Abstract
Eruptive wildfires and any other form of unsteady fire-spread involve a process of dynamical interaction between spread rate $R$ and fireline intensity $I$. If, for example, spread rate changes abruptly, the intensity is then adjusted more slowly as the amount of vegetation that is actively pyrolysing varies through the initiation of flaming in new vegetation and the burning out of previously ignited vegetation.

Using a fairly simple model description for the movement of a narrow zone of pyrolysis through such a vegetation layer, which thus generalises an earlier approach [1], the unsteady dynamical behaviour of a fireline is examined. In its simplest expression, the intensity can be determined as a weighted integral of previous rates of spread from a burnout time into the past up to the present moment. The weighting arises because different parts of a stratified vegetation layer can contribute differently to the overall intensity of an evolving fireline. The problem is closed, dynamically, if the rate of spread is then expressed as a function of the intensity.

By examining a power-law expression for a rate-of-spread law, of the form $R \propto I^{\nu}$, it is found that fires have stable rates of spread in all sublinear cases (having $0 < \nu < 1$). This is the usual nature of fire-spread that is found in the field. But eruptive fire growth is also sometimes observed in the field and this is found to be reproduced by the model in linear cases ($\nu = 1$) or superlinear cases ($\nu > 1$) provided only that a dimensionless ratio (here called the ‘Byram number’) exceeds unity in value. These features arise for any realistic form of weighting in the integral that is used to determine the intensity, modified only through relatively modest differences of detail.

Introduction
Most existing models for the spread of wildfires implicitly assume that the fire spreads in a quasi-steady manner, having a rate of spread $R$ that is fully determined under any given conditions of wind, vegetation and topography. Widely used simulation programs such as BEHAVE [2] and FARSITE [3] adopt this approach, making use of the formulations of Rothermel [4] and Albini [5] (themselves built on experimental correlations) to provide spread rates for fires that have evolved towards an equilibrium rate of spread.

Some commonly used formulae relating properties of fires can only hold under steady conditions. For example, the fireline intensity $I$ of a line fire is defined as the chemical energy from the vegetation that is released as heat through combustion, per unit transverse length along the fireline. For a fire that spreads at a rate of $R$ m s$^{-1}$, consuming a fuel load of $m$ kg m$^{-2}$ and releasing an energy of $Q$ kJ kg$^{-1}$ as the vegetation is ultimately
oxidised, a simple energy balance identifies the fireline intensity as $I = QmR$ kW m$^{-1}$. This formula, generally called Byram’s fireline intensity [6], is frequently used as if it was the definition of fireline intensity whereas it is only correct under steady conditions.

Interestingly, in his original article, Byram [7] was not so categorical and presented an alternative formula for fireline intensity in the form $I = d\dot{q}$ (not Byram’s notation) where \( d \text{ m} \) represents the ‘depth’ of the flaming region, or its length from front to back of the fire, and \( \dot{q} \text{ kW m}^{-2} \) is dubbed the ‘reaction intensity’ within the flaming region. This formula is valid for both steadily and unsteadily spreading firelines.

In a later article, Albini [8] made clear that he would not have expected the quasi-steady spread-rate formulations to extend into general circumstances. His objective in this much neglected paper was to offer an explanation for the unduly large, almost resonant response in spread rate and intensity that can be encountered when fires are subjected to some unsteady winds. His article discusses important features that must interact in any unsteady evolution. For example, he noted that the time-scale for gaseous mixing-controlled oxidation and the response time over which the spread rate \( R \) of a fire adjusts itself to variations in the intensity \( I \) are likely to be relatively short when compared with the ‘burnout time’ \( \tau_b \) s of the vegetation. On the other hand, the intensity of a fire with (say) an abruptly altered spread rate should evolve over a time-scale that is of the order of the burnout time.

