Periodic Disturbance Rejection of Nonlinear Systems via Output Feedback with Neural Network Approximation

Xiafei Tang, Zhengtao Ding
Control Systems Centre, School of Electrical and Electronic Engineering, The University of Manchester, Sackville Street Building, Manchester M13 9PL, United Kingdom
E-mail: shivatang@hotmail.com
E-mail: zhengtao.ding@manchester.ac.uk

Abstract: In this paper neural network (NN) is applied for rejecting periodic disturbances in output feedback nonlinear system. The NN adopted here is Adaptive Radial Basis Function Neural Network (ARBFNN). The parameters of the system, except the high gain frequency, and disturbance are assumed to be unknown. We also postulate that the uncertainty of the output feedback system are bounded by an existing unknown constant polynomial and then adaptive technique can be employed. All of the unknown parameters in the system are dealt with by adaptive control techniques. Control design via backstepping approach is used for this high order system case. The uniform stability is guaranteed through Lyapunov analysis and the tracking error is restricted to an acceptable small region around the origin. An example is included to demonstrate the feasibility of the proposed theory.

Key Words: Neural network, Disturbance rejection, Nonlinear system

1 INTRODUCTION

Output regulation of nonlinear system has been an active research topic for the last decade [1]. When the disturbance is generated by unknown linear exosystem, an adaptive internal model is employed to estimate the desired input to tackle the influence of disturbance in a similar structure to the state observer. The global stability is satisfied and the desired feedforward control input is reformulated to reject unwanted uncertainty in the output feedback nonlinear system [2]. However not all disturbance signals can be modeled by an output of linear exosystem. If the disturbance is generated from a nonlinear exosystem, a filtered transformation is employed and an internal model, based on high gain observer, is used to provide the desired feedforward control input for uncertainty supression [3]. This global stable result is based on some crucial assumptions. There are some recent results on asymptotic rejection of general periodic disturbance for nonlinear systems. For instance, the nonlinear system with half period symmetric matched or unmatched general periodic disturbance can be asymptotically suppressed by bringing in a new half-period integration and a delay operator [4] [5]. Another example is that the internal model is introduced to regenerate the periodic disturbance which can be expanded as a combination of sinusoidal signals in Fourier series [6]. Nevertheless these results are only deal with general periodic disturbance with a known frequency.

For a physical system, disturbances often have complicated frequency components and their periods are not always known. As an online approximation method, neural network can be used to deal with output regulation of nonlinear system which has multi-frequency disturbances. It is well known that neural networks can provide good approximation to continuous functions, including which can be expanded to Fourier series over a compact set. Common neural networks include Fuzzy Neural Network (FNN) [11], Multi-layer Neural Network (MNN) [7] and Radial Basis Function Neural Network (RBFNN) [8]. In terms of NN application, the observer based on FNN control for non-affine nonlinear system was established [11] and developed for disturbed nonlinear systems [12]. RBFNN was employed for pure feedback nonlinear system with small gain theorem [8]. MNN was applied in low triangular nonlinear system with time varying disturbance [10]. But system states are unmeasurable in most of the practical cases. It is reasonable to consider output-feedback nonlinear system and ARBFNN which appears in [14] for the first time. Therein ARBFNN is applied to approximate the boundary of the uncertainties in the output feedback stochastic systems.

Motivated by the aforementioned discussion, we introduce NN to solve disturbance suppression problem of high order nonlinear output-feedback system. ARBFNN is used to approximate the desired forwarding input for system invariant central manifold. Meanwhile, the uncertainty functions are polynomials of output and disturbances with unknown constant coefficients. For such functions, two inherent important inequalities will be satisfied [4]. These inequalities are provided in section 6. In addition, system parameters except the high gain frequency are unknown for this high order case. Hence the adaptive backstepping scheme is used to design the final control based on their estimations [15]. The closed-loop uniform stability of our system in the paper is guaranteed by Lyapunov analysis. For the existence of neural network error, σ modification is applied in the adaptive law and the tracking error converges to a small set around the origin.

The presentation of this paper is organized as follows. Sections 2, 3 and 4 introduce the state transformation and internal model. In section 5 the approximation by neural networks is presented. In sections 6 and 7 we complete the control design and its stability analysis. A simulation example is provided in section 8 to support the theory. Section 9 concludes this paper and points out the future
The system is minimum phase, i.e., the polynomial of $\omega$ dimensional system is the measurement output and $\omega$ are bounded periodic disturbances.

