A New Keynesian Model with Heterogeneous Price Setting

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Abstract

The Calvo contract pricing mechanism has become the most widely accepted microfoundation to the NK Phillips curve but unfortunately predicts that all firms in the economy face the same probability of price change. To better explain the stylized fact this paper relaxes the homogeneous firm assumption in the Calvo contract, to provide a macroeconomic explanation more consistent with recently available microeconomic evidence that suggests firms face differing probabilities of price change. A simple New Keynesian dynamic stochastic general equilibrium (DSGE) model with nominal rigidities and habit in consumption for the US is estimated using Bayesian techniques and finds evidence of a flexible price sector of around 6% and a sticky price sector of between 55% and 70% depending on model specification.

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1 Introduction

The Calvo contract pricing mechanism has become the most widely accepted microfoundation to the NK Phillips curve but unfortunately predicts that all firms in the economy face the same probability of price change. To better explain the stylized fact this paper relaxes the homogeneous firm assumption in the Calvo contract microfoundation, to provide a macroeconomic explanation more consistent with recently available microeconomic evidence. A simple New Keynesian dynamic stochastic general equilibrium (DSGE) model is presented with sticky prices and habit in consumption, for the US area. The model type is chosen for the economical use of theoretical innovation, and introduces a stylized aggregate pricing mechanism to account for recent micro level evidence that suggests that, contrary to the Calvo prediction, firms face differing probabilities of price change. The resulting aggregate price mechanism captures these differing probabilities via heterogeneous price sectors; a flexible price sector, a sticky price sector and traditional Calvo sector. A set of restricted models, with the habit parameter set to zero, are then estimated using Bayesian techniques and find an estimated size for the flexible price sector and sticky price sector of around 6% and 55% respectively. For the unrestricted set of models, with non zero habit consumption, the results consistently predict a flexible sector of around 6% and show that controlling for heterogeneity in price setting can improve the overall model fit. Furthermore, for the unrestricted case, the innovations substantially increase the persistence of monetary shocks, a finding consistent with Carvalho (2006).

Modern monetary policy relies on the stabilization of prices through mechanisms employed by national and central banks. It is the banks remit to anticipate shocks to aggregate supply and ultimately act to promote price constancy. A key issue for the central banker when applying mechanisms of price inflation control is the degree of ‘stickiness’ in the economy, or the lag to which prices react to a monetary policy shock. The modern breed of dynamic macro models need to explain the stylized fact
so that the central bank can answer questions of relevance to the conduct of monetary policy. One of the most qualified criticisms of the standard New Keynesian (NK) model is its inability to generate as much inflation persistence as that displayed in the stylized fact. Attempts to address this problem include numerous ad hoc adjustments to the NK Phillips curve using competing micro foundations, such as the time and state dependent pricing mechanisms of Calvo (1983), Taylor (1980) and Rotemberg (1982), even the addition of a backward looking element to compliment the rational expectations argument in the Phillips curve to increase price rigidity and address this issue, see Gali and Gerlter (1999) and Sheedy (2007) for a microfounded hybrid NK Phillips curve.

Recent evidence from the Inflation Persistence Network, IPN\(^1\), namely Euro micro data, has questioned the ability of all the structural models to explain the micro level behaviour, see Angeloni (2005). Specifically that, according to these facts, unconditional hazard functions of price changes are decreasing in the duration of price contracts. Contrary to these findings the widely accepted Calvo contract pricing mechanism, whilst appealing in its tractability and aggregate approximation, predicts a flat hazard rate of price change, suggesting that all firms face the same probability of price change regardless of contract duration due to the assumption of homogeneous firm types. There is a growing micro level literature that provides evidence of downward or upward sloping hazard functions of price change that challenge this prediction, discussed in the following subsection. None of these studies find evidence of the Calvo predicted flat hazard rate of price change.

The relaxation of the homogeneous firm assumption in the Calvo contract is motivated by the recent microeconomic literature on price changing behaviour in an attempt to provide a macroeconomic model more consistent with this recent evidence. This paper presents a baseline New Keynesian dynamic stochastic general equilibrium model, essentially a derivative of Gail (2002), with sticky prices and

\(^1\)The Inflation Persistence Network, (IPN), is a team of Eurosystem economists undertaking joint research on inflation persistence. The IPN is chaired by Frank Smets, European Central Bank
habit in consumption, for the US, which is estimated using Bayesian techniques with three key economic variables: output, prices and the nominal interest rate. To drive output from its natural rate we introduce three shocks to monetary policy, productivity and government expenditure. The key innovation is the simple relaxation of Calvo’s assumption of homogeneous firm types using a stylized aggregate price mechanism, or more specifically the inclusion of heterogeneous aggregate price sectors in the model. It is well documented that the modern breed of macro models require, at least, some explanation of the existence of a flexible price sector, or description of firms that change prices more frequently, an issue discussed in Smets and Wouters (2007).

To the baseline model, an inclusion of a flexible price sector, so described, intuitively reduces the ability of that model to explain inertia displayed by the data and raises the enquiry of just how that inertia should be represented in the modern breed of macro models. There is an emerging literature which concerns itself with the subject of the source of inertia, unfortunately too voluminous to mention here. However one such recent argument concerns itself with whether, or not, we should rely on nominal or real rigidities to provide the bulk of inertia in modern macro models, see Blanchard and Gali (2010) and Riggi and Tancioni (2010) for an extensive discussion. This recent deliberation within the macro literature, combined with the discussion over the source of inertia, motivates us not to concern ourselves with the amount of persistence that a particular microfoundation can provide us with but to recognize that the micro foundations introduced to our models must accurately describe the micro data they attempt to model, rather than solely provide a theoretical underpinning to innovations that provide a better fit to the data for which the models attempt to emulate. Accordingly we estimate two sets of models; a restricted set with habit parameter set to zero which relies solely on the Calvo explanation to explain persistence, outside of the monetary rule, and an unrestricted set with non zero habit consumption allowing output persistence to reduce the reliance on the
Calvo parameter.

The preference for a Bayesian approach to our estimation of the model is that it can utilize prior information, or beliefs, to characterize the posterior distribution of the models structural estimated parameters, a distinct advantage over other methods of estimating these structural model parameters, and additionally provide us with a posterior odds analysis to imply probabilities that can be assigned to competing models, even where models are not nested, although in this analysis the models are nested. We find that a simple baseline model that incorporates heterogeneous price setting can improve overall fit, or the ability to describe the inertia within the data, depending on the specification chosen. Our estimates predict a flexible price sector in the US of around 6% and a sticky price sector of around 55-70%. Although a model without the flexible price sector is preferred initially over the baseline case, the inclusion of habit in consumption to our model reverses this result so that the model with a flexible price sector is preferred over the baseline with habit. In both cases the estimated size of the flexible price sector is around 6%. The rest of this section discusses the literature on price change and the innovation in our paper. Section 2 sets out a simple baseline New Keynesian model with sticky prices and habit in consumption which is developed into a DSGE with shocks to productivity, demand and monetary policy. Section 3 describes our Bayesian methodology for estimation of the models structural parameters and comparison of two sets of competing models; The restricted set (Models 1 to 3), with habit parameter restricted to zero; Model 1 with a flexible and sticky price sector, Model 2 with only a sticky price sector and Model 3, a baseline without innovation. The unrestricted set (Models 1H to 3H); share the same specifications respectively but with the habit parameter unrestricted. The final section discusses the results of our estimation, the model comparison, and opportunities for further research.
1.1 The Literature on Price Change

The recent micro level research results provide us with an informed direction to improve model building beyond the answers to the Lucas critique, a source already commonly employed by recent business cycle literature for the formation of priors and calibrations before estimation of competing closed macro models. Most of the micro literature on price change, to date, uses the focus of the slope of the hazard function in price change. The hazard function in price duration could be defined as the probability, at a particular time $t$, of a firm resetting its price as a function of the time since its last price change. If prices become more likely to change, the older they become, then the hazard function of price change would become upward sloping, the latter outcome providing Sheedy (2007) with the motivation for a micro founded hybrid New Keynesian Phillips curve. Angeloni et al (2005), suggest that new micro evidence collected by the Eurosystem via the IPN seriously challenges the most commonly utilized assumptions in the current micro founded macro models. Unconditional hazard functions of price changes are decreasing in the duration of price spells, a fact which poses problems for both the standard state and time dependent model explanations. The standard time and state dependent models; Calvo, (1983), Taylor, (1999) or Rotemberg, (1982) often rely on adjustments in order to generate sufficient inflation persistence, suggesting that these structural models can merely explain moderate persistence during periods of monetary stability. This unresolved question, that the macro evidence can not distinguish between the different micro foundations, does not lend to the successful development of new theory. Angeloni et al (2005) conclude that a model that can pass the test of a literal description of the aggregated micro behaviour.

