Lattice Hamiltonian approach to the Schwinger model

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Outline

1. The Schwinger model on lattice
2. Strong coupling expansion (SCE)
3. Ground state energy
4. Mass gaps
5. Chiral condensate
6. Oscillations of chiral condensate
7. Summary & outlook
The Schwinger model

Hamiltonian of the Schwinger model in the Kogut-Susskind staggered discretization [1,2]:

\[
\mathcal{H} = -\frac{i}{2a} \sum_{n=1}^{M} \left( \phi^+(n) e^{i\theta(n)} \phi(n+1) - \phi^+(n+1) e^{-i\theta(n)} \phi(n) \right)
\]

\[
+ m \sum_{n=1}^{M} (-1)^n \phi^+(n) \phi(n) + \frac{ag^2}{2} \sum_{n=1}^{M} L^2(n)
\]

• \(\phi(n)\) — single-component fermion field on a circle with \(M\) sites
• \(\theta(n) = agA_1(n)\) — gauge field variable related to the Abelian vector potential
• \(L(n) = E(n)/g\) — variable related directly to the electric field
• \(m\) — fermion mass
• \(a\) — lattice spacing
• \(g\) — gauge coupling constant

The Schwinger model

Hamiltonian of the Schwinger model in lattice representation after the Jordan-Wigner transformation [3]:

$$\mathcal{H}_{JW} = -\frac{1}{2a} \sum_{n=1}^{M} \left( \sigma^+(n)e^{i\theta(n)}\sigma^-(n+1) + \text{h.c.} \right)$$

$$+ \frac{m}{2} \sum_{n=1}^{M} \left( 1 + (-1)^n\sigma^3(n) \right) + \frac{ag^2}{2} \sum_{n=1}^{M} L^2(n)$$

- $\sigma^i(n)$ — Pauli matrices residing on the sites
- $L(n)$ — gauge field excitations defined between sites $n$ and $n+1$
- $e^{\pm i\theta(n)}$ — ladder operators for gauge field excitations

Spin matrices $\sigma^3(n), \sigma^\pm(n)$

$\sigma^3(n) = \pm 1$

Gauge field excitations $L(n), e^{i\theta(n)}$

$L(n) = 0, \pm 1, \pm 2, \ldots$

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Strong coupling expansion on the Schwinger model

Rewrite the Hamiltonian in a dimensionless form:

\[ W = \frac{2}{ag^2} \mathcal{H}_{JW} = W_0 + xV \]

\[ \sum_{n=1}^{M} \left( \sigma_n^+ e^{i\theta(n)} \sigma_{n+1}^- + \text{h.c.} \right) \]

\[ \frac{m}{ag^2} \sum_{n=1}^{M} \left( 1 + (-1)^n \sigma_n^3 \right) + \sum_{n=1}^{M} L^2(n) \]

• If \( x \equiv \beta = \frac{1}{a^2 g^2} \) is small, we can treat \( W_0 \) as an unperturbed Hamiltonian and \( V \) as a perturbation.

• SCE creates the truncated basis of \( W \)

Start with ground state of \( W_0, |0\rangle \)

Create states by acting with \( V \)

Separate states according to their unperturbed energy

Ortho-normalize

Include in the truncated basis

\[ |0\rangle \rightarrow V|0\rangle \]

\[ |\tilde{1}\rangle \rightarrow |1\rangle \]

\[ |\tilde{1}'\rangle \rightarrow |1'\rangle \]

\[ \left\{ |0\rangle, |1\rangle \right\} \]
Observables

- Ground state energy:
  \[ E_0 = \frac{\omega_0}{2Mx} \quad \text{as } a \to 0, M \to \infty, \quad \frac{1}{\pi} \]

- Scalar mass gap \((m = 0)\):
  \[ \frac{M_S}{g} = \frac{\omega_1 - \omega_0}{2\sqrt{x}} \quad \text{as } a \to 0, M \to \infty, \quad \frac{2}{\sqrt{\pi}} \]

- Vector mass gap \((m = 0)\):
  \[ \frac{M_V}{g} = \frac{\omega_0^V - \omega_0}{2\sqrt{x}} \quad \text{as } a \to 0, M \to \infty, \quad \frac{1}{\sqrt{\pi}} \]

- Chiral condensate (chiral order parameter):
  \[ \frac{\langle \bar{\psi} \psi \rangle_0}{g} = \frac{\sqrt{x}}{2M} \langle 0 | \sum_{n=1}^{M} (-1)^n \sigma^3(n) | 0 \rangle \]
Eigenvalue flow with the order of strong coupling expansion, $N$