This comes about, in its simplest terms, through an adjustment of the flame-depth \( d \). For example, if a fire has been spreading steadily for some time at a speed \( R_1 \) under a mild wind, it would have developed a flame depth of \( d_1 = \tau_b R_1 \) and generated a fireline intensity of \( I_1 = QmR_1 = d_1\dot{q} \); it follows that \( \dot{q} = Qm/\tau_b \) so that \( I_1 = d_1Qm/\tau_b \). However, if a sudden wind change increases the spread-rate to \( R_2 \), the intensity would not also increase instantaneously to \( QmR_2 \). Instead, the leading edge of the flaming region would initially move at a faster speed than the trailing edge, progressively increasing the flame depth \( d \) until it reaches a new equilibrium value of \( d_2 = \tau_b R_2 \) after a burnout time \( \tau_b \) has passed (when the trailing edge then advances at the speed \( R_2 \)). Along with this, the intensity \( I = dQm/\tau_b \) would also steadily increase towards its new equilibrium value, as \( d \) increases towards \( d_2 \). In the interim, the value of the ratio \( QmR_2/I = d_2/d \) would exceed unity.

Recent work [1] has taken the ideas of Albini [8] further and has shown that the way in which intensity feeds back into spread-rate can lead to quite different evolutions of fire spread, depending on whether or not the feedback is sublinear. Under sublinear conditions, which would involve having \( R \propto I^\nu \) (for \( 0 < \nu < 1 \)) the fire always evolves towards a steady equilibrium spread-rate. But under linear or superlinear conditions the spread rate accelerates whenever a quantity, which is dubbed the ‘Byram number’ in [1], is greater than one. The Byram number, which is defined simply as

$$B = QmR/I$$

is clearly exactly equal to one under steady flame-spread conditions (when \( I = QmR \)). Whenever it is not equal to one, the flame must then be propagating in an unsteady way. The value of the Byram number \( B \) is therefore a good indicator of unsteadiness in fire spread.

So far, the articles that have examined unsteady behaviour in this way [1, 8] have all involved simplified situations in which pyrolysis rates or reaction intensities are taken to be constant. That is, quantities such as \( \dot{q} \) would actually have their overall mean value \( Qm/\tau_b \) and would remain constant throughout the burnout time. This might be a reasonable approach if the vegetation were to consist only of a single component. But
if, for example, the vegetation were to contain a range of fine and coarse fuels, each with a different partial fuel load, the fine fuels are likely to be completely pyrolysed over a shorter period of time than the coarse fuels. Thus it would be realistic for the mass-loss rate from the vegetation (and consequently the rate at which vegetation contributes to the gaseous exothermic chemistry) to vary with time from the moment that the vegetation first begins flaming.

This article examines situations of this type, in which the fireline intensity depends on the total rate at which mass is lost within the flaming zone, including situations in which this mass-loss rate is not constant over the burnout time $\tau_b$. Although the problem can be formulated in different ways (as in [1]) only one relatively simple approach is presented.

**Model**

In order to focus on a straightforward but illustrative model, a good starting point is to consider a bed of vegetation that has a vertically stratified structure, of height $h$, as illustrated in Figure 1. The fuel thus consists of different densities $\varrho(y)$ at different heights $y$, integrating to the total fuel load $m$ between $y = 0$ and $y = h$.

Keeping the approach simple but realistic, the vegetation is taken to pyrolyse rapidly in a narrow region around an interface $y = \eta(t, x)$. The fuel should thus be relatively fine (for this model) so that its pyrolysis time is short when compared with the time taken for the interface to spread through the vegetation layer. This spreading is taken to occur at a burning speed $S(y)$ that depends only on the local vegetation properties, so that it too varies with vertical height $y$. The flame begins where $x = X(t)$ and $\eta(t, X) = h$, with the leading edge of the flame moving forwards at the spread rate $dX/dt = R(t)$. This description is a modest development from earlier models [1, 8] and it leads to the partial differential equation for the interface $y = \eta(t, x)$

$$ \eta_t = -S(1 + \eta_x^2)^{1/2} \quad \text{for} \quad 0 \leq X(t) - x \leq d(t) \quad \text{with} \quad \eta(t, X(t) - d(t)) = 0 \quad (1) $$

so that $S$ acts as a normal rate of advancement of the interface and $d$ is the flame depth.

Any small element $dx$ within the flaming region encounters a mass-loss rate of vegetation of $\varrho S (1 + \eta_x^2)^{1/2} dx$, as the interface advances. If the calorific value of the pyrolysis vapours produced is $Q$, which could also (in principle) vary with height, and if all of these

Figure 1: A schematic illustration of an unsteadily evolving fireline in which a thin region of pyrolysis at the interface $y = \eta(t, x)$ spreads into unburnt vegetation at a burning speed $S$. 
vapours are burnt relatively quickly to release their chemical energy, the fireline intensity can then be expressed as

\[ I = \int_{X(t)-d(t)}^{X(t)} Q \rho S \left( 1 + \eta_x^2 \right)^{1/2} \, dx. \] (2)

In principle, for any known variation of the spread rate \( R(t) \) with time, the evolution of \( \eta(t,x) \) with time can be followed, from any given initial pyrolysis surface \( \eta(0,x) \), and the evolution of the intensity \( I \) can be calculated.

