of cooperative control has been very active over the past decade and it was immediately evident that distributed control and communication networks play an important role in stability analysis. The output feedback consensus problem, or more precisely, the output synchronization problem was first studied in [8], and a solution for weakly minimum phase nonlinear systems with relative degree one was presented. Later, [9] extended the result to heterogeneous cases even with uncertainties. The output feedback consensus problem that we consider is to have all the outputs naturally converge to a common trajectory (not necessarily constant) which is entirely determined by the subsystems themselves as well as the graph properties. Although similar approaches were presented in [1], [10] and [11] using a state-space representation, this work can be distinguished from these works via the following aspects: (a) a much simpler D.C gain condition for robust output feedback consensus is given, while the aforementioned works mainly build on the existence of a matrix or matrices such that via the following aspects: (a) a much simpler D.C. gain condition for similar approaches were presented in [1], [10] and [11] using a state-theory of the subsystems themselves as well as the graph properties. Although similar approaches were presented in [1], [10] and [11] using a state-space representation, this work can be distinguished from these works via the following aspects: (a) a much simpler D.C. gain condition for robust output feedback consensus is given, while the aforementioned works mainly build on the existence of a matrix or matrices such that

(c) this work also captures the result of [1] regarding the robustness of the multi-agent systems is never better than that of single agent systems. Recently, a robust consensus problem for heterogeneous multi-agent systems was discussed in [12]. However, the agents considered are constrained to second-order systems, which is just an example of NI systems and the consensus algorithm is based on full state information which is infeasible in most cases, whereas here we handle output feedback. Another work [13] addressed an output consensus problem of heterogeneous uncertain linear multi-agent systems. However, this work requires the following assumptions: (a) [13] makes a minimum phase assumption on all plants which allows the use of high gain control whereas the NI systems in this work are not necessarily minimum phase; (b) [13] only studies a class of unmodelled dynamics but does not explicitly tackle $\Omega_2$ external disturbances whereas this work studies both; (c) again, [13] deals with an output synchronization problem to a limited class of trajectories, such as constant, sinusoidal and diverging signals which are polynomial functions of time due to technical reasons whereas this work studies a consensus problem naturally converging to an unspecified trajectory.

This technical note is motivated by applications in which the system goal cannot be accomplished by a single NI system due to limitations in its capability, such as coverage or precision. This in turn requires the coordination of multiple NI systems, which in this work involves output feedback consensus under external disturbances and model uncertainty. In this technical note, a homogeneous network of NI systems and a fixed communication topology are assumed. The ith NI system is described in the s-domain:

$$y_i = P(s)u_i, i = 1, \ldots, n,$$  (1)

where $P(s)$ is the transfer function (generally MIMO), $y_i \in \mathbb{R}^{m_i \times 1}$ and $u_i \in \mathbb{R}^{n_i \times 1}$ are the output and input of the system with the dimension $m_i \geq 1$, $n_i > 1$ is the number of agents. Then, an elegant problem formulation, using the Laplacian matrix and Kronecker product, is adopted such that the output feedback consensus problem is cast into a robust stability problem, which can be solved via NI systems theory as detailed in [4], [14] and [15]. The contributions of this technical note can be summarized as follows: (a) it provides a novel viewpoint where consensus problems can be studied as internal stability problems, (b) it only exploits output feedback information as opposed to the full state feedback which is common in the literature, (c) it gives a class of consensus protocols that can be tuned for performance and/or robustness, (d) it provides a robustness guarantee via NI systems theory, and (e) it characterises the convergence sets.

