Lecture Notes in Chemistry

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50

D. Mukherjee (Ed.)

Aspects of Many-Body Effects in Molecules and Extended Systems

Proceedings of the Workshop-Cum-Symposium
Held in Calcutta, February 1–10, 1988

Springer-Verlag
Berlin Heidelberg New York London Paris Tokyo
Editor

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Printed in Germany

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2151/3140-543210
QUANTUM FLUID DYNAMICS: AN EXTENDED COUPLED CLUSTER TREATMENT

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Abstract: The extended coupled cluster method (ECCM) is applied to the zero-temperature condensed Bose fluid for the general case of a non-uniform and time-dependent condensate wavefunction. The principal aim is to derive the appropriate balance equations of fluid dynamics for such local observables as the number density, current density and energy density. This is achieved by coupling the system to scalar and vector gauge fields, so that the theory may be formulated in a completely gauge-invariant fashion to take fully into account the underlying $U(1)$ symmetry imposed by number conservation. The ECCM formalism is based on the equations of motion for a set of linked-cluster amplitudes which characterise the system completely. The off-diagonal one- and two-body density matrices are studied in terms of these amplitudes, and the various balance equations of fluid dynamics are thereby derived. Furthermore, we show how these are also exactly satisfied by most practical approximation schemes applied to the otherwise exact ECCM description. The fact that the ECCM amplitudes all strictly obey the cluster property, further implies that our gauge-invariant formalism is capable in principle of a complete fluid-dynamical description of the system at zero temperature, including possible states of topological excitation or deformation and of broken symmetry.

1. INTRODUCTION

We have seen in a companion article in the present volume [1] (hereafter referred to as I) how the extended coupled cluster method (ECCM) [2-4] formally decomposes an arbitrary quantum many-body problem based on a Schrödinger dynamics, into a nonlocal classical field theory for a set of interacting, n-body, classical (i.e., c-number) fields $(\sigma_n, \tilde{\sigma}_n)$. In the coordinate-space representation these are simply n-point functions of spatial arguments $(\hat{x}_1, \hat{x}_2, \cdots \hat{x}_n)$ and time, $t$. The lowest-order (one-body) amplitudes for the condensed zero-temperature Bose fluid considered here, are just the condensate wavefunctions, namely the (time-dependent) expectation values of the one-boson creation and destruction operators (or field operators), $a_{\hat{x}}^+$ and $a_{\hat{x}}^-$ respectively, for a particle at space-point $\hat{x}$. Thus, $\tilde{\sigma}_1(\hat{x}) = <a_{\hat{x}}^+>$ and $\sigma_1(\hat{x}) = <a_{\hat{x}}^->$.
These amplitudes \((\sigma_1, \tilde{\sigma}_1)\) are clearly local by definition, since they depend only on a single spatial coordinate. However, we have seen in I how the higher-order amplitudes are also conceptually of a similar quasi-local character due to their linked-cluster or connected property, which guarantees that they obey the cluster property of vanishing as the relative separation of any two of their spatial arguments approaches infinity. For example, \(\tilde{\sigma}_2(x, y) = \langle a_x^+ a_y^+ \rangle = \langle a_x^+ a_y^\dagger \rangle - \langle a_x^\dagger a_y^+ \rangle + 0\) as \(|x-y| \to \infty\). The basic ECCM amplitudes \(\{\sigma_n, \tilde{\sigma}_n\}\), which completely characterise the original quantum many-body problem, may thus be regarded as a set of generalised quasi-local order parameters, by analogy with the more conventional and strictly local (one-body) order parameters of many-body theory. For example, the total energy expectation value, \(\tilde{\mathcal{E}} = \tilde{\mathcal{R}}[\sigma_n, \tilde{\sigma}_n]\), and other arbitrary expectation values in the ECCM, are almost local functionals of these generalised, classical, quasi-local order parameters.

We now show how the ECCM may be applied to a general, spatially non-uniform and non-static, condensed Bose system at zero temperature. The formalism is developed to provide a completely gauge-invariant description of such systems, and hence also a complete description of its quantum fluid dynamics in the zero-temperature limit. We are less interested in the present work in formulating approximation schemes for specific systems, than in investigating how the various balance equations of fluid dynamics can be extracted both in the exact formulation and in rather general classes of approximations to it.