A vector field with elements being polynomial of $y$ of the origin in $\mathbb{R}$ works.

### 2 Notions and System Structure

Consider a single-input-single-output nonlinear system which can be transformed into output-feedback form as follow.

$$
\begin{aligned}
\dot{x} &= A_c x + \phi(y, \omega) + bu, \\
y &= Cx, \\
e &= y - q(\omega)
\end{aligned}
$$

(1)

with

$$
A_c = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix},
C = \begin{bmatrix}
1 \\
0 \\
\vdots \\
0 \\
b_ho
\end{bmatrix},
b = \begin{bmatrix}
0 \\
\vdots \\
0 \\
b_ho
\end{bmatrix},
$$

where state $x \in U$, input $u \in \mathbb{R}^m$, $y \in \mathbb{R}$ is the output, $e$ is the measurement output and $\omega = [\omega_1, \ldots, \omega_n]^T \in \mathbb{R}^n$ are bounded periodic disturbances. $U$ is a neighborhood of the origin in $\mathbb{R}^n$. It is assumed that $\phi(y, \omega)$ is a smooth vector field with elements being polynomial of $y$ and $\omega$ and $\phi(0, \omega) = 0$ for zero dynamics. $q(\omega)$ is an unknown polynomial of $\omega$ with $\rho \geq 1$.

#### Assumption 2.1

The system is minimum phase, i.e., the polynomial $B(s) = \sum_{i=\rho}^n b_i s^{i-1}$ is Hurwitz and the high frequency gain $b_\rho$ is known.

The problem we are going to solve is to find a finite dimensional system

$$
\begin{aligned}
\dot{\mu} &= \nu(\mu, e(t)), \mu \in \mathbb{R}^{\rho-1}, \\
u &= u(\mu, e(t))
\end{aligned}
$$

(2)

for any initial value of $x$, $\omega$ and $\mu$ in there define field, $x(t)$, $\omega$ and $\mu$ is bounded $\forall t \geq 0$ and the tracking error $e$ will converge into a small set.

### 3 State Transform

System (1) has relative degree $\rho > 0$. Then the filter below is introduced to reduce the relative degree to 1.

$$
\begin{aligned}
\dot{\xi}_1 &= -\lambda_1 \xi_1 + \xi_2, \\
\dot{\xi}_2 &= -\lambda_2 \xi_2 + \xi_3, \\
\vdots \\
\dot{\xi}_{\rho-1} &= -\lambda_{\rho-1} \xi_{\rho-1} + u
\end{aligned}
$$

(3)

where $\lambda_i > 0$ for $i = 1, \ldots, \rho - 1$ are designed parameters. Design the filter transform state as

$$
\tilde{z} = \bar{x} - [d_1 \ldots d_{\rho-1}]\xi
$$

(4)

where $\xi = [\xi_1 \ldots \xi_{\rho-1}]^T$, $d_i \in \mathbb{R}^n$ for $i = 1 \ldots \rho - 1$. The value of $d_i$ is given by

$$
\begin{aligned}
d_1 &= [A_c + \lambda_1 I]d_1, \\
d_i &= [A_c + \lambda_i I]d_{i+1}, \quad i = 1, \ldots, \rho - 2, \\
d_{\rho-1} &= b_\rho
\end{aligned}
$$

(5)

then system (1) is transferred to

$$
\begin{aligned}
\dot{\bar{z}} &= A_c \bar{z} + \phi(y, \omega) + d\xi_1, \\
y &= C\bar{z}
\end{aligned}
$$

(6)

where $d = [A_c + \lambda_1 I]d_1 = [d_1 \ldots d_\rho]$ and $d_1 = b_\rho$.

In system (6), $\xi_1$ can be treated as the new input. This system is relative degree one after state transformation and its state $\bar{z}$ is unknown. For the convenience of applying internal model theory, another state $z = [z_1, \ldots, z_{n-1}]^T \in \mathbb{R}^{n-1}$ is introduced to transform system (6) by setting

$$
z = \bar{z}_{2:n} = \frac{d_{2:n}}{d_1} y
$$

where $(\cdot)_{2:n}$ denotes a vector or matrix formed by the $2_{nd}$ row to the $n_{th}$ row.