The inability of the standard state and time dependent models to explain sufficiently inflation persistence without adjustment may lie with the cause of that persistence. The standard models describe nominal rigidities but there may exist also real rigidities. Real rigidities are not under consideration in this paper, how-
ever there is growing commentary in this area, see Choudhary, Karlsson and Zoega (2007) for a micro discussion. Blanchard and Gali (2010), offer a real wage rigidity as a labour market friction in the baseline NK model in an attempt to address one of the main drawbacks of the standard NK model, namely that the model has an inability to generate inflation inertia beyond that inherited from the output gap. Riggi and Tancioni (2010) criticise the Blanchard and Gali real wage rigidity for generating excess real wage smoothness, caused by the constancy of the parameter representing the degree of wage stickiness. They call for the introduction of nominal wage rigidities; although we already have several studies that encompass these nominal rigidities such as the popular model of Smets and Wouters (2003) and its derivatives.

This most recent debate, on the location of inertia and the arguments for the use of real vs nominal rigidities to describe it, encourages us therefore, not to concern ourselves with the amount of persistence that a particular micro-foundation can provide us with but to recognize that the micro foundations introduced to our models must accurately describe the micro data it attempts to model. Other micro literature that implements a hazard function approach include Dias et al (2005) who estimate a hazard function for Portugal, but find that the frequency of price change tends to depend on sectoral heterogeneity, as some firms depend on state dependent factors and some on time dependent. Dias et al, using an estimation of hazard function approach, suggest that, from an economic point of view, state dependent rules are clearly more attractive than time dependent rules as they assume agents base their decisions on a cost-benefit analysis. The main problem for time dependent rules is that, by nature, they are ad hoc hypotheses and therefore unrealistic. A simple time dependent rule cannot provide a reasonable approximation to the data and thus state dependent models are required to fully characterize price setting behaviour of Portuguese firms. They find that the significant state variables are inflation, demand and size of previous price change. Controlling for this heterogeneity is one way to
tackle this bias when estimating the hazard function.

Angeloni et al (2005) consider survey evidence in respect of:

- Size of Firm
- Explicit/Implicit Contracts
- Sectoral Differences

They find that these factors are all significant heterogeneous factors behind price rigidities. Instead of the model of monopolistic competition a la Dixit and Stiglitz, they call for more complex tractable market structures. Carvalho (2006) shows that allowing for sectoral heterogeneity produces larger and more persistent effects from monetary shocks than would be the case in a homogeneous firm price setting economy; and that accordingly, an identical firm model would require a price changing frequency of up to three times higher than the average heterogeneous economy to approximate these dynamics.

The Calvo and Taylor contracts predict a constant hazard rate, when they are, according to the new evidence, in fact decreasing or increasing in the duration of price spells. Aucremanne and Dhyne (2005) and Dias, Robalo Marques and Santos Silva (2005) show that these unconditional hazard functions become flatter when one controls for heterogeneity.

Alvarez, Burriel and Hernando (2005) show that a mixture of pure Calvo with different probabilities of price adjustment provide a good estimation of declining hazard rates. They assume an economy made up of several types of Calvo agents; a flexible group of Calvo agents (price duration 1 month), an intermediate set of Calvo Agents (10 months), one group of sticky agents (3 years), and one group with an annual Calvo type price setting mechanism (18 months). Although this paper applies a more parsimonious approach within a New Keynesian framework, the work of Alvarez et al to address this issue provides a significant motivation for the model
in this paper. There is a growing literature providing microeconomic evidence on
the slope of the hazard function, in duration of price change, such as

and Campbell and Eden (2007), for the US, all find in favour of a downward sloping
hazard function. Baumgartner et al (2005) find, in an Austrian study, that the
aggregate hazard function for all price spells is decreasing with time, although they
also find strong evidence of state dependent or heterogeneous effects on price change,
an issue addressed by Carvalho (2006). Nakamura and Steinsson (2008), for the US,
find that hazard functions are predominantly downward sloping for the first few
months with a significant twelve month spike, which is more pervasive in producer
data, find that the longer a nominal price remains unchanged the less likely it is
to change, a finding consistent with a downward sloping hazard function. Contrary
to these findings are those of Gotte et al (2005) and Cecchetti (1986) who find
strong evidence of upward sloping hazard functions. Another study in this area by
Nakumura and Stiensson (2007) finds that hazard functions are largely flat with a
spike at about one year suggesting a cluster of annual price changes. As far as the
author is aware there are no findings of a purely horizontal hazard function of price
change, predicted by the widely adopted Calvo contract.

The model introduced in the following section introduces the notion of flexi
price firms mixed with, sticky prices and the standard Calvo agent. The adoption
of this innovation ultimately and predictably changes the level of persistence that
can be explained by the Calvo contract but is an attempt to encompass the recent
microeconomic evidence and studies on price change.

1.2 Relaxing the Assumption of Homogeneity

Drawing monetary policy conclusions from the presented micro evidence requires
structural models consistent with that evidence but these models should also be
analytically tractable.

Angeloni et al (2005) suggest a basic Calvo model extended to describe the heterogenic factors above would not be a bad approximation. The difficulty is highlighting the important micro features, or states, that affect the macro outcome. Alvarez et al (2005) show that a good description of the declining hazard rate can be achieved by mixing heterogeneous Calvo type agents. Carvalho (2006) suggests a model with multiple sectors with different degrees of stickiness.

The innovation of this paper is to relax Calvo’s assumption of the homogeneous firm through stylization of the aggregate price mechanism to include three firm types; the first type facing perfect price flexibility, a second with fixed or sluggish price change and the remainder awaiting the standard Calvo signal to change to their optimal price. By introducing heterogeneous firm types via the aggregate pricing mechanism we can not impose a non horizontal hazard function of price change, but can accommodate and estimate three sectors with different probabilities of price change, with the aim of encompassing the micro fact.

One such proposal to capture the differing probabilities of price change within the standard New Keynesian framework is presented below where a proportion $\zeta$ of firms face perfect price fluidity, a portion $\eta$ face sticky prices, and the remainder $(1 - \zeta - \eta)$ follow a Calvo type price setting.
2 A Simple New Keynesian Model

Our model is essentially a derivative of Gail (2002), with no capital and a perfectly competitive labour market. A cashless economy where homogeneous goods are produced by a final goods sector using CES technology. Price inertia in our model is explained using nominal rigidities a la Calvo (1983) and output inertia by habit formation in consumption. In the model output is driven from its natural rate by shocks to monetary policy, productivity and government expenditure. Our innovation is the addition of flexible and sticky price sectors, alongside the convenient Calvo explanation, to the aggregate price mechanism to control for the differing probabilities of price change highlighted by the recent evidence shown in the micro level literature on price change.

2.1 Preferences

The economy consists of a continuum of representative infinitely lived households whose instantaneous utility function is separable in consumption $C_t(i)$ and labour supply $N_t(i)$. As a result the first order condition for consumption growth will be independent of labour supply effects, as is consistent with the observed relative stability of labour supply in the US. We use habit formation in consumption as a real persistence mechanism and to reduce the reliance on the Calvo contract explanation of inertia displayed in the US data. Following the results of Levine, Pearlman and Yang (2008) the inclusion of persistence in labour supply is omitted to avoid over enrichment of the model.

The instantaneous utility function is given by

$$U(C_t, N_t) = \left( \frac{(C_t - hC_{t-1})^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right)$$

where $\sigma$ is the coefficient of relative risk aversion of households or the inverse of the
intertemporal elasticity of substitution and $\varphi$ is the elasticity of work effort with respect to the real wage or the inverse of the Frisch elasticity of labour supply. The parameter $h$ represents the proportion of habitual consumption or desire to herd. Households seek to maximize

$$E_0 \sum_{t=0}^{\infty} \left( \frac{(C_t - hC_{t-1})^{1-\sigma}}{1 - \sigma} - \frac{N_t^{1+\varphi}}{1 + \varphi} \right)$$  \hspace{1cm} (1)$$

subject to an intertemporal budget constraint of the form

$$\int_{0}^{1} P_t(i) C_t(i) di + Q_t B_t \leq B_{t-1} + W_t N_t + T_t$$  \hspace{1cm} (2)$$

where $P_t(i)$ is the price level, $C_t(i)$ the consumption of differentiated good, $i$, respectively. Households hold their wealth in the form of securities and accordingly $Q_t$ represents the price of the riskless, one period, bond; $B_t$ is the holdings of risk free bonds at the beginning of period $t$. $W_t$ is the nominal wage and $N_t$ a measure of households employed. $T_t$ is the lump sum component of net dividend income and taxation.

Maximising (1) subject to (2) yields the familiar Euler equations:

$$\frac{N_t^{\varphi}}{(C_t - hC_{t-1})^{-\sigma}} = \frac{W_t}{P_t}$$  \hspace{1cm} (3)$$

$$Q_t = \beta E_t \left[ \frac{(C_{t+1} - hC_t)}{(C_t - hC_{t-1})} \right]^{\sigma} \left[ \frac{P_t}{P_{t+1}} \right]$$  \hspace{1cm} (4)$$

The first equation represents the household’s labour supply decision, equating the real wage with the marginal rate of substitution between consumption and work effort. The second is the familiar Keynes-Ramsey rule which relates the expected future path of consumption to the real interest rate.
2.2 Technology

Each firm produces a differentiated good $i$ with labour the sole input and identical exogenous technology $A_t$ assumed to evolve over time. Aggregate supply, then, evolves according to the product:

$$Y_t(i) = A_t N_t(i)^{1-\alpha}$$  \hspace{1cm} (5)

In this CES production function the parameter $\alpha$ is the elasticity of output with respect to labour. We also assume that the labour market is perfectly competitive and wages are fully flexible.