Notation: $\mathbb{R}^{m \times n}$ and $\mathbb{C}^{m \times n}$ denote the sets of $m \times n$ real and complex matrices respectively, $I_n$ is the $n \times n$ identity matrix and $L_1$ is the $n \times 1$ vector with all elements being 1. Given $M \in \mathbb{R}^{m \times n}$, $M > (\leq) 0$ means $M$ is positive (negative) definite and $M \preceq (\succeq) 0$ means $M$ is positive (negative) semi-definite. $\lambda(M)$ denotes the largest eigenvalue of $M$ when $M$ has only real eigenvalues and $\sigma(M)$, $\sigma_i(M)$ represent the maximum and minimum singular values of $M$ respectively. $N(M)$ denotes the null space of $M$. $M^T$ and $M^*$ are the transpose and the complex conjugate transpose of $M$, respectively. In addition, given $s \in \mathbb{C}$, $\Re(s)$ is the real part of $s$. Given $a_1, a_2 \in \mathbb{C}$, $\text{diag}(a_1, a_2) = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$. Finally, given $a \in \mathbb{R}^{n \times 1}$, $\text{ave}(a)$ is the average operation of all elements of $a$. $\text{OLHP}$ is short for open left half plane and $\text{MMOS}$ is short for multi-input and multi-output.

Preliminaries of graph theory: A graph can be mathematically expressed by $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ is a nonempty finite set of $n$ nodes and an edge set $E \subseteq V \times V$ is used to model the communications links among nodes. The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, where $a_{ii} = 0$ and $a_{ij} \neq a_{ji}$.
of NI and SNI (short for Strictly Negative-Imaginary) systems: uncertainty is considered. To this end, let us first recall the definitions for networked NI systems under external disturbances and NI model $\bar{P}$ is undirected and connected:

$$L_n \geq 0, \text{null}(L_n) = \text{span}\{1_n\}. \quad (2)$$

II. ROBUST OUTPUT FEEDBACK CONTROL PROTOCOL

In this section, a class of output feedback consensus protocols for networked NI systems under external disturbances and NI model uncertainty is considered. To this end, let us first recall the definitions of NI and SNI (short for Strictly Negative-Imaginary) systems:

**Definition 1:** ([15]) A square, real, rational, proper transfer function $P(s)$ is NI if the following conditions are satisfied:
1) $P(s)$ has no pole in $\text{Re}[s] > 0$;
2) $\omega_d > 0$ such that $j\omega$ is not a pole of $P(s)$, $j(P(j\omega) - P(\bar{P}(j\omega)^n)) \geq 0$;
3) If $s = j\omega_0$ with $\omega_0 > 0$ is a pole of $P(s)$, then it is a simple pole and the residue matrix $K = \lim_{s \to j\omega_0} (s - j\omega_0)P(s)$ is Hermitian and positive semi-definite;
4) If $s = 0$ is a pole of $P(s)$, then $\lim_{s \to 0} s^kP(s) = 0, \forall k \geq 3$ and $P_2 = \lim_{s \to 0} s^2P(s)$ is Hermitian and positive semi-definite.

It can be observed that Definition 1 captures the definitions of NI systems in [4] and [14]. Examples of NI systems can be found in [6] and include single-integrator systems, double-integrator systems, undamped and damped flexible structures, to name a few typically considered in the consensus literature.

**Definition 2:** ([4]) A square, real, rational, proper transfer function $P_i(s)$ is SNI if the following conditions are satisfied:
1) $P_i(s)$ has no pole in $\text{Re}[s] > 0$;
2) $\omega_d > 0, j(P_i(j\omega) - P_i(\bar{P}(j\omega)^n)) > 0$.

Homogeneous NI agents are defined in the $s$-domain in the form of (1). Since $P(s)$ is in general a MIMO plant, the Laplacian matrix describing the network interconnection is modified via a Kronecker product to $L_n \otimes I_m$ and the total networked plant under consideration is depicted in Fig. 1 with

1) output consensus is achieved, i.e., $y_i \to y_\infty, \forall i \in \{1, \cdots, n\}$ for a family of plant dynamics with no external disturbance, where $y_\infty$ is the final convergence trajectory.
2) $y_\infty$ is perturbed by additive $\epsilon_0[0, \infty]$ signals when $\epsilon_0[0, \infty]$ disturbances are present on both input and output.