#### Remark 3.1

From Assumption (2.1), there is $B(s)$ Hurwitz, then $D(s) := \sum_{i=\rho}^n d_i s^{i-1} = B(s) \prod_{i=1}^{\rho-1} (s + \lambda_i)$ is Hurwitz with designed $\lambda_i$.

After three times state transformation the new system will be

$$
\begin{aligned}
\dot{\bar{z}} &= D\bar{z} + \psi(y, \omega), \\
y &= z_1 + \psi_y(y, \omega) + b_\rho \xi_1
\end{aligned}
$$

(7)

where $D$ is given by

$$
D = \begin{bmatrix}
-d_{2:n} & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-d_{2:n} & 0 & \ldots & 0
\end{bmatrix}
$$

and

$$
\begin{aligned}
\psi(y, \omega) &= D\frac{d_{2:n}}{d_1} y + \phi_{2:n}(y, \omega) - \frac{d_{2:n}}{d_1} \phi_1, \\
\psi_y(y, \omega) &= \frac{d_2}{d_1} y + \phi_1(y, \omega)
\end{aligned}
$$

Matrix $D$ is hurwitz from $D(s)$ hurwitz. It is not difficult to find $\psi(0, \omega) = 0$ and $\psi_y(0, \omega) = 0$.

### 4 Internal Model

From output regulation theory, the assumption below is needed for this case of output regulation [16].

#### Assumption 4.1

There is an invariant manifold $\pi(\omega) \in \mathbb{R}^{n-1}$ and desired input $\alpha(\omega)$ satisfying

$$
\begin{aligned}
\frac{d\pi(\omega)}{dt} &= D\pi(\omega) + \psi(q(\omega), \omega), \\
\frac{d\pi(\omega)}{dt} &= \pi_1(\omega) + \psi_y(q(\omega), \omega) + b_\rho \alpha
\end{aligned}
$$

(8)

Based on invariant manifold theory, define a new state transform as

$$
\bar{z} = z - \pi
$$
The final model is

\[
\begin{align*}
\dot{z} &= D\bar{z} + \bar{\psi}, \\
\dot{c} &= \bar{z}_1 + \bar{\psi}_y + b_p(\xi_1 - \alpha), \\
\dot{\xi}_1 &= -\lambda_1\xi_1 + \xi_2, \\
\dot{\xi}_2 &= -\lambda_2\xi_2 + \xi_3, \\
\end{align*}
\]

where

\[
\begin{align*}
\bar{\psi}(e, \omega) &= \psi(y, \omega) - \psi(q(\omega), \omega) \\
\bar{\psi}_y(e, \omega) &= \psi_y(y, \omega) - \psi_y(q(\omega), \omega)
\end{align*}
\]

\section{5 RBFNN Approximator}
Motivated by previous study [14], RBFNN will be employed to approximate the desired input \(\alpha(e)\) on a compact set \(\Omega\) and \(\hat{\alpha} = \alpha - \tilde{\alpha}\).

\[
\alpha = W^TS(e) + \delta(e)
\]

where \(S(e) = [s_1(e), \ldots, s_l(e)]^T : \Omega \rightarrow R^l\) is a known smooth vector function with neural network node number \(l > 1\). Gaussian function

\[
s_i(e) = \exp[-\frac{(e - \mu_i)^2}{\eta^2}]
\]

is the basis function here. The centre \(\mu_i \in \Omega\) and the width \(\eta > 0\). The desired weighting vector \(W = [w_1, \ldots, w_l]^T\) is defined as

\[
W := \arg \min_{W \in R^l} \{\sup_{e \in \Omega}|\alpha - W^TS(e)|\}
\]

\(\delta(e)\) is the NN inherent approximation error and satisfying \(|\delta| \leq \delta\), which is the minimum upper bound of \(\delta\) and can be decreased by increasing the number of \(r\) and \(q\). The approximation of \(\alpha\) can be written as

\[
\hat{\alpha} = \hat{W}^TS(e).
\]

\section{6 Adaptive Backstepping Control Design}
Control design of system (9) starts from \(\xi_1\). The backstepping technique is used to find our final control \(u\). Suppose \(\hat{\xi}_1 = b_p\hat{\xi}_1\) is the desired value of \(\xi_1\) and \(\hat{\xi}_1 = \xi_1 - \xi\), then from system (9)

\[
\dot{\hat{\xi}} = \hat{z}_1 + \hat{\psi}_y + b_p\hat{\xi}_1 + \hat{\xi}_1 - b_p\alpha.
\]