All firms face the isoelastic demand schedule

$$C_t(i) = \left( \frac{P_t(i)}{P_t} \right)^{-\varepsilon} C_t$$

The parameter $\varepsilon$ is the elasticity of substitution between differentiated goods, or the elasticity of demand and $P_t \equiv \left( \int_0^1 P_t(i)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}}$ is the Dixit-Stiglitz aggregate price index.

In an environment of monopolistic competition firms choose a price to maximise discounted future profits

$$\max_{P_t} \sum_{k=0}^{\infty} \theta^k E_t [Q_{t,t+k}(P^{*}_t Y_{t+k/t} - \Psi_{t+k}(Y_{t+k/t}))]$$

The expression $\Psi(\cdot)$ represents the implicit form of the firms’s cost function, $P^{*}_t$ the firm’s target price and $Q_t$ the discount factor for nominal payoffs. Note that the constraint faced by firms is identical and therefore the price that they target will be the same.
subject to the demand constraint

\[ Y_{t+k|t} = \left( \frac{P^*_t}{P_{t+k}} \right)^{-\varepsilon} C_{t+k} \]

which is derived from the maximization of the Dixit Stiglitz consumption index subject to to any given level of expenditure.

The familiar solution to this problem can be expressed as:

\[
\frac{P^*_t}{P_{t-1}} = M \sum_{k=0}^{\infty} \theta^k E_t \left[ Q_{t,t+k} Y_{t+k|t} M C_{t+k|t} \Pi_{t-1,t+k} \right] \sum_{k=0}^{\infty} \theta^k E_t \left[ Q_{t,t+k} Y_{t+k|t} \right] \tag{6}
\]

Equation (6) expresses the relative target price for the optimizing firm as a weighted average of the expected path of current and future marginal cost.

Marginal cost can be defined as the ratio of the real wage to the marginal product of labour, written explicitly as:

\[ MC_t = \frac{W_t}{(1 - \alpha)AP_t N_t^{-\alpha}} \]

For later use it is convenient to rewrite this expression, using (3) and (5):

\[ MC_t = \frac{N_{t}^{\alpha+\alpha}}{(1 - \alpha)A(C_t - hC_{t-1})^{-\alpha}} \tag{7} \]

### 2.3 An Alternative Aggregate Price Level

Nominal rigidities are introduced in a manner consistent with much of the recent business cycle literature, using the model of Calvo (1983). Firms may reset their prices only when they receive a randomly set signal, generated with constant probability \(1 - \theta\) in any given period. Thus the remainder of firms that are unable to change price in any period can be considered the portion of sticky price firms in the
economy. The aggregate price level thus satisfies

\[ P_t = \left[ \theta(P_{t-1})^{1-\varepsilon} + (1 - \theta)(P_t^*)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}} \] (8)

For later use, dividing through by \( P_{t-1} \), this expression becomes convenient to write as

\[ \Pi_t^{1-\varepsilon} = \theta + (1 - \theta) \left( \frac{P_t^*}{P_{t-1}} \right)^{1-\varepsilon} \]

The aim of this paper is to relax the assumption of homogeneous firms. By introducing a further two sectors to the baseline pure Calvo case we can accommodate the recently discussed evidence: imagine one sector of firms facing flexible price in all time periods and another with prices fixed to the previous period. The remainder face the typical Calvo type price mechanism, we can not impose a downward or upward sloping aggregate hazard function of price duration upon the model and so the probability of price change will remain constant but now include three different probabilities. The notion of including different price sectors in a New Keynesian framework is not new. In fact, one such work involving the inclusion of a flexible and sticky price sector is described in Aoki (2001), who introduces the concept of a twin sector economy using a dynamic sticky price model in two sectors of production. To avoid model enrichment this work takes a more parsimonious approach to capture the heterogeneous behaviour displayed by firms in the micro literature. This delineation can be achieved through a simple stylization of the aggregate pricing mechanism. Gali and Gertler (1999) split the aggregate price mechanism in this manner to provide an explanation for a set of firms with backward looking behaviour to compliment the standard Calvo agents in a model of this type.

To capture heterogeneity in price setting this imagine a portion (\( \zeta \)) of firms outside the Calvo set up which face perfect price fluidity, and a further set of firms \( \eta \) that face fixed prices and the remaining proportion of firms facing the Calvo type price setting as \((1 - \zeta - \eta)\). It is convenient to assume that \( \zeta \in (0, 1] \) and \( \eta \in (0, 1] \).
Allowing this framework to capture these micro facts at the aggregate level, the aggregate price mechanism now becomes:

\[ P_t = \left[ \zeta (P_t^*)^{1-\varepsilon} + \eta (P_t-1)^{1-\varepsilon} + (1-\zeta - \eta) (\theta(P_t-1)^{1-\varepsilon} + (1-\theta)(P_t^*)^{1-\varepsilon}) \right]^{\frac{1}{1-\varepsilon}} \]

Note that in the trivial case that \( \zeta = \eta = 0 \), this price mechanism collapses down to the baseline, later referred to as Model 3 and 3H, which is the case that would result from the aggregate price mechanism given previously.

Rearranging and dividing through by \( P_t^{1-\varepsilon} \) yields a similar expression for later use

\[ \Pi_t^{1-\varepsilon} = (\eta + (1-\zeta-\eta)\theta) + (\zeta + (1-\zeta-\eta)(1-\theta)) \left( \frac{P_t}{P_{t-1}^*} \right)^{1-\varepsilon} \]  

(9)

2.4 Equilibrium

The market clearing conditions in the goods market and labour market are given as:

\[ Y_t = C_t + G_t \]

and

\[ Y_t = \frac{A_t N_t}{D_t} \]

where \( G_t \) is government expenditure, also interpreted as the exogenous element of aggregate demand, and \( A_t \) productivity, the exogenous element of aggregate supply. In our model both variables are responsible for driving output away from its natural rate.

Finally market clearing in the labour market implies

\[ N_t = \int_0^1 N_t(i) di \]
Assume further that government expenditure is met by lump sum taxes, then, by Walras’ Law we can dispense with the government budget constraint or bond market equilibrium condition.

### 2.5 The Log Linearized Model

A log linearized zero-inflation steady state, where lowercase variables describe proportional deviations from the deterministic steady state is given below:

Combine the linearized versions of equation (6) and (9) to get the Phillips curve:

\[
\pi_t = \frac{\beta \theta}{(\eta + \theta)(1 - \zeta)} E_t(\pi_{t+1}) + \frac{(\zeta + (1 - \zeta - \eta)(1 - \theta))}{(\eta + \theta)(1 - \zeta)} (1 - \theta \beta) \Theta \hat{mc}_t
\]

The inflation equation shows how our parameters of interest \( \zeta \) and \( \eta \) co-govern the sensitivity of inflation to changes in marginal cost alongside the price stickiness parameter in the Calvo contract \( \theta \). A larger value for \( \zeta \) increases the slope of the Phillips curve whereas larger values of \( \eta \) decrease the slope, as we would expect given the nature of the sectors that the parameters represent. It is worth noting that when \( \zeta \) and \( \eta \) are both equal to zero this inflation equation collapses down to the baseline New Keynesian Phillips Curve which would result from the standard aggregate pricing mechanism given by equation (8).

Marginal cost is given from the linearization of equation (7)

\[
mc_t = \frac{\sigma}{(1 - h)}(c_t - hc_{t-1}) + (\varphi + \alpha)n_t - a_t
\]

and equation (5) for aggregate supply

\[
n_t = \frac{1}{1 - \alpha} (y_t - a_t)
\]

See the appendix for the derivation of the consumer’s Euler equation from (4).
\[ c_t = \frac{h}{(1-h)c_{t-1}} - \frac{1}{(1-h)}E_t[c_{t+1}] - \frac{1}{\sigma(1+h)}[i - E_t[\pi_{t+1}] - \rho] \]

and from the goods market clearing condition we obtain

\[ y_t = c_y c_t + (1 - c_y)g_t \quad \text{where} \quad c_y = \frac{C}{Y} \]

In order to introduce a simple analysis of monetary policy and to provide a channel for a monetary shock to drive output from its natural rate we use a simple Taylor rule. As is consistent with the literature and to generate persistence from monetary shocks we include a smoothing parameter \( \rho_e \), following from Clarida et al (2000).

\[ i_t = \rho_e i_{t-1} + (1 - \rho_e)(\phi_\pi \pi_t + \phi_\pi \tilde{\pi}_t) + e_t \]

The linearized model is completed with three exogenous shocks: government expenditure, productivity and monetary. All processes follow first-order autoregressive processes with i.i.d normal disturbances:

\[ g_{t+1} = \rho_g g_t + \epsilon_{g,t+1} \]

\[ a_{t+1} = \rho_a a_t + \epsilon_{a,t+1} \]

\[ e_{t+1} = \rho_e e_t + \epsilon_{e,t+1} \]
3 Bayesian Estimation

Bayesian estimation offers a useful tool to estimate and evaluate dynamic stochastic general equilibrium models. The aim of implementing this methodology is to characterize the posterior distribution of the models parameters conditional on prior beliefs of the estimated parameters, a distinct advantage over other methods of estimating these structural models.