It can be seen that the output $y$ reaches consensus when $\bar{y} \to 0$ via the properties of the Laplacian given in (2). This formulation actually converts the output consensus problem to an internal stability problem which is easier to tackle and investigate the robustness property via standard control theoretic methods. We now impose the following standing assumption:

**Assumption 1:** $G$ is undirected and connected.

The following preliminary lemmas are needed:

**Lemma 1:** ([16]) Let $\lambda_j$ and $\gamma_k, j = 1, \cdots, n, k = 1, \cdots, m$, be eigenvalues of matrices $\Lambda_n \otimes m$ and $\Gamma_{m \times m}$ respectively, the eigenvalues of $\Lambda \otimes \Gamma$ are $\lambda_j \gamma_k$.

Note that Lemma 1 also applies to the singular values [16].

**Lemma 2:** Given $\Lambda \in \mathbb{R}^{n \times n}$ and $\Gamma \in \mathbb{R}^{m \times m}$, then

$$\text{null}(\Lambda \otimes \Gamma) = \{a \otimes b : b \in \mathbb{R}^{m \times 1}, a \in \text{null}(\Lambda)\} \cup \{c \otimes d : c \in \mathbb{R}^{n \times 1}, d \in \text{null}(\Gamma)\}.$$

Proof: the proof simply follows from the definition of null space and the properties of Kronecker product.

The following lemma states that the augmented networked plant $\bar{P}(s) = L_n \otimes P(s)$ is NI if and only if every single system $P(s)$ is NI.

**Lemma 3:** $\bar{P}(s)$ is NI if and only if $P(s)$ is NI. **Proof:** First note that $L_n \geq 0$ due to Assumption 1 in (3). Then, the sufficiency and necessity are straightforward by applying Lemma 1 to Definition 1.

Since Lemma 3 requires positive semi-definiteness of $L_n$, this work cannot be applied to directed graphs. The output $\bar{y}$ to 0 if internal stability is achieved for $\bar{P}(s)$ with some controller. From [4], [14] and [15], the following internal stability results are summarized:

**Lemma 4:** Given an NI transfer function $P(s)$ and an SNI transfer function $P_i(s)$ with $P_2 = \lim_{s \to 0} s^2P(s), P_1 = \lim_{s \to \infty} s(sP(s) - \bar{P})$ and $P_3 = \lim_{s \to \infty} (sP(s) - \frac{P_1}{s})$, the positive feedback interconnection $\bar{P}(s)P_3(s)$ is internally stable if and only if any of the following conditions is satisfied:

1) $\lambda(\bar{P}(0)P_2(0)) < 1$ when $P(s)$ has no pole(s) at the origin, $P(\infty)P_2(\infty) = 0$ and $P_3(\infty) \geq 0$;
2) $P_3(0)J < 0$ when $P(s)$ has pole(s) at the origin and is strictly proper, $P_2 \neq 0, P_3 \neq 0, N(P_2) \subseteq N(P_3^T)$, where $P_3 = J^T$ with $J$ having full column rank;
3) $F^T P_3(0)F_1 < 0$ when $P(s)$ has pole(s) at the origin and is strictly proper, $P_2 \neq 0, P_3 \neq 0, N(P_3^T) \subseteq N(P_2)$, where $P_1 = F_1V_1^T$ with $F_1$ and $V_1$ having full column rank and $V_1F_1 = I$.