In practice, the spread-rate \( R \) itself must depend on the intensity \( I \) in some way, as well as depending on external conditions such as wind and slope. This dependence is not yet known for unsteady evolutions although Albini [8] proposed one, as yet untested formula. Following [1] a working hypothesis that helps to uncover generic forms of fireline evolution is arrived at by adopting either of the two simple power-law relations

\[ \frac{R}{R_s} = \left( \frac{I}{Q_m R_s} \right)^\nu \quad \text{or} \quad R = B \times \frac{I}{Q_m} \] (3)

the latter being the linear version of a power law (with constant Byram number \( B \)) for the case in which the exponent \( \nu \) takes the value \( \nu = 1 \). In cases for which \( \nu \neq 1 \), the spread rate can be considered to have a steady-state value \( R_s \). If \( Q \) varies within the vegetation layer, then \( \bar{Q} \) in the formulae (3) should be assigned the mass-weighted mean value of \( Q \).

**Further simplification for cases in which \( R \gg S \)**

The essential features of this model are most easily uncovered in situations for which the spread-rate \( R \) is always significantly greater than the burning speed \( S \). The value of \( \eta_x^2 \) in equations (1) and (2) is then very small. Neglecting it means that the interface is always dominated by its downwards burning, now taking the simpler form \( \eta_t = -S(\eta) \), at a fixed value of \( x \), with \( \eta = h \) at the moment when the burning begins. If \( \zeta(\eta) \) is defined, for \( 0 \leq \zeta \leq \tau_b \), such that

\[ \zeta = \int_0^h \frac{dy}{S(y)} \quad \text{with} \quad \tau_b = \int_0^h \frac{dy}{S(y)} \]

then \( \zeta_t \equiv 1 \), at a fixed value of \( x \), so that \( \zeta \) represents a time-like coordinate, measuring the time elapsed after flaming is initiated at any position \( x \). Also, since \( \eta \) is a straightforwardly decreasing function of \( \zeta \), having \( \eta_\zeta = -S(\eta) \), properties such as \( S, \rho \) and (if it is not constant) \( Q \) can all be considered to vary as functions of \( \zeta \) up to the burnout time \( \tau_b \).

Moreover, since differentiating \( \zeta_t \equiv 1 \) gives \( \zeta_x t \equiv 0 \), the partial derivative \( \zeta_x \) must stay practically constant at any position \( x \) as time progresses. That is, at a time \( t \) and position \( x \), the value of \( \zeta_x(t,x) \) is the same as its value at the earlier time \( t - \zeta \), when the pyrolysing interface began burning downwards. Also, since \( \eta_\zeta \) begins with the value \( \eta_x = S/R \) at \( \eta = h \), for \( S \ll R \), it follows that \( \zeta = \eta_x \times d\zeta/d\eta = -1/R(t - \zeta) \). The fireline intensity formula (2) can then be transformed into the simpler expression

\[ I = \int_0^{\tau_b} \dot{q}(\zeta) R(t - \zeta) \, d\zeta \quad \text{with} \quad \dot{q}(\zeta) = Q(\eta) \rho(\eta) S(\eta) \] (4)

which would reduce to the form studied in [1] if \( Q \rho S \) were to stay constant. That is, \( I \) would take the form \( I = \dot{q} \, d(t) \) and changes in fireline intensity would then be determined purely by changes in flame depth \( d \). In effect, the ‘reaction intensity’ \( \dot{q}(\zeta) \) in equation (4)
provides a weighting factor, so that different parts of the flame depth are allowed to contribute differently to the intensity $I$.

As a model, equation (4) allows for a considerable degree of flexibility in the way that a stratified vegetation layer feeds into fireline intensity under conditions in which the burning is predominantly downwards. If the spread rate also happens to be steady (having $R$ constant) the intensity becomes

$$I = R \int_{0}^{\tau_b} \dot{q}(\zeta) \, d\zeta = \bar{Q} m R$$

which implicitly identifies a formula for the mean or mass-weighted calorific value $\bar{Q}$.