The posterior distribution is obtained by employing the Bayes rule:

\[ p(\theta / Y^T) = \frac{L(Y^T|\theta)p(\theta)}{\int L(Y^T|\theta)p(\theta)d\theta} \propto L(Y^T|\theta)p(\theta) \]

gives the Bayesian relationship between the posterior density, \( p(\theta / Y^T) \), the unconditional sample density, \( \int L(Y^T|\theta)p(\theta)d\theta \), and the prior density, \( p(\theta) \). The posterior density evolves from a weighted average of prior non sample information and the conditional densities. These weights are related to the variances of the prior distributions and the data. A tighter prior, therefore, will result in a more constrained, and perhaps less informative, estimation. The parameters are estimated by maximizing the likelihood function and then combining with the prior distributions of the parameters in the model, to form the posterior density functions. The posterior distributions are then optimized using Monte-Carlo Markov Chain (MCMC) simulation techniques. Under the Bayesian perspective, both the posterior distribution and the likelihood function can be utilized to obtain a probabilistic interpretation of the estimated parameters. Another advantage of this methodology is the ability to make model comparisons, even where the models are not nested, using posterior odds analysis, conveying relative probabilities to competing models. We make use of log likelihood race statistics to compare our model fit with the case of the baseline model, where \( \zeta = \eta = 0 \).

A number of structural parameters are kept fixed during the estimation in order to identify them separately. Obviously our estimation results are sensitive to this
calibration; but we justify this by assuming these values are estimated, equivalently, with a prior that exhibits a zero standard deviation. For our calibrated values we proceed in a manner which is consistent with quarterly data observations. For the preference parameters in our model we assume values commonly found in the business cycle literature. The discount factor $\beta$ is set to 0.99 which is congruous with a real interest rate of about 4%. The elasticity of labour supply $\varphi$ is set to unity, following Christiano et al (2005), which is between the more commonly used values in DSGE models and those estimates in the micro labour literature. The elasticity of demand, $\varepsilon$, is set to unity, following Christiano et al (2005), which is between the more commonly used values in DSGE models and those estimates in the micro labour literature. The elasticity of demand, $\varepsilon$, is a crucial parameter in our analysis as it primarily governs the sensitivity of inflation to marginal cost. Ellis (2006) provides empirical evidence of this parameter which is rather sensitive to model specification and remarks that assuming a constant value may be too restrictive, an observation addressed by Smets and Wouters (2003) who model the elasticity as a time varying stochastic process. However, our analysis assumes a constant markup and accordingly, we set this parameter to 6, following Blanchard and Gali (2010) although this is markedly lower than that used by Krause et al (2008) who set it high to address the sensitivity to marginal cost. The labour income share in the production function $(1 - \alpha)$ is set at 0.30. The distinguishing parameters of Models 1 and 1H are the proportion of flexi price firms, $\zeta$, and the proportion of sticky price firms, $\eta$. Both of these parameters are arbitrarily set with prior means of 0.30 and standard deviation 0.2 representing a relatively loose prior. The posterior mean of these parameters will provide us with an estimate of the size of the flexi and sticky price sectors in the US.

The data used for the estimation is US quarterly macro economic time series: real GDP, GDP deflator and the nominal interest rate from 1970:1 to 2004:1. As the log linearised steady state solution represents deviations from their natural rate, time series are detrended using a linear trend and converted to quarterly rates. The choice of prior distributions for the Bayesian estimation of DSGE models matters both for posterior values and for model comparison. The views on priors varies considerably
among commentators and, unfortunately, the facts described by the aggregate data are unable to discriminate amongst these views. One approach, suggested by Del Negro and Schorfheide (2008), to aid with this discrimination is the use of micro data studies, but work still needs to be done to show how the facts displayed by the micro data should relate to the macro picture, where much micro deviation is washed out in aggregation. As a result we are left to draw on the existing literature for the prior specification. The means and standard errors of the technology and government spending shocks are set with a mean of 0.85 and standard error 0.07, (monetary shock 0.75, 0.15). The corresponding innovations are harmonised, as in Smets and Wouters (2007) and consequently share a mean of 0.25 (monetary shock 0.05) and standard error of 2.00 representing fairly loose priors. The risk aversion parameter we also follow with a mean of 1.50 and a standard error of 0.375. For monetary policy we follow Levine, Pearlman and Yang (2008) so that the interest rate smoothing parameter has a mean of 0.75, an inflation feedback consistent with a robust FED response to inflation of 1.70 and an output feedback mean of 0.50.

We estimate the following model variants: The restricted models with $h = 0$ (Models 1 to 3), Model 1 with the parameters of interest, $\zeta > 0$ and $\eta > 0$, Model 2 with $\zeta = 0 \eta > 0$ and the baseline case, Model 3, with $\zeta = \eta = 0$. The unrestricted models (Models 1H to 3H) follow the same set up but with $h \neq 0$. The estimation is carried out in DYNARE (Matlab version) programme, see Juillard (2006) and the resulting posterior means and confidence internals can be found in Table 2. The estimated risk aversion parameter is greater than one in all unrestricted models, particularly Model 1H (1.558) and Model 2H (1.618) as is consistent with empirical evidence. The parameters characterizing monetary policy are stable across all models and close to the prior means; $\rho_\pi$ in particular describing a predictably strong response from the Federal Reserve to the deviation of inflation from target. The interest rate smoothing parameter $\rho_i$ is substantial as is the persistence of the productivity and demand shocks, consistent across all models. The monetary shocks
are much less persistent than expected in all restricted models, questioning their ability to contribute towards fluctuations in the business cycle, but considerably more persistent for the unrestricted models with innovation and habitual consumption, (Models 1H and 2H), a finding consistent with Carvalho (2006). The estimate for the size of the sticky price sector is between around 55% and 70% depending on model specification. The estimate for the flexible price sector is around 6.5% and consistent across the restricted and unrestricted models (Models 1 and 1H). The bottom line of Table 2 reports the log marginal density of the estimation of each model, indicating a preference for Model 2H, with habitual consumption and sticky price sector. The most striking result, however, is given when comparing the Models with full innovation and the baseline case. Without habit the baseline (Model 3) is preferred to the model with full innovation (Model 1). With habit this result is reversed and the model with full innovation is preferred (Model 1H). In summary, the innovation improves the model fit when provided with an alternative channel to emulate the inertia displayed in the data.

4 Conclusion

In this paper we have discussed the Calvo contract pricing mechanism, which has become the most widely accepted microfoundation to the NK Phillips curve, and its prediction of a flat hazard rate caused by the assumption of homogeneous firm types. To better explain the stylized fact displayed in the micro data this paper relaxes this homogeneous firm assumption.

With this aim in mind we have estimated a baseline New Keynesian DSGE model adapted to account for a flexible price sector, similar to that described in Aoki (2001), albeit with a different motivation; and a fixed price sector to account for firms that are unable to change price in each period. The inclusion of these sectors is motivated by recent literature that suggests that firms face differing probabilities of
price change in an economy as discussed in Alvarez et al (2005) and Carvalho (2006), amongst others. These innovations in the aggregate price mechanism also allow for the control of heterogeneity in price setting implied by this recent microeconomic literature with the additional benefit of providing an estimate of the size of these sectors. The results predict the size of the flexible price sector to be around 6.5% and the size of the sticky price sector to be around 55% to 70%. The estimate of the former being consistent across both model specifications.

The inclusion of a flexible price sector in the restricted model gives an entirely predictable outcome of a worsening of fit, when compared to the baseline case, as the model specification relies solely on sticky prices to explain the inertia displayed in the data. Most interestingly, this result is reversed when accounting for persistence in output and a model with both innovations is preferred over its baseline counterpart, suggesting an important role for the sector in the allocation of inertia in the model. Furthermore, controlling for heterogeneity in this manner, provides a result consistent with Carvalho (2006), who consequently finds greater persistence in monetary shocks. Microfoundations to general equilibrium models should fit the data that they are trying to predict rather than solely provide an answer to the Lucas critique and an improved overall model fit. This work is a tentative and parsimonious attempt to address this problem.
References


[34] Taylor, J., 1979 *Staggered Wage Setting in a Macro Model*, The American Economic review, Vol. 69, No.2

### A Tables

<table>
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<tr>
<th>Description</th>
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Table 1: Parameterisation of the Model
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Table 2: Results from Bayesian Estimation
B Derivation of the Model

B.1 Households

Households maximise:

\[ E_0 \sum_{i=0}^{\infty} \beta^i U(C_t, N_t) \]

where \( C_t \) and \( N_t \) represent consumption and labour in period \( t \) and \( \beta \) is the discount factor.