The main extension of the theory necessary to achieve this goal is its formulation in a completely gauge-invariant fashion to take properly into account the underlying \(U(1)\) gauge symmetry imposed by particle number conservation. After a brief review of the ECCM in Sec. 2, we show in Sec. 3 how this may be achieved by coupling the system to appropriate external scalar and vector gauge fields \(\phi(x, t)\) and \(\vec{A}(x, t)\). We show that these fields can be thought of as providing differential or local Galilean transformations which vary with time and position in the system, thereby setting the system into the general state of motion desired.

The equations of motion of the basic ECCM amplitudes \(\{\sigma_n, \tilde{\sigma}_n\}\) are derived in Sec. 4, where we show also that at the lowest level of truncation which puts to zero all amplitudes except \(\sigma_1\) and \(\tilde{\sigma}_1\), the formalism simply degenerates to the mean field description of Gross and Pitaevskii [5-7]. The evolution of the one-body density matrix and its gauge-transformation properties are investigated in Sec. 5, and its physical content in leading to the appropriate balance equations of quantum fluid dynamics is demonstrated explicitly in Sec. 6.

2. THE ECCM REVIEWED

In this Section we gather together the main results that we need from I, to which the reader is referred for the omitted details. The description in I was given in terms of a very general configuration-space indexing, and we now take the opportunity to specialise to the coordinate-space representation that is most convenient for
present purposes. We consider a system of bosons at zero temperature described by
the usual single-particle creation and destruction operators, $a_x^\dagger$ and $a_x$ respectively, which obey the standard canonical commutation relation,

$$[a_x, a_y^\dagger] = \delta^{(3)}(x-y) \equiv \delta(x-y),$$

(1)

and where we indicate position-space vectors and their corresponding differential
volume elements as $x = (x^1, x^2, x^3)$ and $dx$ respectively, etc. As the model state
or cyclic vector $|\phi>$ we now choose the bare vacuum state, so that $a_x |\phi> = 0$ for
all $x$. In a Hilbert space of definite total particle number, $N$, the identity operator $I$
has the following useful resolution in terms of the basic single-particle operators,

$$I = \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \cdots \int dx_n a_{x_1}^\dagger \cdots a_{x_n}^\dagger |\phi><\phi| a_{x_1} \cdots a_{x_n}. $$

(2)

The exact time-dependent states $|\psi(t)>$ and $<\psi'(t)|$ of our many-bosons system
obey the time-dependent Schrödinger equations,

$$H|\psi(t)> = i \frac{\partial}{\partial t} |\psi(t)> ; \quad <\psi'(t)|H = -i \frac{\partial}{\partial t} <\psi'(t)|,$$

(3)
in terms of the Hamiltonian $H$. We may also define an action functional,

$$A = \mathcal{A}[\psi, \psi'] = \int dt <\psi'(t)|(i\hbar/\partial t - H)|\psi(t)>,$$

(4)

whose stationarity with respect to (independent) variations in the wavefunctions
$<\psi'(t)|$ and $|\psi(t)>$ leads respectively to the Schrödinger equations (3). The exact
states may be defined in terms of the model (vacuum) state in terms of the time-
development operators $U(t)$ and $\tilde{U}(t)$ defined as,

$$|\psi(t)> = U(t)|\phi>; \quad U(t) \equiv e^{a(t)} e^{S(t)} e^{-\tilde{S}(t)},$$

(5)

$$<\psi'(t)| = <\phi|\tilde{U}(t); \quad \tilde{U}(t) \equiv e^{-\tilde{a}(t)} e^{\tilde{S}(t)} e^{-S(t)},$$

where $a(t)$ and $\tilde{a}(t)$ are c-number scale factors, and $S(t)$ and $\tilde{S}(t)$ are op-
ernators which contain only creation pieces and destruction pieces respectively with
respect to the model state $|\phi>$, so that $<\phi|S(t) = 0 = \tilde{S}(t)|\phi>$. Explicit normalisa-
tion is guaranteed for all times $t$,

$$<\psi'(t)|\psi(t)> = 1,$$

(6)

by making the specific choice $\tilde{a}(t) = a(t)$. Once this choice is made the scale fac-
tors play no further role in our formalism.