**Taylor approximation**

A revealing way of examining equation (4) in situations where the spread-rate $R$ does not vary too rapidly with time, is to consider a Taylor expansion in the form

$$R(t - \zeta) = R(t) - \zeta R'(t) + \frac{1}{2} \zeta^2 R''(t) - \frac{1}{6} \zeta^3 R'''(t) + \cdots$$

giving

$$I = \bar{Q} m \left( R(t) - \frac{1}{2} \tau_1 R'(t) + \frac{1}{6} \tau_2^2 R''(t) - \frac{1}{24} \tau_3^3 R'''(t) + \cdots \right)$$

with

$$\bar{Q} m = \int_{0}^{\tau_b} \dot{q}(\zeta) \, d\zeta \quad \text{and} \quad \tau_k = \frac{k + 1}{\bar{Q} m} \int_{0}^{\tau_b} \zeta^k \dot{q}(\zeta) \, d\zeta$$

so that the time-scales $\tau_1$, $\tau_2$, etc. arise from different moments of $\dot{q}(\zeta)$. It is readily noted that if $\dot{q}$ is constant then all of these times are exactly equal to the burnout time $\tau_b$.

Several forms of variation of $\dot{q}$ with $\zeta$ and their resulting moments are shown in Table 1. As should be expected, distributions of $\dot{q}(\zeta)$ that are biased towards $\zeta = 0$ lead to reduced values of each of the moments, while a bias towards $\zeta = \tau_b$ leads to increased moments.

**Table 1: Some of the time-scales $\tau_k$ arising from different distributions of $\dot{q}(\zeta)$.**

<table>
<thead>
<tr>
<th>$\dot{q} / \bar{Q} m$</th>
<th>$\frac{1}{\tau_b}$</th>
<th>$\frac{3(\tau_b-\zeta)^2}{2\tau_b^3}$</th>
<th>$\frac{3\sqrt{\tau_b-\zeta}}{2\tau_b^{3/2}}$</th>
<th>$\frac{2(\tau_b-\zeta)}{\tau_b^2}$</th>
<th>$\frac{3(\tau_b-\zeta)^2}{\tau_b^3}$</th>
<th>$\frac{6\zeta(\tau_b-\zeta)}{\tau_b^5}$</th>
<th>$\frac{2\zeta}{\tau_b}$</th>
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</thead>
<tbody>
<tr>
<td>sketch</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>e</td>
<td>f</td>
<td>g</td>
</tr>
<tr>
<td>case:</td>
<td>$\tau_1 / \tau_b$</td>
<td>$\tau_2 / \tau_b$</td>
<td>$\tau_3 / \tau_b$</td>
<td>$\tau_4 / \tau_b$</td>
<td>$2\tau_4 / \tau_b$</td>
<td>$3\tau_4 / \tau_b$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>$\frac{3}{4}$</td>
<td>$\frac{4}{5}$</td>
<td>$\frac{2}{3}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{3}$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$\frac{3}{5}$</td>
<td>$\frac{24}{35}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{3}{10}$</td>
<td>$\frac{9}{16}$</td>
<td>$\frac{5}{8}$</td>
<td>$\frac{3}{2}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2}$</td>
<td>$\frac{64}{105}$</td>
<td>$\frac{2}{5}$</td>
<td>$\frac{1}{5}$</td>
<td>$\frac{4}{5}$</td>
<td>$\frac{8}{5}$</td>
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<td></td>
<td>$\frac{3}{7}$</td>
<td>$\frac{128}{231}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{7}$</td>
<td>$\frac{5}{7}$</td>
<td>$\frac{5}{3}$</td>
<td>$\frac{5}{3}$</td>
</tr>
</tbody>
</table>
Figure 2: Relationships between $\frac{dR}{dt}$ and $R$ as approximated by equation (5) for selected values of $\nu \neq 1$. In this formulation, the case $\nu = 1$ reduces to $\frac{dR}{dt} \equiv 0$.