However, in order to introduce, \( i \), differentiated goods, \( C_t \) is now a consumption index given by

\[ C_t \equiv \left( \int_0^1 C_t(i)^{1-\frac{1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon - 1}} \]

where \( \varepsilon \) is the elasticity of substitution, or the elasticity of demand.

The budget constraint now takes the form

\[ \int_0^1 P_t(i) C_t(i) di + Q_t B_t \leq B_{t-1} + W_t N_t + T_t \]

To derive the demand equations we maximise consumption, (2) subject to any given level of expenditure, say

\[ \int_0^1 P_t(i) C_t(i) di = Z_t \]

Formally we write the Lagrangian as:

\[ L = \left( \int_0^1 C_t(i)^{1-\frac{1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon - 1}} - \lambda \left( \int_0^1 P_t(i) C_t(i) di - Z_t \right) \]

\[ L_c = \frac{\varepsilon}{\varepsilon - 1} \left( 1 - \frac{1}{\varepsilon} \right) (C_t)^{\frac{1}{\varepsilon}} C_t(i)^{1-\frac{1}{\varepsilon}} - \lambda P_t(i) C_t(i) \]

The first order condition

\[ \frac{\partial L}{\partial C_t(i)} = 0 \]
produces only one first order condition in $\lambda$

\[ C_t(i) \left(-\frac{1}{\varepsilon} (C_t)^{\frac{1}{\varepsilon}} \right) = \lambda P_t(i) \]

for a second good, $j$, we can write the above again

\[ C_t(j) \left(-\frac{1}{\varepsilon} (C_t)^{\frac{1}{\varepsilon}} \right) = \lambda P_t(j) \]

to substitute out $\lambda$ which produces an Euler equation:

\[ C_t(i) = C_t(j) \left( \frac{P_t(i)}{P_t(j)} \right)^{-\varepsilon} \]

Substituting this into the expression for consumption expenditure, gives

\[ \int_0^1 P_t(i) C_t(j) \left( \frac{P_t(i)}{P_t(j)} \right)^{-\varepsilon} di = Z_t \]

Extracting constants and rearranging

\[ C_t(j)^{\varepsilon} \int_0^1 P_t(i)^{(1-\varepsilon)} di = Z_t \]

Simplifying

\[ C_t(j)^{\varepsilon} P_t^{(1-\varepsilon)} = Z_t \]

\[ C_t(j) \left( \frac{P_t}{P_t(j)} \right)^{-\varepsilon} = \frac{Z_t}{P_t} \]

or

\[ C_t(j) = \left( \frac{P_t(j)}{P_t} \right)^{-\varepsilon} \frac{Z_t}{P_t} \]
restating in good $i$ gives optimal consumption of good, $i$.

\[ C_t(i) = \left( \frac{P_t(i)}{P_t} \right)^{-\varepsilon} \frac{Z_t}{P_t} \]

Substituting into the definition of the consumption index gives

\[ C_t = \left( \int_0^1 \left( \left( \frac{P_t(i)}{P_t} \right)^{-\varepsilon} \frac{Z_t}{P_t} \right)^{1-\frac{\varepsilon}{\varepsilon+1}} \right)^{\frac{\varepsilon+1}{\varepsilon}} \]

Extracting constants and simplifying

\[ C_t = \left( P_t^{\varepsilon-1}Z_t \right)^{1-\frac{\varepsilon}{\varepsilon+1}} \int_0^1 ((P_t(i))^{-\varepsilon})^{1-\frac{\varepsilon}{\varepsilon+1}} di \]

\[ = P_t^{\varepsilon-1}Z_t \int_0^1 P_t(i)^{-\varepsilon} di \]

\[ C_t = P_t^{-1}Z_t \]

\[ Z_t = P_tC_t \]

or the level of expenditure in the economy is equal to the price level multiplied by aggregate consumption.

As $Z_t$ is the given level of expenditure it follows that

\[ \int_0^1 P_t(i)C_t(i)di = P_tC_t \]

Finally, substituting into the expression for optimal consumption of good, $i$, yields

\[ C_t(i) = \left( \frac{P_t(i)}{P_t} \right)^{-\varepsilon} \frac{P_tC_t}{P_t} \]
which simplifies to the consumers demand equation

\[ C_t(i) = \left( \frac{P_t(i)}{P_t} \right)^{-\varepsilon} C_t \]  

(10)

Households maximise

\[ E_0 \sum_{i=0}^{\infty} \beta^i U(C_t, N_t) \]

subject to

\[ P_tC_t + Q_tB_t \leq B_{t-1} + W_tN_t + T_t \]

which can be re-written

\[ C_t + \frac{Q_tB_t}{P_t} \leq \frac{B_{t-1}}{P_t} + \frac{W_tN_t}{P_t} + \frac{T_t}{P_t} \]

Formally we can write the Lagrangian

\[ L = E_t \left( \sum_{i=0}^{\infty} \beta^i [U(C_{t+i}, N_{t+i})] - \lambda \left[ C_{t+i} + \frac{Q_{t+i}B_{t+i}}{P_{t+i}} - \frac{B_{t+i-1}}{P_{t+i}} - \frac{W_{t+i}N_{t+i}}{P_{t+i}} - \frac{T_{t+i}}{P_{t+i}} \right] \right) \]

with first order conditions

\[ L_c = E_t \left[ \beta^i U_{c,t+i} + \lambda_{t+i} \right] = 0 \]

\[ L_n = E_t \left[ \beta^i U_{n,t+i} - \lambda_{t+i} \frac{W_{t+i}}{P_{t+i}} \right] = 0 \]

\[ L_b = E_t \left[ \lambda_{t+i} \frac{Q_{t+i}}{P_{t+i}} - \lambda_{t+i+1} \frac{1}{P_{t+i+1}} \right] = 0 \]

Equating \( L_c \) and \( L_n \) gives the Euler equation for labour supply

\[ - \frac{U_{n,t}}{U_{c,t}} = \frac{W_t}{P_t} \]
Substituting $L_c$ into $L_b$ gives the Euler equation for consumption

$$E_t \left[ \beta^{t+1} U_{c,t+1} \frac{1}{Q_t P_{t+1}} \right] = E_t \left[ \beta^t U_{c,t} \right]$$

or

$$Q_t = \beta E_t \left[ \frac{U_{c,t+1} P_t}{U_{c,t} P_{t+1}} \right]$$

Assuming a period utility of

$$U(C_t, N_t) = \frac{(C_t - hC_{t-1})^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}$$

The consumer’s optimality conditions thus become

$$\frac{N_t^\varphi}{(C_t - hC_{t-1})^{-\sigma}} = \frac{W_t}{P_t}$$

To linearise the first optimality condition rearrange the first optimality condition

$$\frac{W_t}{P_t} = N_t^\varphi(C_t - hC_{t-1})^\sigma$$

Taking logs of both sides

$$w_t - p_t = \varphi \log N_t + \sigma \log(C_t - hC_{t-1})$$
If we write \( z_t = w_t - p_t \) then a simple Taylor first order approximation yields

\[
z_t \approx \varphi \log \bar{N} + \sigma \log(\bar{C} - h\bar{C}) \\
+ \frac{\partial z_t}{\partial \log N_t} \left[ \log N_t - \log \bar{N} \right] \\
+ \frac{\partial z_t}{\partial \log C_t} \left[ \log C_t - \log \bar{C} \right] \\
+ \frac{\partial z_t}{\partial \log C_{t-1}} \left[ \log C_{t-1} - \log \bar{C} \right]
\]

Evaluating the derivatives at the steady state

\[
\frac{\partial z_t}{\partial \log N_t} = \frac{\partial z_t}{\partial N_t} \frac{\partial N_t}{\partial \log N_t} = \left[ \frac{1}{\bar{N}} \right] \bar{N} = \varphi
\]

\[
\frac{\partial z_t}{\partial \log C_t} = \frac{\partial z_t}{\partial C_t} \frac{\partial C_t}{\partial \log C_t} = \left[ \frac{\sigma}{1 - h} \right] \bar{C} = \sigma \frac{1}{1 - h}
\]

\[
\frac{\partial z_t}{\partial \log C_{t-1}} = \frac{\partial z_t}{\partial C_{t-1}} \frac{\partial C_{t-1}}{\partial \log C_{t-1}} = \left[ \frac{-h}{1 - h} \right] \bar{C} = -\sigma \frac{h}{1 - h}
\]

Substituting the derivatives into \( z_t \), yields

\[
z_t \approx \varphi \log \bar{N} + \sigma \log(\bar{C} - h\bar{C}) \\
+ \left[ \varphi \right] \left[ \log N_t - \log \bar{N} \right] \\
+ \left[ \sigma \frac{1}{1 - h} \right] \left[ \log C_t - \log \bar{C} \right] \\
+ \left[ -\sigma \frac{h}{1 - h} \right] \left[ \log C_{t-1} - \log \bar{C} \right]
\]