Since the nonlinear function in \(\psi\) and \(\psi_y\) are polynomial with \(\psi(0, \omega) = 0\) and \(\psi_y(0, \omega) = 0\), here \(\omega\) is bounded and unknown parameters are constants, we have

\[
\begin{align*}
|\hat{\psi}| &< \bar{r}_z(|e| + |e|^p), \\
|\hat{\psi}_y| &< \bar{r}_y(|e| + |e|^p)
\end{align*}
\]

where \(p\) is a known parameter which is equal or greater than the largest degree of \(e\) and \(\omega\) in \(\hat{\psi}\) and \(\hat{\psi}_y\). Design \(\tilde{\xi}_1\) as

\[
\tilde{\xi}_1 = -c_0e - \tilde{\lambda}_0(e + e^{2p-1}) + b_p\hat{W}^TS(e)
\]

where \(c_0 > 0\) and \(\tilde{\lambda}_0\) and \(\hat{W}^T\) and are adaptive parameters with initial value of zero. So the error dynamics will be

\[
\dot{c} = \tilde{z}_1 + \tilde{\psi}_y - c_0\hat{e} - \tilde{\lambda}_0(e + e^{2p-1}) + b_p\hat{\xi}_1 + b_p\hat{W}^TS(e) - b_p\delta
\]

The adaptive laws are given by

\[
\begin{align*}
\dot{\xi}_1 &= b_p[-c_0e - \tilde{\lambda}_0(e + e^{2p-1}) + b_p\hat{W}^TS(e)] + b_p, \\
\dot{\xi}_r &= \lambda_{r-1}\xi_{r-1} - \xi_{r-1} + \frac{\partial \hat{z}_1}{\partial \psi}b_p[\hat{W}^TS(e)] \\
&+ \frac{\partial \hat{z}_1}{\partial \psi}(\hat{\psi}_y - \hat{z}_1) - \frac{\partial \hat{z}_1}{\partial y}(\hat{W}^TS(e)) + b_p\hat{W}^TS(e) + b_p\hat{W}^TS(e) - b_p\delta
\end{align*}
\]

where \(\gamma_r\) is positive designed parameter. \(\gamma_{we} \in R^*\) is positive definite defined matrix. The control \(u = \xi_1\) when relative degree \(\rho = 1\). In this case the relative degree is greater than then 1, the adaptive backstepping technique is then invoked in and we will have stabilizable functions as

\[
\begin{align*}
\dot{\xi}_1 &= b_p[-c_0e - \tilde{\lambda}_0(e + e^{2p-1}) + b_p\hat{W}^TS(e)] + b_p, \\
\dot{\xi}_r &= \lambda_{r-1}\xi_{r-1} - \xi_{r-1} + \frac{\partial \hat{z}_1}{\partial \psi}b_p[\hat{W}^TS(e)] \\
&+ \frac{\partial \hat{z}_1}{\partial \psi}(\hat{\psi}_y - \hat{z}_1) - \frac{\partial \hat{z}_1}{\partial y}(\hat{W}^TS(e)) + b_p\hat{W}^TS(e) + b_p\hat{W}^TS(e) - b_p\delta
\end{align*}
\]

\section{7 Stability Analysis}
In this section we will derive the uniform boundedness of all variables and the approximation error will converge to a small set around the origin.

\[
V = \beta V_z + V_{\hat{e}} + V_{\bar{W}} + V_{\tilde{\lambda}_0}
\]
\[
\begin{aligned}
V_t &= z^TPDz \\
V_e &= \frac{1}{2}e^2 \\
V_c &= \frac{1}{2}z^TPDz \\
V_l &= \frac{1}{2}T_n^{-1}l_0 \\
V_l &= \frac{1}{2}T_n^{-1}l_0
\end{aligned}
\]

Matrix \( D \) is Hurwitz so there exists a positive definite \( PD \) satisfying
\[
D^TPD + PD = -I
\]