Note that the above result is actually a robust stability result because an NI plant $P(s)$ can be perturbed by any unmodelled dynamics $\Delta(s)$ such that the perturbed plant $P_\Delta(s)$ which then replaces the nominal plant $P(s)$ in Lemma 4 retains the NI system property and still fulfills any one of the conditions of Lemma 4. Similarly, $P_i(s)$ can be perturbed to any SNI controller subject to 1), 2), 3). Henceforth, we do not distinguish between $P(s)$ and $P_\Delta(s)$ for simplicity of notation, though it is stressed that $P(s)$ could be the resulting perturbed dynamics of some simpler nominal plant. There is clearly a huge class of permissible dynamic perturbations to the nominal dynamics as conditions 1), 2) and 3) impose a restriction on $P(s)$ only at the frequency $\omega = 0$ or on the associated residues of $P(s)$ at $\omega = 0$ and the NI class has no gain or order restriction [4]. A few examples of permissible perturbations are $a_{ij} = 1 \text{ if } (v_i, v_j) \in E \text{ and } 0 \text{ otherwise. The in-degree of node } i \text{ is defined as } d_i = \sum a_{ij} \text{ and } D = \text{diag}\{d_1, d_2, \cdots, d_n\} \in \mathbb{R}^{n \times n} \text{ is the in-degree matrix. The Laplacian matrix of graph } G \text{ is given by } L_n = D - A. \text{ A sequence of successive edges of } E \text{ in the form of } ((v_i, v_k), (v_k, v_j), \ldots, (v_n, v_j)) \text{ is defined as a path from node } i \text{ to node } j. \text{ An undirected graph is said to be connected if there is a path from node } i \text{ to node } j \text{ for all the distinct nodes } v_i, v_j \in V. \text{ It is well-known that } L_n \text{ has the following properties when the graph is undirected and connected:}$

$$L_n \geq 0, \text{null}(L_n) = \text{span}\{1_n\}. \quad (2)$$

![Fig. 1. Networked NI systems](image-url)
Theorem 1: Given a graph $\mathcal{G}$ which satisfies Assumption 1 and models the communication links for networked homogeneous NI systems, and given any SNI control law $P_s(s)$, robust output feedback consensus is achieved via the protocol

$$U_{i,\infty} = \hat{P}_i(s)\tilde{y} = C_u(s)y = (L_n \otimes P_s(s))y$$

shown in Fig. 2, or in a distributed manner for each agent $i$,

$$u_i = P_s(s)\sum_{j=1}^{n}a_{ij}(y_j - y_i),$$

under any external disturbances $d_1, d_2 \in \mathbb{L}_2[0,\infty)$ and any model uncertainty which retains the NI system property of the perturbed plant $P(s)$ if and only if $P(s)$ and $P_s(s)$ satisfies 1), 2) or 3) in Lemma 4 except that

$$\tilde{\lambda}(P(0)P_s(0)) < \frac{1}{\tilde{\lambda}(L_n)}$$

replaces $\lambda(P(0)P_s(0)) < 1$ in case 1).

Proof: Before presenting the consensus result, let us first prove the internal stability of $[\hat{P}(s), \hat{P}_i(s)]$ using Lemma 4. From Fig. 2, we have $P(s) = L_n \otimes P(s)$ which has been shown to be NI in Lemma 3 and it is straightforward to see $P_i(s) = I_m \otimes P(s)$ is SNI since $P_i(s)$ is SNI.

($\Rightarrow$) Sufficiency: From Lemma 4, we can conclude that $[\hat{P}(s), \hat{P}_i(s)]$ is internally stable since

1) when $P(s)$ has no pole(s) at the origin, $\hat{P}(s)$ has no pole(s) at the origin as well. Also, $\hat{P}(\infty)P(\infty) = (L_n \otimes P(\infty))(I_m \otimes P(\infty)) = (L_n \otimes P(\infty))P(\infty) = 0$ and $P(\infty) = L_n \otimes P(\infty) = 0$ due to Lemma 1 as well as the pair of $[P(s), P_i(s)]$ satisfies condition 1) of Lemma 4. Finally, $\tilde{\lambda}(P(0)P_i(0)) = \tilde{\lambda}(L_n \otimes P_i(0)) < 1$ since $\tilde{\lambda}(P(0)P_i(0)) < \frac{1}{\tilde{\lambda}(L_n)}$ due to Lemma 1.

2) when $P(s)$ has pole(s) at the origin, $\hat{P}(s)$ has pole(s) at the origin as well. In case of $P_2 = 0, P_1$, it is straightforward to see $N(L_n \otimes P_2) \notin N(P_2)$ and $N(L_n \otimes P_1) \notin N(P_1)$ due to Lemma 2 and $N(P_2) \subseteq N(P_1)$.