Simplifying

\[
w_t - p_t = \varphi n_t + \left( \frac{\sigma}{1 - h} \right) c_t - \left( \frac{\sigma h}{1 - h} \right) c_{t-1}
\]

---

\(^2\)If \( Z_t = F(Q_t, Y_t, MC_t, \Pi_t) \) with steady state values \( Q, Y, MC, \Pi \) then a Taylor approximation to \( Z_t \) can be given by

\[
F(Q_t, Y_t, MC_t, \Pi_t) \approx F(Q, Y, MC, \Pi) + F_Q(Q_t - Q) + F_Y(Y_t - Y) + F_{MC}(MC_t - MC) + F_{\Pi}(\Pi_t - \Pi)
\]
from which we can write the linearised labour supply optimality condition.

\[ w_t - p_t = \varphi n_t + \frac{\sigma}{1 - h} (c_t - hc_{t-1}) \]

To linearise the second optimality condition, first rearrange the second optimality condition:

\[ \Pi_{t+1} = \frac{\beta}{Q_t} E_t \left[ \left( \frac{(C_{t+1} - hC_t)}{C_t - hC_{t-1}} \right)^{-\sigma} \right] \]

Taking logs of both sides

\[ E_t[\pi_{t+1}] = \log \beta - \log Q_t - \sigma \log E_t [(C_{t+1} - hC_t)] + \sigma \log E_t [(C_t - hC_{t-1})] \]

If we write \( z_t = E_t[\pi_{t+1}] \) then a simple Taylor first order approximation yields

\[ z_t \approx \log \beta - \log \bar{Q} - \sigma \log E_t [(\bar{C} - h\bar{C})] + \sigma \log E_t [(\bar{C} - h\bar{C})] \]

\[ + E_t \frac{\partial z_t}{\partial \log Q_t} [\log Q_t - \log \bar{Q}] \]

\[ + E_t \frac{\partial z_t}{\partial \log C_{t+1}} [\log C_{t+1} - \log \bar{C}] \]

\[ + E_t \frac{\partial z_t}{\partial \log C_t} [\log C_t - \log \bar{C}] \]

\[ + E_t \frac{\partial z_t}{\partial \log C_{t-1}} [\log C_{t-1} - \log \bar{C}] \]

Evaluating the derivatives at the steady state

\[ \frac{\partial z_t}{\partial \log Q_t} = \frac{\partial z_t}{\partial Q_t} \frac{\partial Q_t}{\partial \log Q_t} = \left[ -1 \frac{1}{\bar{Q}} \right] \bar{Q} = -1 \]

\[ \frac{\partial z_t}{\partial \log C_{t+1}} = \frac{\partial z_t}{\partial C_{t+1}} \frac{\partial C_{t+1}}{\partial \log C_{t+1}} = \left[ -\sigma \frac{1}{(1 - h)\bar{C}} \right] \bar{C} = -\sigma \frac{1}{1 - h} \]

\[ \frac{\partial z_t}{\partial \log C_t} = \frac{\partial z_t}{\partial C_t} \frac{\partial C_t}{\partial \log C_t} = \left[ -\sigma \frac{-h}{(1 - h)\bar{C}} + \sigma \frac{1}{(1 - h)\bar{C}} \right] \bar{C} = \sigma \frac{1 + h}{1 - h} \]
\[
\frac{\partial z_t}{\partial \log C_{t-1}} = \frac{\partial z_t}{\partial C_{t-1}} \frac{\partial C_{t-1}}{\partial \log C_{t-1}} = \left[ \sigma \frac{-h}{(1-h) \bar{C}} \right] \bar{C} = -\sigma \frac{h}{1-h}
\]

Substituting the derivatives into \( z_t \), and noting that the third and fourth terms sum to zero, yields

\[
z_t \cong \log \bar{\beta} - \log \bar{Q} + E_t[-1][\log Q_t - \log \bar{Q}] + E_t[-\sigma \frac{1}{1-h}][\log C_{t+1} - \log \bar{C}] + E_t[\sigma \frac{1+h}{1-h}][\log C_t - \log \bar{C}] + E_t[-\sigma \frac{h}{1-h}][\log C_{t-1} - \log \bar{C}]
\]

Simplifying and noting that \( E(c_t) = c_t \) and \( E(c_{t-1}) = c_{t-1} \) at \( t+1 \)

\[
E_t[\pi_{t+1}] = -\rho + i - (\sigma \frac{1}{1-h})E_t[c_{t+1}] + (\sigma \frac{1+h}{1-h})c_t - (\sigma \frac{h}{1-h})c_{t-1}
\]

from which we can write the linearised consumers optimality condition.

\[
c_t = \frac{h}{(1-h)}c_{t-1} - \frac{1}{(1-h)}E_t[c_{t+1}] - \frac{1}{\sigma (1+h)}[i - E_t[\pi_{t+1}] - \rho]
\]

**B.2 The Firm**

Each firm produces a differentiated good, \( i \), with identical technology, \( A_t \), represented by the production function

\[
Y_t(i) = A_t N_t(i)^{1-a}
\]  

(11)

All firms face the isoelastic demand schedule given previously by (1)

\[
C_t(i) = \left( \frac{P_t(i)}{P_t} \right)^{-\epsilon} C_t
\]
Each firm may reset its price with constant probability $1 - \theta$ in any given period, a la Calvo (1983). Thus

$$P_t = [\theta(P_{t-1})^{1-\varepsilon} + (1 - \theta)(P_t^*)^{1-\varepsilon}]^{\frac{1}{1-\varepsilon}}$$

Dividing through by $P_{t-1}$ yields

$$\Pi_t^{1-\varepsilon} = \theta + (1 - \theta) \left( \frac{P_t^*}{P_{t-1}} \right)^{1-\varepsilon} \quad (12)$$

Define $\pi_t = \log \Pi_t$ and $p_t = \log P_t$ then we can linearise the aggregate price index around a zero inflation steady state.

Alternatively imagine a portion ($\zeta$) of firms outside the Calvo set up which face perfect price fluidity, and a further set of firms $\eta$ that face fixed prices and the remaining proportion of firms facing the Calvo type price setting as $(1 - \zeta - \eta)$. It is convenient to assume that $\zeta \in (0, 1]$ and $\eta \in (0, 1]$.

Allowing this framework to capture heterogeneous price setting at the aggregate level, the aggregate price mechanism now becomes:

$$P_t = [\zeta(P_t^*)^{1-\varepsilon} + \eta(P_{t-1})^{1-\varepsilon} + (1 - \zeta - \eta) \left( \theta(P_{t-1})^{1-\varepsilon} + (1 - \theta)(P_t^*)^{1-\varepsilon} \right)]^{\frac{1}{1-\varepsilon}}$$

Rearranging and dividing through by $P_t^{1-\varepsilon}$ yields a similar expression for later use

$$\Pi_t^{1-\varepsilon} = (\eta + (1 - \zeta - \eta)\theta) + (\zeta + (1 - \zeta - \eta)(1 - \theta)) \left( \frac{P_t^*}{P_{t-1}} \right)^{1-\varepsilon} \quad (13)$$

Define $\pi_t = \log \Pi_t$ and $p_t = \log P_t$ then we can linearise the aggregate price index around a zero inflation steady state

$$\pi_t = (\zeta + (1 - \zeta - \eta)(1 - \theta))(p_t^* - p_{t-1}) \quad (14)$$
B.3 Optimal price Setting

Firms choose a price to maximise

\[
\max_{P_t^*} \sum_{k=0}^{\infty} \theta^k E_t \left[ Q_{t,t+k} (P_t^* Y_{t+k|t} - \Psi_{t+k}(Y_{t+k|t})) \right]
\]

subject to the demand constraint

\[
Y_{t+k|t} = \left( \frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} C_{t+k}
\]

(15)

note that costs, \( \Psi_{t+k}(Y_{t+k|t}) \) are written as an implicit function of \( Y_{t+k} \)

also that

\[
Y_{t+k|t}' = -\varepsilon Y_{t+k|t} (P_t^*)^{-1}
\]

simplifying

\[
Y_{t+k|t}' = -\varepsilon Y_{t+k|t} (P_t^*)^{-1}
\]

The first term in our argument, \( P_t^* Y_{t+k|t} \), can be written as

\[
P_t^* Y_{t+k|t} = P_t^* \left( \frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} C_{t+k}
\]

with its derivative

\[
\frac{\partial P_t^* Y_{t+k|t}}{\partial P_t^*} = (1 - \varepsilon) \left( \frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} C_{t+k}
\]

or simply

\[
\frac{\partial P_t^* Y_{t+k|t}}{\partial P_t^*} = (1 - \varepsilon) Y_{t+k|t}
\]