So we have
\[
\dot{V} = \beta(-z^TPDz + z^2PDP\tilde{\psi} + c\tilde{e} + c\tilde{\psi} - c_0e^2 \\
-\dot{l}_0(e^2 + e^2) + eb_{P}\tilde{\xi}_1 - eb_{P}\tilde{\psi} e)
\]

where the unknown disturbance is given by
\[
\begin{aligned}
\tilde{e} &= \tilde{z} = \tilde{\psi} \\
\psi &= \tilde{\psi}
\end{aligned}
\]

Then
\[
\begin{aligned}
\tilde{\psi} &= \tilde{e} + 2e\omega_2 - e\omega_1 \\
\xi &= \tilde{z} + e + e\omega_1 + \xi - W^TS(e) - \delta \\
\dot{\xi} &= -\dot{\xi}
\end{aligned}
\]

The control \( u \) will be as below with designed constant parameters \( c_0 = 4, c_1 = 3 \) and \( h_1 = 2 \).
\[
\begin{aligned}
u &= \xi - 3\tilde{\psi} + \tilde{\psi}e + \tilde{\psi}(\xi - W^TS(e)) + \tilde{\psi} \frac{\partial \dot{\xi}}{\partial W} \\
&+ \tilde{\psi} \frac{\partial \dot{\xi}}{\partial l_0} - h_1\xi \frac{\partial \delta}{\partial l_0}
\end{aligned}
\]
where

\[
\begin{align*}
\dot{\xi}_1 &= -4e - \dot{l}_0(e + e^{2p-1}) + \hat{W}^T S(e) \\
\frac{\partial e}{\partial l} &= -4 - \dot{l}_0(2p - 1)e^{2p-2} + \hat{W}^T \frac{\partial S(e)}{\partial e} \\
\frac{\partial \hat{W}}{\partial \hat{W}} &= S(e)^T \\
\frac{\partial \hat{W}}{\partial \hat{l}} &= -(e + e^{2p-1}) \\
\frac{\partial S(e)}{\partial e} &= \left[\frac{\partial s_1(e)}{\partial e}, \ldots, \frac{\partial s_l(e)}{\partial e}\right]^T \\
\frac{\partial s_i(e)}{\partial e} &= \exp\left(-\frac{(e - \mu_i)^2}{\eta^2}\right) - \frac{2}{\eta^2} (e - \mu_i), i = 1, \ldots, l
\end{align*}
\]  \tag{23}

The adaptive gains are defined as \( \gamma_w = I \) and \( \gamma_l = 5 \) with \( \sigma_1 = 0.01 \) and \( \sigma_2 = 0.001 \). If \( p = 4 \), the inequality (13) will be satisfied. Adaptive laws are

\[
\begin{align*}
\dot{\hat{W}} &= -\gamma_w e S(e) + \gamma_w \frac{\partial e}{\partial \hat{W}} \frac{\partial S(e)}{\partial e} - \sigma_1 \gamma_w \hat{W} \\
\dot{l}_0 &= \gamma_l (e^2 + e^8) - \sigma_2 \gamma_l \hat{l}_0
\end{align*}
\]  \tag{24}

The output will be in a compact set \([-4, 4]\) so the centre \( \mu_i \) will be in \([-4, 4]\). Suppose we have node number \( l = 51 \) and width \( \eta = 1 \). The simulation result is shown in Fig. 1. The input and norm of adaptive parameters are shown in Fig. 2.

It can be seen from the Fig. 3 that the tracking error \( e \) converge to a neighborhood around the origin (\(|e| < 0.12\)). The neural network approximation can match the ideal forwarding input well after 10s. It also shows the boundedness of adaptive parameters. The system error can be reduced by changing design parameters. If the gains are selected as \( \gamma_w = 100I, \sigma_1 = 0.0001 \) and others are kept. The simulation result of revised gain has been shown by Fig. 4 to Fig. 6. Comparing those figure with the results we have before, it can be seen that the error is smaller (\(|e| < 0.05\)). Simultaneously the amplitude of final control is larger.

9 Conclusions and Future Works

In this paper, ARBFNN technique is selected for nonlinear system suppression. After three times state transformation, the system model turns to the final system form then neural network is applied to approximate desired forward input. This method can reject general periodic disturbances without known their periods. In the future, the work is focus on neural networks disturbance rejection of more com-
complicated nonlinear systems. The non-periodic disturbance is also investigated.

REFERENCES