Remark 1: It can be seen that the condition in inequality (6) is stricter than that in the inequality of case 1) of Lemma 4 due to the network interconnection. If originally $P(0)$ was such that $0 < \lambda(P(0)P_i(0)) < 1$, the controller $P_i(s)$ needs to be tuned for smaller eigenvalues in order to satisfy inequality (6). On the other hand, if $\lambda(P(0)P_i(0)) < 0$, there is no need to tune further.

Fig. 2. Closed-loop system with SNI controllers.
and a minimal realization of the $i$th SNI controller $P_i(s)$,
\begin{equation}
\begin{aligned}
\dot{x}_i^{(m+1)} &= \bar{A}_i^{(m+1)}x_i^{(m+1)} + \bar{B}_i^{(m+1)}u_i^{(m+1)}, \\
\end{aligned}
\end{equation}
where $p$ and $q$ are the dimensions of the states of the NI plant and the SNI controller, respectively. The closed-loop system of Fig. 2 is given as
\begin{equation}
\begin{aligned}
\dot{x}_i &= \begin{bmatrix} I_n & A_i & \mathcal{L}_n \otimes B \mathcal{D} \mathcal{C} \\ A_i & \bar{B} \mathcal{D} \mathcal{C} \mathcal{L}_n \otimes BC \\ \mathcal{L}_n & \mathcal{L}_n \otimes BC \\ \mathcal{L}_n & \mathcal{L}_n \otimes B \mathcal{D} \mathcal{C} \end{bmatrix} x_i \\
&= \Phi T x_i
\end{aligned}
\end{equation}
\begin{equation}
\dot{x}_i = \Phi T x_i.
\end{equation}
(11)

The spectrum of $\Psi$ is of importance since it will determine the equilibria. In particular, in this work, the eigenvalues of $\Psi$ on the imaginary axis will determine the steady-state behaviour. To this end, the following lemma is given to characterise the spectrum of $\Psi$.

**Lemma 5:** Let $\lambda_i^* \in \mathbb{C}$ be the $i$th eigenvalue of $\mathcal{L}_n$, associated with eigenvector $v_i^*$. The spectrum of $\Psi$ is given by the union of spectra of the following matrices:
\begin{equation}
\begin{aligned}
\Psi_i &= \begin{bmatrix} \bar{A} + \lambda_i^* \mathcal{L}_n \otimes B \mathcal{D} \mathcal{C} & \lambda_i^* \mathcal{L}_n \otimes B \mathcal{D} \mathcal{C} \\ A_i & \bar{B} \mathcal{D} \mathcal{C} \mathcal{L}_n \otimes BC \\ \mathcal{L}_n & \mathcal{L}_n \otimes BC \\ \mathcal{L}_n & \mathcal{L}_n \otimes B \mathcal{D} \mathcal{C} \end{bmatrix}.
\end{aligned}
\end{equation}

Furthermore, let $[v_i^* \ v_j^T]^T$ be an eigenvector of $\Psi_i$. Then, the corresponding eigenvalue of $\Psi$ is $\lambda_i^* + \lambda_j^*$, where $\lambda_i^*$ and $\lambda_j^*$ are the eigenvalues associated with eigenvectors $v_i^*$ and $v_j^*$, respectively.

**Proof:** Let $\lambda_i^*$ be the eigenvalue of $\Psi_i$, and
\begin{equation}
\begin{aligned}
\Psi_i &= \begin{bmatrix} \lambda_i^* \mathcal{L}_n \otimes B \mathcal{D} \mathcal{C} & \lambda_i^* \mathcal{L}_n \otimes B \mathcal{D} \mathcal{C} \\ A_i & \bar{B} \mathcal{D} \mathcal{C} \mathcal{L}_n \otimes BC \\ \mathcal{L}_n & \mathcal{L}_n \otimes BC \\ \mathcal{L}_n & \mathcal{L}_n \otimes B \mathcal{D} \mathcal{C} \end{bmatrix},
\end{aligned}
\end{equation}

which shows that $\lambda_i^*$ is also an eigenvalue of $\Psi$. The associated eigenvector being $v_i^*$ and $v_j^*$.