We note that with the parametrisations of Eq. (5) the states $|\psi(t)>$ and $<\psi'(t)|$
are not eigenstates of particle number. From the outset we thus work in a number non-
conserving formalism of the type introduced by Bogoliubov [7,8] and now familiar in the
condensed Bose systems of concern here. The fundamental ECDM operators $S(t)$ and
\( \tilde{\Sigma}(t) \) have the specific parametrisations,

\[
S(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 \cdots \int dx_n s_n(x_1 \cdots x_n; t) a_{x_1}^+ \cdots a_{x_n}^+ ,
\]

\[
\tilde{\Sigma}(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 \cdots \int dx_n \tilde{s}_n(x_1 \cdots x_n; t) a_{x_1}^n \cdots a_{x_n}^n .
\]

We have seen in I how the entire set of amplitudes \( \{ S_n, \tilde{s}_n \} \) that characterise our problem completely, are linked-cluster amplitudes which obey the cluster property. We may assume without loss of generality that the amplitudes \( \{ S_n, \tilde{s}_n \} \) in Eq. (7) are all totally symmetric under permutations of their arguments. In this case, Eq. (7) shows that they have the explicit form

\[
S_n(x_1 \cdots x_n; t) = \langle \phi | a_{x_1}^n \cdots a_{x_n}^n S(t) | \phi \rangle ,
\]

\[
\tilde{s}_n(x_1 \cdots x_n; t) = \langle \phi | \tilde{\Sigma}(t) a_{x_1}^n \cdots a_{x_n}^n | \phi \rangle .
\]

It is useful to introduce the ECCM double similarity transform \( \hat{A}(t) \) of an arbitrary operator \( A \), as

\[
\hat{A}(t) = e^{\hat{\Sigma}(t)} e^{-S(t)} A e^{S(t)} e^{-\hat{\Sigma}(t)} ,
\]

in terms of which the expectation value of the operator may be written as

\[
\langle A \rangle \equiv \langle \hat{A}(t) \rangle \equiv \langle \psi'(t) | A | \psi(t) \rangle = \langle \phi | \hat{A}(t) | \phi \rangle .
\]

We have seen in I how the ensuing ECCM formalism takes a more symmetrical form if we make a further change of parametrisation from the "bottom" amplitudes \( \{ S_n \} \) to a new set of linked-cluster amplitudes \( \{ s_n \} \) associated with a new creation operator \( \Sigma(t) \), defined as

\[
\Sigma(t) | \phi \rangle \equiv (1 - |\phi \rangle \langle \phi |) e^{\hat{\Sigma}(t)} S(t) | \phi \rangle ,
\]

\[
\Sigma(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 \cdots \int dx_n s_n(x_1 \cdots x_n; t) a_{x_1}^n \cdots a_{x_n}^n .
\]

The resulting ECCM is now completely characterised by the linked-cluster amplitudes \( \{ s_n, \tilde{s}_n \} \), and assumes a much more symmetric form with respect to the corresponding pairs \( \sigma_n \) and \( \tilde{\sigma}_n \). We may again assume with no loss of generality that the amplitudes \( s_n(x_1 \cdots x_n; t) \) are totally symmetric with respect to permutations of the spatial arguments. Hence we have the explicit transformation,

\[
s_n(x_1 \cdots x_n; t) = \langle \phi | a_{x_1}^n \cdots a_{x_n}^n e^{\hat{\Sigma}(t)} S(t) | \phi \rangle .
\]

It is a simple matter to show from Eqs. (7)-(12), for example, that the expectation
values of the fundamental operators $a_x^+$ and $a_x$, namely the so-called condensate wavefunctions [7], are given as

$$<a_x> = \tilde{a}_x(t) = \sigma_1(x; t); \quad <a_x^+> = \tilde{a}_x^+(t) = \tilde{\sigma}_1(x; t),$$

namely, just the lowest-order (one-body) ECCM basic amplitudes.