For example, in the two very different cases $e$ and $g$ shown in the table, the intensity can be written in the following forms, involving derivatives of $R(t)$ and the relevant values of $\tau_1$

$$I = \bar{Q}m \times \begin{cases} R(t) - \frac{1}{2}\tau_1 R'(t) + \frac{1}{5}\tau_1^2 R''(t) - \frac{1}{15}\tau_1^3 R'''(t) + \cdots & \text{(case } e) \\ R(t) - \frac{1}{2}\tau_1 R'(t) + \frac{9}{64}\tau_1^2 R''(t) - \frac{9}{320}\tau_1^3 R'''(t) + \cdots & \text{(case } g) \end{cases}$$

having coefficients that tend to decrease more rapidly with order in cases that are more biased towards $\zeta = \tau_b$ (as in case $g$) partly because $\tau_1$ is greater in such cases. These formulae suggest strongly that the first-moment time $\tau_1$ provides a suitable representative time-scale for evolutionary problems, with relatively mild differences arising in the higher derivatives from different biases in the distribution of $\dot{q}(\zeta)$.

In the context of the power-law relations (3) for the dependence of the spread-rate $R$ on $I$, for $\nu \neq 1$ or $\nu = 1$, this leads to the respective phase-plane relationships

$$\frac{\tau_1}{2R_s} \frac{dR}{dt} \approx \frac{R}{R_s} - (\frac{R}{R_s})^{1/\nu} \quad \text{or} \quad \frac{B\tau_1}{2} \frac{dR}{dt} \approx (B - 1)R \quad (5)$$

if the Taylor expansion is truncated at the first derivative. These formulae are the same and would generate the same dynamical behaviour on the time-scale of $\tau_1$, for any bias that the function $\dot{q}(\zeta)$ might have; differences would only arise from higher derivative terms that are omitted in (5). Resulting phase-plane diagrams for various values of $\nu \neq 1$ are shown in Figure 2. These diagrams all involve two ‘steady’ spread rates (having $\frac{dR}{dt} = 0$) although one of these is the zero spread rate $R = 0$, which would mean that there is no fire; the other steady spread rate is $R = R_s$.

In cases having $\nu < 1$ (representing sublinear dependence of $R$ on $I$) the sign of $\frac{dR}{dt}$ for any positive spread rate $R > 0$ is always such that the spread rate changes with time towards $R = R_s$; it decreases if $R > R_s$ and increases if $R < R_s$. Thus the steady spread rate $R_s$ should be stable.

For cases having $\nu > 1$ (superlinear dependence of $R$ on $I$) values of $\frac{dR}{dt}$ are negative (so that $R$ decreases) for $R < R_s$ and positive (so that $R$ increases) for $R > R_s$. The steady spread rate is therefore unstable and any spread rate below $R_s$ should evolve towards zero. Above $R_s$ the spread-rate increases.

The linear case, represented by the second equation in (5), shows that the zero spread-rate $R = 0$ is unstable if $B > 1$ and is stable if $B < 1$. That is, if the Byram number is bigger than one then any positive spread-rate grows exponentially. If it is less than one then any spread rate decreases exponentially towards zero.
Nonlinear dependence of spread-rate on intensity

Returning to the fuller set of equations (4) and (3), the stability of the steady solutions can be studied more precisely (without truncation of a Taylor expansion) in the form

$$R = R_s + \varepsilon e^{\alpha t/\tau_1}$$

taking $\varepsilon$ to be arbitrarily small if $\nu \neq 1$ and taking $R_s = 0$ in the linear case $\nu = 1$. Neglecting small terms, it is found that the dimensionless growth rate coefficient $\alpha$ satisfies the same equation in either case, namely

$$\gamma \int_0^{\tau_b} \frac{\dot{q}(\zeta)}{Q m} e^{-a \zeta/\tau_1} d\zeta = 1 \quad \text{or} \quad \gamma \int_0^{\tau_b} \dot{q}(\zeta) e^{-a \zeta/\tau_1} d\zeta = \int_0^{\tau_b} \dot{q}(\zeta) d\zeta$$

with $\gamma = \nu$ for $\nu \neq 1$ or $\gamma = B$ for $\nu = 1$. Details of how $\alpha$ changes with $\gamma$ must depend on the actual distribution $\dot{q}(\zeta)$, but it is straightforward to observe that $\alpha$ must be negative if $\gamma < 1$, and positive if $\gamma > 1$. In the case when $\dot{q}$ is constant, it is easily seen that

$$\gamma = \alpha / (1 - e^{-\alpha})$$

with other distributions of $\dot{q}(\zeta)$ providing qualitatively similar results, as is plotted in Figure 3 for each of the distributions presented in Table 1.