It is well known in [19] that there is only one zero eigenvalue in $\mathcal{L}_n$, $\lambda^*_0 = 0$, when the graph $\mathcal{G}$ satisfies Assumption 1. In this case, $\psi_i$ has eigenvalues $\lambda_A$ and $\lambda_j$ associated with eigenvectors $v_j$, and
\begin{equation}
\begin{aligned}
(\lambda_A I_n - \lambda_j)\mathcal{L}_n \otimes B \mathcal{D} \mathcal{C} v_j
\end{aligned}
\end{equation}
respectively since $\psi_i = \begin{bmatrix} \bar{A} & 0 \\ \bar{B} \mathcal{D} \mathcal{C} A_i & \bar{B} \mathcal{D} \mathcal{C} \mathcal{L}_n \otimes BC \\ \mathcal{L}_n & \mathcal{L}_n \otimes BC \\ \mathcal{L}_n & \mathcal{L}_n \otimes B \mathcal{D} \mathcal{C} \end{bmatrix}$ is a Hurwitz matrix by Assumption 1. Thus, the steady state of the closed-loop system (11) in general depends only on the eigenvalues of $\Psi$ on the imaginary axis as shown in the following theorem.

**Theorem 2:** Given the closed-loop system in (11), the steady state can be expressed in the general form
\begin{equation}
\begin{aligned}
\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} &\rightarrow \infty \begin{bmatrix} w_1, & \ldots, & w_n \end{bmatrix} e^{J^r t} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} \begin{bmatrix} \bar{x}(0) \\ \dot{\bar{x}}(0) \end{bmatrix},
\end{aligned}
\end{equation}

where $J^r$ is the Jordan block associated with $\lambda_i^*$ eigenvalues of $\Psi$ on the imaginary axis denoted by $\lambda_A$, $v_j$, and $v_i$, are the right and left eigenvector of $\Psi$ associated with $\lambda_i^*$, given by
\begin{equation}
\begin{aligned}
w_j &= \begin{bmatrix} 0 & 1 \end{bmatrix} v_j, \quad v_i = \begin{bmatrix} 1 & 0 \end{bmatrix} \bar{v}_i,
\end{aligned}
\end{equation}

where $k = 1, \ldots, n$, $w_i^T$ and $v_i^T$ are the generalised right and left eigenvectors of $A$ associated with $\lambda_A$.

**Proof:** It is straightforward that
\begin{equation}
\begin{aligned}
\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} &\rightarrow \infty \begin{bmatrix} w_1, & \ldots, & w_n \end{bmatrix} e^{J^r t} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} \begin{bmatrix} \bar{x}(0) \\ \dot{\bar{x}}(0) \end{bmatrix},
\end{aligned}
\end{equation}

where $w_i = [v_1^T v_2^T]^T$, $v_i^T$, $w_i$, and $v_i$ are the generalised right and left eigenvectors of $A$ associated with $\lambda_A$.

It can be found, without loss of generality, that the right and left eigenvectors of $\Psi$ associated with the eigenvalues on imaginary axis are given in (15). Thereby, the steady state generally converges to
\begin{equation}
\begin{aligned}
\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} &\rightarrow \infty \begin{bmatrix} w_1, & \ldots, & w_n \end{bmatrix} e^{J^r t} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} \begin{bmatrix} \bar{x}(0) \\ \dot{\bar{x}}(0) \end{bmatrix},
\end{aligned}
\end{equation}

where $k = 1, \ldots, n$, $w_i^T$ and $v_i^T$ are the generalised right and left eigenvectors of $A$ associated with $\lambda_A$.