An arbitrary expectation value may now be wholly expressed in terms of the basic ECCM amplitudes, $\bar{A} = \bar{A}[\sigma_n, \tilde{\sigma}_n]$. The action of Eq.(4) takes the explicit form,

$$\mathcal{A} = \int dt [\phi(E(t)E(t)\phi^* - \bar{H}] = \int dt [\phi(E(t)E(t)\phi^* - \bar{H}]$$

$$= i \int dt \sum_{n=1}^\infty \frac{1}{n!} \int dx_1 \cdots \int dx_n \tilde{\sigma}_n(x_1 \cdots x_n; t) \tilde{\sigma}_n(x_1 \cdots x_n; t) - \int dt \bar{H}[\sigma_n, \tilde{\sigma}_n].$$

The fundamental equations of motion for the basic ECCM amplitudes, which come either from extremising the action of Eq. (14) or directly from the Schrödinger equation, are

$$i \tilde{\sigma}_n(x_1 \cdots x_n; t) = \frac{\delta \bar{H}}{\delta \sigma_n(x_1 \cdots x_n; t)},$$

$$-i \sigma_n(x_1 \cdots x_n; t) = \frac{\delta \bar{H}}{\delta \sigma_n(x_1 \cdots x_n; t)},$$

where, for an arbitrary operator $A(t)$, the functional derivatives of its expectation value $\bar{A} = \bar{A}[\sigma_n, \tilde{\sigma}_n; t]$ are defined, with respect to infinitesimal changes, in accord with the earlier notation and the permutation symmetry of the basic ECCM amplitudes, as

$$\delta \bar{A} = \frac{\delta A}{\delta t} \delta t + \sum_{n=1}^\infty \frac{1}{n!} \int dx_1 \cdots \int dx_n \left[ \frac{\delta A}{\delta \sigma_n(x_1 \cdots x_n; t)} \delta \sigma_n(x_1 \cdots x_n; t) + \frac{\delta A}{\delta \tilde{\sigma}_n(x_1 \cdots x_n; t)} \tilde{\sigma}_n(x_1 \cdots x_n; t).$$

Finally, it was shown in I how the expectation value of the commutator of two arbitrary operators $A$ and $B$, may be expressed as,

$$<\psi'(t) | [A, B] | \psi(t)> = i(\bar{A}, \bar{B}),$$

in terms of a generalised Poisson bracket, defined as,

$$i(\bar{A}, \bar{B}) = \sum_{n=1}^\infty \frac{1}{n!} \int dx_1 \cdots \int dx_n \left[ \frac{\delta A}{\delta \sigma_n(x_1 \cdots x_n; t)} \frac{\delta B}{\delta \sigma_n(x_1 \cdots x_n; t)} - \frac{\delta A}{\delta \tilde{\sigma}_n(x_1 \cdots x_n; t)} \frac{\delta B}{\delta \tilde{\sigma}_n(x_1 \cdots x_n; t)} \right].$$

In order to simplify the subsequent notation, we shall henceforth often omit the time-label from, and thus leave implicit the time-dependence of, both our basic ECCM amplitudes $\{\sigma_n, \tilde{\sigma}_n\}$ and other dynamical variables which are constructed from them.
3. INCORPORATION OF THE EXTERNAL GAUGE FIELDS

The ECCM formalism described above and more fully in I is now applied to a system of \( N \) identical uncharged bosons, of mass \( m \) each, interacting via pairwise forces and described by a Hamiltonian, \( h_{(0)} \).

\[
h_{(0)} = -\frac{1}{2m} \sum_{j=1}^{N} \sum_{j'}^{N} \frac{v^2}{j} \sum_{i=1}^{j-1} \sum_{i'=1}^{j-1} v(x_i - x_{j'}) .
\]

Although the Hamiltonian conserves particle number, our ECCM states do not, and we therefore work instead with the grand-canonical Hamiltonian, \( H_{(0)} \), defined in terms of the chemical potential \( \mu \) as,

\[
H_{(0)} = h_{(0)} - \mu N .
\]

In order to preserve the underlying \( U(1) \) symmetry of number conservation we still wish to formulate the theory in a gauge-invariant fashion. We do this by introducing the two external gauge fields [9], namely the scalar and vector potentials, \( \phi(x,t) \) and \( \vec{A}(x,t) \) respectively, which are familiar from electrodynamics where they arise ultimately in similar fashion due to the underlying charge conservation.