These expressions help us to understand how we can write the F.O.C as

\[
\sum_{k=0}^{\infty} \theta^k E_t \left[ Q_{t,t+k} \left( (1 - \varepsilon) Y_{t+k|t} + \varepsilon Y_{t+k|t} (P_t^*)^{-1} \Psi_{t+k}'(Y_{t+k|t}) \right) \right] = 0
\]
Multiplying through by \( P_t^*/(1 - \varepsilon) \)

\[
\sum_{k=0}^{\infty} \theta^k E_t \left[ Q_{t,t+k} \left( P_t^* Y_{t+k|t} + \frac{\varepsilon}{1 - \varepsilon} Y_{t+k|t} \Psi_{t+k} (Y_{t+k|t}) \right) \right] = 0
\]

and simplifying

\[
\sum_{k=0}^{\infty} \theta^k E_t \left[ Q_{t,t+k} Y_{t+k|t} \left( P_t^* + \frac{\varepsilon}{1 - \varepsilon} \Psi_{t+k} (Y_{t+k|t}) \right) \right] = 0
\]

If the frictionless mark up, \( \mathcal{M} = \frac{\varepsilon}{\varepsilon - 1} \) and nominal marginal cost is expressed as \( \psi_{t+k|t} \equiv \Psi_{t+k} (Y_{t+k|t}) \) then

\[
\sum_{k=0}^{\infty} \theta^k E_t \left[ Q_{t,t+k} Y_{t+k|t} \left( P_t^* + \mathcal{M} \psi_{t+k|t} \right) \right] = 0 \quad (16)
\]

Dividing through by \( P_{t-1} \) and letting \( \Pi_{t,t+k} \equiv P_{t+k} / P_t \)

\[
\sum_{k=0}^{\infty} \theta^k E_t \left[ Q_{t,t+k} Y_{t+k|t} \left( \frac{P_t^*}{P_{t-1}} + \mathcal{M} C_{t+k|t} \Pi_{t-1,t+k} \right) \right] = 0 \quad (17)
\]

where \( MC_{t+k|t} = \psi_{t+k|t} / P_{t+k} \) is real marginal cost

Around the steady state we have that \( P_t^*/P_{t-1} = 1 \) and \( \Pi_{t-1,t+k} = 1 \), ie constant prices, \( P_t^* = P_{t+k} \) from which it follows that, in the steady state, \( Y = Y_{t+k|t} \), \( MC = MC_{t+k|t} \) and \( Q_{t,t+k} = \beta^k \)

To linearise the firm’s price decision first we rewrite:

\[
\frac{P_t^*}{P_{t-1}} = \mathcal{M} \frac{\sum_{k=0}^{\infty} \theta^k E_t \left[ Q_{t,t+k} Y_{t+k|t} C_{t+k|t} \Pi_{t-1,t+k} \right]}{\sum_{k=0}^{\infty} \theta^k E_t \left[ Q_{t,t+k} Y_{t+k|t} \right]}
\]

We can linearise the firm’s optimal price by first taking logs of both sides

\[
p_t^* - p_{t-1} = \log \mathcal{M} + \log \sum_{k=0}^{\infty} \theta^k E_t \left[ Q_{t,t+k} Y_{t+k|t} MC_{t+k|t} \Pi_{t-1,t+k} \right] - \log \sum_{k=0}^{\infty} \theta^k E_t \left[ Q_{t,t+k} Y_{t+k|t} \right]
\]
If we write \( z_t = p_t^* - p_{t-1} \) then a simple Taylor first order approximation \(^3\) yields

\[
z_t \equiv \log\mathcal{M} + \log \sum_{k=0}^{\infty} \theta^k E_t \left[ \beta^k Y \frac{1}{\mathcal{M}} \right] - \log \sum_{k=0}^{\infty} \theta^k E_t \left[ \beta^k Y \right] + \sum_{k=0}^{\infty} E_t \frac{\partial z_t}{\partial \log Q_{t,t+k}} \left[ \log Q_{t,t+k} - \log \beta^k \right] + \sum_{k=0}^{\infty} E_t \frac{\partial z_t}{\partial \log Y_{t+k|t}} \left[ \log Y_{t+k|t} - \log Y \right] + \sum_{k=0}^{\infty} E_t \frac{\partial z_t}{\partial \log MC_{t+k|t}} \left[ \log MC_{t+k|t} - \log \frac{1}{\mathcal{M}} \right] + \sum_{k=0}^{\infty} E_t \frac{\partial z_t}{\partial \log \Pi_{t-1,t+k}} \left[ \log \Pi_{t-1,t+k} - 0 \right]
\]

Evaluating the derivatives at the steady state

\[
\frac{\partial z_t}{\partial \log Q_{t,t+k}} = \frac{\partial z_t}{\partial Q_{t,t+k}} \frac{\partial Q_{t,t+k}}{\partial \log Q_{t,t+k}} = \left[ \frac{\theta^k Y \frac{1}{\mathcal{M}}}{\sum_{k=0}^{\infty} \theta^k \beta^k Y \frac{1}{\mathcal{M}}} - \frac{\theta^k Y}{\sum_{k=0}^{\infty} \theta^k \beta^k Y} \right] \beta^k = 0
\]

\[
\frac{\partial z_t}{\partial \log Y_{t+k|t}} = \frac{\partial z_t}{\partial Y_{t+k|t}} \frac{\partial Y_{t+k|t}}{\partial \log Y_{t+k|t}} = \left[ \frac{\theta^k \beta^k \frac{1}{\mathcal{M}}}{\sum_{k=0}^{\infty} \theta^k \beta^k Y \frac{1}{\mathcal{M}}} - \frac{\theta^k \beta^k}{\sum_{k=0}^{\infty} \theta^k \beta^k Y} \right] Y = 0
\]

\[
\frac{\partial z_t}{\partial \log MC_{t+k|t}} = \frac{\partial z_t}{\partial MC_{t+k|t}} \frac{\partial MC_{t+k|t}}{\partial \log MC_{t+k|t}} = \left[ \frac{\theta^k \beta^k Y \frac{1}{\mathcal{M}}}{\sum_{k=0}^{\infty} \theta^k \beta^k Y \frac{1}{\mathcal{M}}} - \frac{\theta^k \beta^k Y}{\sum_{k=0}^{\infty} \theta^k \beta^k Y} \right] \frac{1}{\mathcal{M}} = (1 - \theta \beta)(\theta^k \beta^k)^4
\]

\[
\frac{\partial z_t}{\partial \log \Pi_{t-1,t+k}} = \frac{\partial z_t}{\partial \Pi_{t-1,t+k}} \frac{\partial \Pi_{t-1,t+k}}{\partial \log \Pi_{t-1,t+k}} = \left[ \frac{\theta^k \beta^k Y \frac{1}{\mathcal{M}}}{\sum_{k=0}^{\infty} \theta^k \beta^k Y \frac{1}{\mathcal{M}}} - \frac{\theta^k \beta^k Y}{\sum_{k=0}^{\infty} \theta^k \beta^k Y} \right] 1 = (1 - \theta \beta)(\theta^k \beta^k)
\]

Substituting the derivatives into, and noting that the first three terms in \( z_t \) sum to zero yields

\[
p_t^* - p_{t-1} = \sum_{k=0}^{\infty} (1 - \theta \beta)(\theta^k \beta^k) E_t \left[ \log MC_{t+k|t} - \log \frac{1}{\mathcal{M}} \right] + \sum_{k=0}^{\infty} (1 - \theta \beta)(\theta^k \beta^k) E_t \left[ p_{t+k} - p_{t-1} \right]
\]

\(^3\) If \( Z_t = F(Q_t; Y; MC_t; \Pi_t) \) with steady state values \( Q, Y, MC, \Pi \) then a Taylor approximation to \( Z_t \) can be given by \( F(Q_t, Y; MC_t; \Pi_t) \equiv F(Q, Y; MC; \Pi) + F_Q(Q_t - Q) + F_Y(Y_t - Y) + F_{MC}(MC_t - MC) + F_{\Pi}(\Pi_t - \Pi) \)

\(^4\) using \( \sum_{k=0}^{\infty} \theta^k = \frac{1}{1 - \theta} \)
Simplifying
\[ p_t^* - p_{t-1} = (1 - \theta \beta) \sum_{k=0}^{\infty} (\theta \beta)^k E_t \left[ mc_{t+k|t} - \log \frac{1}{M} + p_{t+k} - p_{t-1} \right] \]

Define \( mc_{t+k|t} \equiv mc_{t+k|t} - mc \) as the deviation of log marginal cost from its steady state value, \( mc = -\mu \), where \( \mu \equiv \log M \) is the log of the desired gross markup, then
\[ p_t^* - p_{t-1} = (1 - \theta \beta) \sum_{k=0}^{\infty} (\theta \beta)^k E_t \left[ \hat{mc}_{t+k|t} + p_{t+k} - p_{t-1} \right] \]

**B.4 Equilibrium**

Market clearing in the goods market requires:
\[ Y_t(i) = C_t(i) + G_t \]