In other words, this confirms that the steady spread rate $R_s$ is stable if $R$ depends sublinearly on $I$ ($\nu < 1$) and it is unstable if $R$ depends superlinearly on $I$ ($\nu > 1$). If $R$ varies linearly with $I$ then the spread-rate grows exponentially if the Byram number exceeds one and decays exponentially towards zero if it is less than one.

Dynamically stable fire spread

Thus relatively simple arguments show that the nature of any unsteady fire behaviour is closely linked to the manner in which the spread-rate $R$ varies as a function of the fireline intensity $I$. The fact that most fires do spread in a quasi-steady manner under constant or slowly varying conditions, possibly after an unsteady initial transient, means that $R$ does normally vary sublinearly with $I$.

One might therefore expect that, under given conditions of (say) wind $w$ and slope $\theta$, the fire would have a steady spread-rate of $R_s(w, \theta)$, which might be determined using Rothermel’s [4] or any other formulation. Under unsteady conditions, when Byram's
formula for fireline intensity is not satisfied, the spread-rate \( R \) might then be determined by a formula of the nature

\[
R(I, w, \theta) = R_s(w, \theta) \times \left( \frac{I}{Q_m R_s(w, \theta)} \right) ^ \nu
\]  

(7)

for a suitable power \( \nu \) that is less than one. Using this formula to calculate the Byram number, it is found that

\[
B = \frac{\dot{Q}_m R}{I} = \left( \frac{\dot{Q}_m R_s}{I} \right) ^ {1-\nu} = \left( \frac{R_s}{R} \right) ^ {(1-\nu)/\nu}
\]

so that the Byram number is greater than one (as a result of which the fire accelerates) whenever \( R < R_s \) or \( I < \dot{Q}_m R_s \) and it is less than one (so that the fire decelerates) when \( R > R_s \) or \( I > \dot{Q}_m R_s \).

Although the form of the spread-rate law (7) is fairly speculative at this time (and it differs in significant respects from a formula offered by Albini [8]) such a law would nonetheless model unsteady evolutions towards stable quasi-steady spread rates, when solved in combination with equation (4) for suitable distributions of \( \dot{q}(\zeta) \). Under conditions in which the spread rate \( R \) is not very significantly larger than the burning speed \( S \), the somewhat more complex model equations (1) and (2) would need to be solved in place of (4), but the same manner of dynamical behaviour should still be reproduced.

**Eruptive fire spread**

When modelling eruptive fires, there is no starting point in the literature on quasi-steady fire spread that can suggest the most suitable model for the dependence of spread rate \( R \) on the intensity \( I \). The evidence of the arguments presented earlier (and in [1]) shows that the dependence must be either linear or superlinear during any eruptive phase of a fire.

As a fire erupts, its spread-rate and intensity both grow progressively in time (requiring that the Byram number stays greater than one). It has already been seen that an exponential growth arises when \( R \) depends linearly on \( I \), having the basic form

\[
R = R_i e^{\alpha t/\tau_1} \quad \text{and} \quad X = \frac{\tau_1 R_i}{\alpha} e^{\alpha t/\tau_1} \quad \text{so that} \quad R = \frac{\alpha}{\tau_1} X
\]

since \( dX/dt = R \). It follows that the spread-rate \( R \) then grows linearly with the distance travelled, having a constant of proportionality that is the same as the growth-rate factor in the exponential. From an experimental point of view this would enable an estimate to be made for the parameter \( \alpha \). In turn, the Byram number could be estimated from equations (6) and some knowledge of the time-variation of the reaction intensity \( \dot{q}(\zeta) \).

If the spread-rate law was genuinely superlinear, having \( \nu > 1 \), then at some stage, the spread-rate at the front of the fire \( R(t) \) would greatly dominate over earlier spread-rates \( R(t-\zeta) \) except when \( \zeta \) is close to zero. The integral in (4) would then be dominated by values of \( \dot{q}(\zeta) \) that are close to \( \zeta = 0 \) (apart from distributions of \( \dot{q} \) in the form of cases \( f \) and \( g \), which are unlikely to be realistic). Changing variables using \( \zeta = t-\varsigma \) in (4) and proposing a solution of the form \( R \sim A/(t_\infty - t)^\mu \) when the spread rate \( R \) is very large, leads to the approximate solutions