The scalar and vector fields are coupled in the usual fashion to the system via the density operator \( \rho(r) \) and the current operator \( \vec{j}(r) \) respectively,

\[
\rho(r) = \rho(r,r) \quad \rho(r,r') = a_r^+ a_{r'},
\]

\[
\vec{j}(r) = \lim_{r' \to r} \frac{i}{2m} (\vec{\phi}_{r-r'} \rho(r,r')).
\]

In this way we are led to the grand-canonical Hamiltonian, \( H \), for the system coupled to the external gauge fields,

\[
H = \frac{1}{2m} \sum_{j=1}^{N} [\vec{\phi}(x_j)^2 + \sum_{j=1}^{N} [\phi(x_j) - \mu] \sum_{j=1}^{N} \sum_{i=1}^{j-1} v(x_i - x_{j'}) ] .
\]

It may be equivalently written in second-quantised form as

\[
H = H_{(0)} + \int dr [\phi(r) + (1/2m)\vec{A}^2(r)] \rho(r) - \vec{A}(r) \cdot \vec{j}(r)
\]

\[
H_{(0)} = \frac{1}{2m} \int dr (\vec{\phi}_r a_r^+ )(\vec{\phi}_r a_r) + \frac{1}{2} \int dr dr' v(r-r') a_r^+ a_{r'} a_{r'} a_r - \mu \int dr a_r^+ a_r .
\]

The whole rationale behind the coupling of the system to the gauge fields is that the system (and observer) can be put into arbitrary (relative) motion by suitable choices of the gauge fields. In this way they may be thought of as inducing local or differential Galilei transformations in the system. Particularly simple examples are provided by the two cases of: (i) uniform motion with velocity \( \vec{V}_0 \), which is simply a global Galilei transformation that may be induced by the choice \( \vec{A}(r) = m \vec{V}_0 \cdot \vec{r} \), \( \phi(r) = -i m V_0^2 \), and which is easily seen to lead in the case of translationally
invariant systems to the total wavefunction of the system simply acquiring an extra multiplicative phase factor, \( \exp(it\vec{V}_0\cdot\vec{P}) \), where \( \vec{P} \) is the total momentum operator which is the generator of spatial translations; and (ii) uniform rotation with angular velocity \( \vec{\Omega} \), which may be induced by the choice \( \vec{A}(r) = m\vec{r} \times \vec{r} \), \( \phi(r) = -\frac{i}{2m}|\vec{\Omega}|^2 \), and which is correspondingly easily seen to lead in the case of rotationally-invariant systems to the total wavefunction of the system simply acquiring an extra multiplicative phase factor, \( \exp(it\vec{\Omega}\cdot\vec{L}) \), where \( \vec{L} \) is the total angular momentum operator which is the generator of spatial rotations.

The more general effect of a local \( U(1) \) gauge transformation is for the total wavefunction to pick up an extra overall phase factor,

\[
|\psi(x_1 \cdots x_N)\rangle \rightarrow |\psi'(x_1 \cdots x_N)\rangle = \exp \left[ -i \sum_{j=1}^{N} \Lambda(x_j, t) \right] |\psi(x_1 \cdots x_N)\rangle ,
\]

where \( \Lambda(r, t) \equiv \Lambda(r) \) is some arbitrary gauge parameter or phase field. Just as in standard electrodynamics, it is easy to see from Eq. (22) that the transformed wavefunction \( |\psi'\rangle \) satisfies a similar Schrödinger equation (3) to that for \( |\psi\rangle \),

\[
i(\partial/\partial t)|\psi(t)\rangle = H|\psi(t)\rangle \leftrightarrow i(\partial/\partial t)|\psi'(t)\rangle = H'|\psi'(t)\rangle ,
\]

but with a transformed Hamiltonian, \( H = H[\phi, \vec{A}] + H' = H'[\phi', \vec{A}'] = H[\phi', \vec{A}'] \), of the same form as that in Eq. (20) but with gauge fields transformed as,

\[
\phi' \equiv \phi + \alpha \Lambda/\partial t ; \quad \vec{A}' \equiv \vec{A} - \vec{v}_A .
\]