Example for: $M = 8$, $x \equiv \beta = 2500$

Lattice Hamiltonian approach to the Schwinger model saturated!
Ground state energy

\[ E_0 = \frac{\omega_0}{2Mx} \quad m = 0 \]
Scalar mass gap

\[ \frac{M_S}{g} = \frac{\omega_1 - \omega_0}{2\sqrt{x}} \]

\( m = 0 \)
Vector mass gap

\[ \frac{M_V}{g} = \frac{\omega_0^V - \omega_0}{2\sqrt{x}} \]

\( m = 0 \)
Chiral condensate

Lattice Hamiltonian approach to the Schwinger model

$m = 0$
Comparison with MPS results

- Ground state energy and mass gaps – massless model:

<table>
<thead>
<tr>
<th>Observable</th>
<th>SCE+ED</th>
<th>MPS [4]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_0$</td>
<td>-0.3183098860(2)</td>
<td>-0.318338(24)</td>
</tr>
<tr>
<td>$M_S/g$</td>
<td>1.12837916711(1)</td>
<td>1.1279(12)</td>
</tr>
<tr>
<td>$M_V/g$</td>
<td>0.5641895845(9)</td>
<td>0.56421(9)</td>
</tr>
</tbody>
</table>

- Chiral condensate - massless case:

<table>
<thead>
<tr>
<th>$x$</th>
<th>SCE+ED</th>
<th>MPS [5]</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>-0.189878819389204</td>
<td>-0.19025255847009401</td>
<td>0.00037</td>
</tr>
<tr>
<td>25</td>
<td>-0.187519020840406</td>
<td>-0.18796879340592226</td>
<td>0.00045</td>
</tr>
<tr>
<td>30</td>
<td>-0.185829589660617</td>
<td>-0.18620821935803569</td>
<td>0.00038</td>
</tr>
<tr>
<td>cont.</td>
<td>-0.16(1)</td>
<td>-0.159930(8)</td>
<td></td>
</tr>
</tbody>
</table>

- C.c. - massive $m = 0.125$ (this is after subtracting log divergence [6]):

<table>
<thead>
<tr>
<th>$x$</th>
<th>SCE+ED</th>
<th>MPS [5]</th>
</tr>
</thead>
<tbody>
<tr>
<td>cont.</td>
<td>-0.091(5)</td>
<td>-0.092023(4)</td>
</tr>
</tbody>
</table>

Oscillations of chiral condensate while changing the SCE order $N$

$Lattice Hamiltonian approach to the Schwinger model$
Oscillations: fitting ansatz

- We have chosen the following fitting function:

\[ \Sigma(N) = \Sigma(N \to \infty) + a \left( \frac{b}{N^3} + e^{-\alpha N} \right) \sin \left( \frac{2\pi}{T} N + \varphi \right) \]

for huge \( x \)

for small \( x \)

- If we can guess the fitting ansatz correctly, we can use small number of points to approximate saturated values, \( \Sigma(N \to \infty) \).
Oscillations: fitting parameters

Phase and period:
- \( T = M \)
- \( \phi = \frac{2\pi}{M} \)
Fitting function: \( \Sigma(N) = \Sigma(N \to \infty) + a \left( \frac{b}{N^3} + e^{-\alpha N} \right) \sin \left( \frac{2\pi}{T} N + \varphi \right) \)
Fitting function: \( \Sigma(N) = \Sigma(N \to \infty) + a \left( \frac{b}{N^3} + e^{-\alpha N} \right) \sin \left( \frac{2\pi}{T} N + \varphi \right) \)

- High \( x \): errors due to huge oscillations
- \( \Sigma(N \to \infty) \) seems to go to zero – because of huge finite volume effects.
Oscillations of the chiral condensate and flux loops

- Every time $N = k M$, we reach the next flux loop in the system
- Period must reflect presence of the flux loops

Final fitting ansatz:

$$
\Sigma(N, M, x) = \Sigma(N \to \infty, M, x) + \left( A(M) \frac{\sqrt{x}}{N^3} + B(M, x)e^{-\alpha(M, x)N} \right) \sin \frac{2\pi}{M} (N + 1)
$$

Example:

- $x = 10^{10}$
- $M = 24$
Summary and outlook

Lattice Hamiltonian results for massless Schwinger model:
- Almost machine precision for GS energy and mass gaps
- Chiral condensate, but we have a problem with FVE

Future:
- Chiral condensate oscillations – further investigation
- How to deal with finite volume effects in the chiral condensate?
- Fitting functions?
- Damped harmonic oscillator for small $\chi$?
Thanks for listening!

Acknowledgements:

• Co-authors:
  - dr Krzysztof Cichy
  - dr Agnieszka Kujawa-Cichy

• I’m funded by:
  - EPSRC
  - Lancaster University
  - Manchester 1824
  - NowNano