**Next, convergence sets of several special cases of NI systems are given in detail:**

**Corollary 1:** In the case that the NI plant is a single-integrator, i.e., $\dot{x}_1 = u_1$, $y_1 = x_1$, the convergence set of (11) is $y_{s+} = -\mathcal{C} \bar{A}^{-T} \text{ave}([\bar{B} \mathcal{D} \mathcal{C}]) + \text{ave}([\bar{B} \mathcal{D} \mathcal{C}])$.

**Proof:** The convergence set can be obtained by noting the eigenvectors $v_j = \begin{bmatrix} 0^T \ \mathbf{1}_m \end{bmatrix}^T$, $v_j = \begin{bmatrix} -\frac{1}{n} \mathcal{C} \bar{A}^{-T} \mathbf{1}_m \ \frac{1}{n} \mathbf{1}_n \end{bmatrix}^T$ and applying (17) in Theorem 2.

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Corollary 2: In the case that the NI plant is a double-integra-
tics as shown in Corollary 1 with the initial condition being
$y_{ss} = -C\dot{A}^{-T}T\cdot \text{ave} (\bar{x}(0)) + \text{ave} (\bar{\xi}(0)) + \text{ave} (\xi(0))t$.

Proof: For double-integrator plants, $n_x = 2 > 1$
for $\lambda(A) = 0$. The convergence set is straightforward by r-
ing $w_j = [0^T \ 1^T \ 0^T \ 0^T]^T$, $v_i = [0^T \ 0^T \ 1^T \ 0^T]^T$, $w_0/[0^T \ 0^T \ 0^T \ 0^T]^T$, $\bar{v}_0 = [-\frac{1}{4}C\dot{A}^{-T}1^T \ 0^T 0^T]^T$ after $\bar{\xi}$
ranging $x = [\bar{\xi}^T \ \bar{\xi}^T]^T$ and applying (14).

Corollary 3: In the case that the NI plant is a damped flex-
state consensus value perturbed by filtered disturbances as shown at
the top left of Fig. 6 under external disturbances.

IV. ILLUSTRATIVE EXAMPLES

In this section, numerical examples of typical NI systems are given
to illustrate the main results of this technical note. A scenario of 3 NI
systems is considered and the communication graph $G$ is given as
in Fig. 3. Therefore, the Laplacian matrix of $G$ can be derived
according to the definition in Section I:

$$L_3 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Fig. 3. Communication topology $G$ and associated Laplacian matrix

A. Multiple Single-Integrator Systems

Suppose that the NI systems have identical single-integrator
amics as shown in Corollary 1 with the initial condition be-
$x(0) = [1 \ 2 \ 3]^T$. The SNI controller is designed as indices
in Theorem 1 to be $\bar{A} = -2, \bar{B} = 1, \bar{C} = 1, \bar{D} = 1$
with the initial condition being $\bar{x}(0) = [0.1 \ 0.2 \ 0.3]^T$. Without
considering disturbances firstly, it can be verified as Corollary 1 that
$y_{ss} = -C\bar{A}^{-T}T\cdot \text{ave}(\bar{x}(0)) + \text{ave}(\bar{x}(0)) = \frac{1}{2} \times 0.2 + 2 = 2.1$, which is
shown at the top left of Fig. 4. If external disturbances are inserted,
robust output feedback consensus is also achieved with the steady
state consensus value perturbed by filtered disturbances as shown at
the top right of Fig. 4. The robust performance of the control law can
be improved by tuning the SNI controller to, for example $\bar{D} = -5$,
which are shown in the bottom left and right of Fig. 4, respectively.

One may notice that when the initial condition of the controller
$\bar{x}(0)$ is set to 0 (a reasonable choice as the controller is determined
by the designer), the convergence set naturally becomes the centroid
of the initial pattern, i.e., $y_{ss} = \text{ave}(\bar{x}(0))$, which in turn implies that
the result for the average consensus protocol in [19] is a special case
of the proposed result. Alternatively, the desired convergence point
can be chosen by properly initialising the SNI controller, which can
be seen as a more general result.