Under the effect of these local Galilei transformations induced in the system by the coupling to the external gauge fields, the (gauge-invariant) physical forces create local translational motions which vary temporally and spatially throughout the system. It is clear that a proper description of the quantum fluid dynamics of the system must be able to separate exactly, and in a gauge-invariant way, between, for example, the local fluid-dynamical kinetic energy density of translation and the intrinsic kinetic energy density in the local rest frame. In the ECCM approach which we consider here, we have seen that any aspect of the system may be described in terms of the linked-cluster amplitudes \( \sigma_n, \tilde{\sigma}_n \). It is apparent from their definitions in Sec. 2 that the mode of action of the gauge transformation of Eq. (24) may equivalently be expressed in terms of its effect on these amplitudes, namely

\[
\sigma_n(x_1 \cdots x_n) \rightarrow \sigma'_n(x_1 \cdots x_n) = \exp \left[ -i \sum_{j=1}^{n} \Lambda(x_j) \right] \sigma_n(x_1 \cdots x_n) ,
\]

\[
\tilde{\sigma}_n(x_1 \cdots x_n) \rightarrow \tilde{\sigma}'_n(x_1 \cdots x_n) = \exp \left[ +i \sum_{j=1}^{n} \Lambda(x_j) \right] \tilde{\sigma}_n(x_1 \cdots x_n) .
\]

In the remainder of this work we show explicitly how the ECCM provides the detailed quantum fluid-dynamical description of the system that we seek.
In order to examine more closely the equations of motion (15) for the basic ECCM amplitudes, we need to elucidate the one- and two-body density matrices necessary to calculate respectively the expectation values of the one- and two-body operators that comprise the Hamiltonian $H$ of Eq. (23). Using Eq. (21) the one-body density matrix is given as

$$
\rho(r,r') = \langle \psi(t) | \rho(r,r') | \psi(t) \rangle ,
$$

(28)

and by further use of the ECCM parametrisations of the states given in Eqs. (5) and (11) it is not difficult to show,

$$
\rho(r,r') = \langle \phi | \Sigma \Sigma_{r}^{+} a_{r} a_{r}^{+} | \phi \rangle 
= \sum_{n=0}^{\infty} \frac{1}{n !} \int dx \cdots \int dx_{n} \Sigma_{n+1}^{+} (r x_{1} \cdots x_{n} ) \Sigma_{n+1} (r' x_{1} \cdots x_{n}) ,
$$

(29)

where the second equality follows from the first by inserting a resolution of the identity $I$ of the form of Eq. (2) in the place so indicated. Using the fact that each of the ECCM amplitudes $\{ \Sigma_{n}, \Sigma_{n+1} \}$ obeys the cluster property, it is interesting to note from Eq. (29) that the one-body density matrix exhibits the off-diagonal long-range order at large relative separations,

$$
\rho(r,r') \xrightarrow{|r-r'| \to \infty} \Sigma_{1}(r) \Sigma_{1}(r') ,
$$

(30)

which is typical of superfluids [10].

In order to evaluate the expectation value $\bar{V}$ of the potential energy, we also need the two-body density matrix, $D(r,r')$, 

$$
\bar{V} = \bar{V}[\Sigma_{n}, \Sigma_{n+1}] = \frac{1}{2} \int dr \int dr' v(r-r') D(r,r') ; D(r,r') = \langle a_{r}^{+} a_{r}^{+} a_{r} a_{r}^{+} | \psi \rangle .
$$

(31)

The two-body density matrix can be similarly decomposed by an appropriate insertion of a unit operator as in Eq.(29),

$$
\langle a_{r}^{+} a_{r}^{+} a_{r} a_{r}^{+} \rangle = \sum_{n=0}^{\infty} \frac{1}{n !} \int dx \cdots \int dx_{n} \phi_{n+2}^{+} (r r_{1} x_{1} \cdots x_{n}) x_{n+2} (r' r_{1} x_{1} \cdots x_{n}) ,
$$

(32)

in terms of the reduced subsystem amplitudes [2,11] $\phi_{n}$ and $x_{n}$,

$$
\phi_{n+2} (r r_{1} x_{1} \cdots x_{n}) = \langle \phi | \Sigma_{r_{1}}^{+} \Sigma_{r_{1}}^{+} \Sigma_{r}^{+} \Sigma_{r}^{+} a_{r} a_{r_{1}} \cdots a_{r_{1}} x_{1} x_{1} \cdots x_{n} | \phi \rangle ,
$$

(33)

$$
\phi_{n+2} (r r_{1} x_{1} \cdots x_{n}) = \langle \phi | a_{r_{1}} a_{r_{1}} \cdots a_{r_{1}} a_{r_{1}} x_{1} x_{1} \cdots x_{n} | \phi \rangle .
$$

It is possible to find explicit expressions for these reduced subsystem amplitudes in
terms of the basic EECM quantities \( \{ \sigma_n, \tilde{\sigma}_n \} \) by using the functional derivative techniques discussed in I. Such expressions are not however needed for present purposes.