\[
R \approx R_s \left( \frac{\dot{Q}_m/\dot{q}(0)}{(\nu-1)(t_\infty - t)} \right) ^ {\nu/(\nu-1)} \quad \text{with} \quad X \approx R_s \left( \frac{\dot{Q}_m}{\dot{q}(0)} \right) ^ {\nu/(\nu-1)} \left( (\nu-1)(t_\infty - t) \right) ^ {1/(\nu-1)}
\]
for describing the behaviour of \( R \) close to a time \( t_\infty \) when \( R \) itself grows towards an infinite value. As a result, when \( R \) is very large, it can be seen that

\[ R \propto X^\nu. \]

This would offer a means of identifying superlinear behaviour in an eruptive fire, through examining if the spread rate increases superlinearly with distance travelled.

In reality, of course, the spread rate \( R \) is unlikely to grow indefinitely in value. Thus a spread-rate law of the form \( R \propto I^\nu \), with \( \nu \geq 1 \), is unlikely to hold for very large values of \( R \) and \( I \). However, because it does describe eruptive growth for \( \nu \geq 1 \), such a law must offer a reasonable approximation for erupting fires over a range of intensities and spread-rates. Of course, further study is needed at this stage to establish what parameters are most relevant for any given eruptive fire and its vegetation conditions.

**Discussion**

Any model for fire-spread that takes into account behaviour that is not quasi-steady has to address two main points.

The first of these is the manner in which fireline intensity \( I \) evolves in response to changes in spread rate \( R \). This must involve a process of accumulation as the amount of vegetation that is recruited into supporting the flames either grows or decreases (as it is burnt out). Byram [7] anticipated this by recognising that one possible expression for fireline intensity should involve the flame-depth \( d \). His other formula for fireline intensity \( I = \bar{Q}mR \) is more limited because it is only valid under conditions of steady or quasi-steady fire spread. In order to highlight deviations from steady fire-spread, it is useful therefore to introduce the dimensionless parameter \( B = \bar{Q}mR/I \) (which we name the Byram number) that takes the value one only under quasi-steady conditions.

In proposing a suitable model for intensity accumulation, some accounting needs to be made of the amount of vegetation actually involved in the flaming process and the rate at which it is losing mass. A procedure for doing this has been offered in this article, which allows for the fact that different parts of the vegetation can contribute differently to the rate of energy release at different stages in the flaming. An earlier article [1] examined the simpler, but more limited situation, in which the vegetation was all taken to contribute at the same rate. The intensity thus emerges as a weighted integral over past rates of spread or, in its fullest formulation, as an integral related to solutions of a partial differential equation for the movement of a surface of pyrolysis, that retains information from earlier rates of spread.

As often happens in modelling situations, the simplest model is seen to still capture the essence of all aspects of the dynamical behaviour. Differences arising from the more general formulation examined here are found to be matters of detail rather than fundamental change.

The second point that needs to be addressed in any quasi-steady modelling of fire spread is the way in which the spread-rate \( \dot{R} \) responds to unsteady changes in fireline intensity \( I \). This relation has barely been touched on in the wildfire literature, with Albini’s work [8] providing a notable exception. By examining a simple power-law relation of the form \( R \propto I^\nu \), as a working hypothesis, different forms of dynamical behaviour are readily identified.

In particular, it is seen that stable quasi-steady spread rates are only possible for spread rate laws that are (at least qualitatively) of the form \( R \propto I^\nu \) for a power \( \nu \) that is
between zero and one. Albini [8] offered one spread-rate law that is of this form, as was shown in [1]. Another example is offered in the spread-rate law (7) which would provide for a direct link between quasi-steady and unsteady formulations for flame spread.

For powers $\nu$ that are either one or greater, the dynamical behaviour becomes altogether different. There is then no longer any non-zero stable spread rate and evolutions either lead to spread rates that decay to zero (if the Byram number is less than one) or towards infinity (if it is greater than one). There is no body of work on quasi-steady fire spread that can be drawn on and extended to suggest suitable spread-rate laws in this range. However, it should be noted that eruptive fire growth is observed in practice [9]–[13] so that linear and superlinear dependence of spread-rate on intensity is not unreasonable. As this article suggests, the manner in which spread rates vary with distance travelled can offer practical clues about an appropriate value for the exponent $\nu$.

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References