B. Multiple Double-Integrator Systems

Suppose that the NI systems have identical double-integrator
dynamics as shown in Corollary 2 with the initial conditions being
$\xi(0) = [1 \ 2 \ 3]^T$, $\bar{\xi}(0) = [0.1 \ 0.2 \ 0.3]^T$. The same SNI controller can
be adopted as in Subsection IV-A. Without considering disturbances
firstly, it can be verified using Corollary 2 that $y_{ss} = \bar{\xi}(\infty) =
-C\bar{A}^{-T}T\cdot \text{ave}(\bar{\xi}(0)) + \text{ave}(\bar{\xi}(0))$ and $\bar{\xi}(\infty) = [0.2 + 2 + 0.2t = 2.1 + 0.2t \ 0.2 + 0.2t =
0.3, 0.3]$ which is exactly as shown at the top of Fig. 5.
If the same disturbances as in Subsection IV-A are inserted, output
consensus is also achieved with the steady state values perturbed
by filtered disturbances as shown at the bottom of Fig. 5. Again,
appropriate choices of the SNI controller can be made to minimise
the effects of external disturbances, which is omitted here due to the
page limitations.

One can also choose the initial condition of the controller to be
$\bar{x}(0) = 0$ to obtain the natural convergence set as $y_{ss} = \xi_{ss} =
\text{ave}(\xi(0)) + \text{ave}(\xi(0))t$ and $\bar{\xi}_{ss} = \text{ave}(\bar{\xi}(0))$. The same conclusion
can hence be drawn as in Subsection IV-A.

C. Multiple Flexible Structures Systems

Suppose that the NI systems are considered as undamped flexible structures as shown for example in Fig. 2 of [4]: $M\ddot{x}_i + C\dot{x}_i + Kx_i = u_i$, $y_i = x_i$, $i = 1, \ldots, 3$, where $x_i = [x_i^T \ \dot{x}_i^T]^T$, $u_i =
[u_i^T \ \dot{u}_i^T]^T$, $M = \text{diag}(m_1, m_2)$, $C = c_1 (c_1 + c_2 \ c_2 + c)$, $K =
[k_1 + k \ \ -k \ \ -k]$, $k_1 = k_2 = k = 1$ and $c_1 = c_2 = c = 0.1$. The initial conditions are given as
$x(0) = [1 \ 2 \ 3 \ 4 \ 5 \ 6]^T$ and $\bar{x}(0) = [0.1 \ 0.2 \ 0.3 \ 0.4 \ 0.5 \ 0.6]^T$.

The SNI controller can be designed as indicated in Theorem 1 to be $\bar{A} = -4I_2, \bar{B} = I_2, \bar{C} = I_2, \bar{D} = 0$ since $\lambda(P(0)) = 1$ and thus
$P(0) = I$ has the same conclusion being $[0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$. Robust output feedback consensus can be achieved as shown at the top of Fig. 6 under external disturbances, which also
validates Corollary 3.

If the NI systems are considered as undamped flexible structures as shown in Fig. 2 of [14], which correspond to the above
damped flexible structure dynamics without the damping term $C$, robust
output feedback consensus can be achieved as shown at the bottom
of Fig. 6 under external disturbances.

V. CONCLUSION

NI systems include a wide range of LTI systems that are commonly
studied in the consensus literature. This class of systems
and corresponding theory also include a large class of dynamical systems that have not been studied in consensus literature to date. The robust output feedback consensus problem for this class of systems is hence of interest. The advantage of using NI systems theory for solving the consensus problem is four-fold: (a) it only uses output feedback information as opposed to full-state feedback information; (b) it provides robustness guarantees w.r.t. $L_2$ external disturbance; (c) it allows tuning of a whole class of SNI control laws for performance; and (d) it bypasses traditional searches for Lyapunov candidate functions. In addition, the characterised convergence set also makes it possible to initialise the controller state to achieve the desired final consensus target.

Future research directions include robust output feedback consensus for networked heterogeneous NI systems as well as the impact of switching topologies and time delays.

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REFERENCES