If
\[ Y_t \equiv \left( \int_0^1 Y_t(i)^{1 - \frac{1}{\gamma}} di \right)^{\frac{\gamma}{\gamma - 1}} \]

then it follows that
\[ \left( \int_0^1 Y_t(i)^{1 - \frac{1}{\gamma}} di \right)^{\frac{\gamma}{\gamma - 1}} = \left( \int_0^1 C_t(i)^{1 - \frac{1}{\gamma}} di \right)^{\frac{\gamma}{\gamma - 1}} + G_t \]

and that aggregate output is equal to aggregate consumption
\[ Y_t = C_t + G_t \]

which linearizes to
\[ y_t = c_y c_t + (1 - c_y) g_t \quad \text{where} \quad c_y = \frac{C}{Y} \]

Accordingly from the consumption Euler (3) and the market clearing condition we
can yield the equilibrium optimality condition

\[ y_t = E_t[y_{t+1}] - \frac{1}{\sigma} [\pi_t - E(\pi_{t+1}) - \rho] \]  

(18)

Equilibrium in the labour market requires that each household provides an amount of labour \( N_t \) which is equal to the sum of labour supplied to each firm.

\[ N_t = \int_0^1 N_t(i)di \]

Using the production function (5) and the consumers demand equation (1)

\[ N_t = \int_0^1 \left( \frac{Y_t(i)}{A_t} \right)^{\frac{1}{\alpha}} di \]

\[ N_t = \int_0^1 \left( \frac{P_t(i)^{-\epsilon}}{Y_t(i) A_t} \right)^{\frac{1}{\alpha}} di \]

\[ N_t = \left( \frac{Y_t}{A_t} \right)^{\frac{1}{\alpha}} \int_0^1 \left( \frac{P_t(i)}{P_t} \right)^{\frac{-\epsilon}{\alpha}} di \]

We can define a measure of price dispersion as

\[ d_t = (1 - \alpha) \log \int_0^1 \left( \frac{P_t(i)}{P_t} \right)^{\frac{-\epsilon}{\alpha}} di \]

thus we can rewrite the previous expression

\[ N_t = \left( \frac{Y_t}{A_t} \right)^{\frac{1}{\alpha}} D_t^{\frac{1}{1-\alpha}} \]

to arrive at the aggregate output equation we can linerise the above expression by rewriting

\[ 1 = \exp \left( \frac{y_t}{1-\alpha} - \frac{a_t}{1-\alpha} + \frac{d_t}{1-\alpha} - n_t \right) \]
Using our linearisation result gives the equilibrium output relation

\[ n_t(1 - \alpha) = y_t - a_t + d_t \]

assuming that price dispersion is close to zero around the steady state we have

\[ y_t = a_t + (1 - \alpha)n_t \]  \hspace{1cm} (19)

B.5 Marginal Cost

Define the economy’s average marginal cost as the difference between the log real wage and the log marginal product of labour.

Remember

\[ Y_t(i) = A_t N_t(i)^{1-\alpha} \]

thus the marginal product of labour, MPN

\[ MPN = (1 - \alpha)A_t N_t(i)^{-\alpha} \]

Taking logs

\[ mpn = a_t - \alpha n_t + \log(1 - \alpha) \]

Marginal Cost

\[ mc_t = (w_t - p_t) - mpn \]

\[ mc_t = (w_t - p_t) - (a_t - \alpha n_t) - \log(1 - \alpha) \]

The aggregate relation, (23) can be written

\[ n_t = \frac{1}{1 - \alpha}(y_t - a_t) \]
\[(a_t - \alpha n_t) = \frac{1}{1 - \alpha} (a_t - \alpha y_t)\]

The real marginal cost for a firm that last reset its price in period, \(t\), can be written

\[mc_{t+k|t} = (w_{t+k} - p_{t+k}) - \frac{1}{1 - \alpha} (a_{t+k} - \alpha y_{t+k|t}) - \log(1 - \alpha) \quad (20)\]

Note we can take our equation for \(mc_t\) forward by \(k\) periods and combine with the above (20) to get

\[mc_{t+k|t} = mc_{t+k} + \frac{\alpha}{1 - \alpha} (-y_{t+k} + y_{t+k|t})\]

From the equilibrium condition and the demand schedule we have

\[(y_{t+k|t} - y_{t+k}) = -\varepsilon (p^* - p_{t+k})\]

so that

\[mc_{t+k|t} = mc_{t+k} + \frac{\varepsilon \alpha}{1 - \alpha} (p^* - p_{t+k}) \quad (21)\]

### B.6 The Phillips Curve

substitute (25) into the firms price setting decision

\[p_t^* - p_{t-1} = (1 - \theta \beta) \sum_{k=0}^{\infty} (\theta \beta)^k E_t \left[ \hat{mc}_{t+k} + \frac{\varepsilon \alpha}{1 - \alpha} (p^* - p_{t+k}) + p_{t+k} - p_{t-1} \right]\]

Collecting terms and rewriting

\[p_t^* - p_{t-1} = (1 - \theta \beta) \sum_{k=0}^{\infty} (\theta \beta)^k E_t [\Theta \hat{mc}_{t+k} + p_{t+k} - p_{t-1}]\]

where \(\Theta = \left(\frac{1 - \alpha}{1 - \alpha + \varepsilon}\right)\)
Rewriting

\[ p_t^* - p_{t-1} = (1 - \theta \beta) \Theta \sum_{k=0}^{\infty} (\theta \beta)^k E_t \hat{m} c_{t+k} + (1 - \theta \beta) \sum_{k=0}^{\infty} (\theta \beta)^k E_t \pi_{t+k} \]

Forward one period

\[ p_{t+1}^* - p_t = (1 - \theta \beta) \Theta \sum_{k=0}^{\infty} (\theta \beta)^k E_{t+1} \hat{m} c_{t+1+k} + (1 - \theta \beta) \sum_{k=0}^{\infty} (\theta \beta)^k E_{t+1} \pi_{t+1+k} \]

Taking expectation and combining these two equations yields

\[ p_t^* - p_{t-1} = \beta \theta E_t (p_{t+1}^* - p_t) + (1 - \theta \beta) \Theta \hat{m} c_t + \pi_t \tag{22} \]

Combine (26) and (18) for the NK Phillips curve

\[ \pi_t = \frac{\beta \theta}{(\eta + \theta)(1 - \zeta)} E_t (\pi_{t+1}) + \frac{(\zeta + (1 - \zeta - \eta)(1 - \theta))}{(\eta + \theta)(1 - \zeta)} (1 - \theta \beta) \Theta \hat{m} c_t \tag{23} \]

note this can be compared to the case of \( \zeta = 0 \), the baseline New Keynesian Phillips curve.

\[ \pi_t = \beta E_t (\pi_{t+1}) + \gamma \hat{m} c_t \tag{24} \]

where \( \gamma = \frac{(1-\theta)(1-\theta \beta)}{\theta} \Theta \)

We can write the Phillips curve in terms of the output gap, rather than marginal cost. To see this we use the production function and our optimality condition, (2).
We can write real marginal cost as

\[ mc_t = (w_t - p_t) - m pn \]

\[ = (\sigma y_t + \varphi n_t) - (y_t - n_t) - \log(1 - \alpha) \]

\[ = (\sigma + \frac{\varphi + \alpha}{1 - \alpha}) y_t - \frac{1 + \varphi}{1 - \alpha} a_t - \log(1 - \alpha) \]

Define the natural rate of output, \( y^*_n \), as the equilibrium output under flexible prices with \( mc = -\mu \) such that

\[ mc = \left( \sigma + \frac{\varphi + \alpha}{1 - \alpha} \right) y^*_n - \frac{1 + \varphi}{1 - \alpha} a_t - \log(1 - \alpha) \]

then

\[ \hat{mc_t} = \left( \sigma + \frac{\varphi + \alpha}{1 - \alpha} \right) (y_t - y^*_n) \]

The log deviation of marginal cost from its steady state is proportional to log deviation of output from its flexible counterpart, or the output gap, \( \tilde{y}_t \)

We can write the DIS by rewriting equation (12) in terms of the output gap

\[ y_t = \frac{h}{(1 - h)} y_{t-1} - \frac{1}{(1 - h)} E_t[y_{t+1}] - \frac{1}{\sigma}(1 - h) \left[ i_t - E_t[\pi_{t+1}] - \rho \right] \]

where the natural rate of interest, \( r^n_t \equiv \rho + \sigma E_t \Delta y^*_t \)

and the NKPC in terms of the output gap

\[ \pi_t = \frac{\theta}{\theta(1 + \zeta)} \beta E_t(\pi t + 1) + \frac{(\zeta + (1 - \zeta)(1 - \theta))}{\theta(1 + \zeta)} (1 - \theta \beta) \Theta \left( \sigma + \frac{\varphi + \alpha}{1 - \alpha} \right) \tilde{y}_t \] (25)

with the Taylor monetary policy rule:

\[ i_t = \rho i_{t-1} + (1 - \rho)(\phi_x \pi_t + \phi_y \tilde{y}_t) + e_t \]