After some simple manipulation of the one-body part of the Hamiltonian in Eq.(23), we thus find that the basic EECM equations of motion (15) may be written as,

\[
\begin{align*}
\imath \frac{d}{dt} \sigma_n(x_1 \cdots x_n) &= \sum_{j=1}^{n} h(x_j) \sigma_n(x_1 \cdots x_{j-1} x_{j+1} \cdots x_n) + \frac{\delta \tilde{V}}{\delta \sigma_n(x_1 \cdots x_n)}; \\
\frac{d}{dt} \tilde{\sigma}_n(x_1 \cdots x_n) &= \sum_{j=1}^{n} \tilde{h}(x_j) \tilde{\sigma}_n(x_1 \cdots x_{j-1} x_{j+1} \cdots x_n) + \frac{\delta \tilde{V}}{\delta \tilde{\sigma}_n(x_1 \cdots x_n)},
\end{align*}
\]

(34)

in terms of the reduced one-body Hamiltonians,

\[
\begin{align*}
h(r) &= -\frac{1}{2m} v^2 + \frac{i}{m} \vec{A}(r) \cdot \vec{v} + \frac{i}{2m} \left[ \vec{v} \cdot \vec{A}(r) \right] + \frac{1}{2m} \vec{A}^2(r) + \phi(r) - \mu; \\
\tilde{h}(r) &= -\frac{1}{2m} v^2 - \frac{i}{m} \vec{A}(r) \cdot \vec{v} - \frac{i}{2m} \left[ \vec{v} \cdot \vec{A}(r) \right] + \frac{1}{2m} \vec{A}^2(r) + \phi(r) - \mu.
\end{align*}
\]

(35)

After the further evaluation of \( \tilde{V} \) and its functional derivatives, Eqs. (34)-(35) constitute a set of exact (nonlinear and non-local) coupled equations for the EECM amplitudes \( \{ \sigma_n, \tilde{\sigma}_n \} \). In practice they will need to be truncated to make further progress. The simplest and most natural hierarchy of approximations is the so-called SUBn scheme in which, at the \( n \)-th level of approximation, all amplitudes \( \{ \sigma_m, \tilde{\sigma}_m \} \) with \( m > n \) are set to zero.

We note that the lowest SUB1 such approximation is just the time-dependent Hartree approximation for the mean fields \( \sigma_1(r) \) and \( \tilde{\sigma}_1(r) \) or, equivalently, the condensate wavefunctions from Eq. (13). After making the further localising approximation of replacing the two-body potential \( v \) by a repulsive delta-function form, this SUB1 approximation becomes identical to the well-known Gross-Pitaevskii approximation [5-7], with the kink-soliton solutions in a one-dimensional geometry and comparable vortex solutions in a cylindrical geometry [7]. We see that the EECM thus provides a formally exact generalisation of the Gross-Pitaevskii treatment. Whereas the latter is essentially applicable only to weakly-interacting condensed Bose fluids, the EECM extends the treatment to include the higher generalised mean fields or generalised order parameters \( \{ \sigma_n, \tilde{\sigma}_n \} \).

5. THE ONE-BODY DENSITY MATRIX

The fundamental equations of motion (34) - (35) now suffice to determine the time evolution of all expectation values in the formalism. In particular, the one-body density matrix of Eq. (29) is now easily seen to obey the equation of motion,
\[ \frac{\partial}{\partial t} \rho(r,r') = \frac{i}{2m} (v_r^2 - v_r'^2) \rho(r,r') + \frac{1}{m} [\hat{A}(r) \cdot \hat{v}_r + \hat{A}(r') \cdot \hat{v}_r', \rho(r,r')]
+ \frac{i}{2} (v_r - v_r') + \frac{1}{2m} \{ \hat{A}^2(r) - \hat{A}^2(r') \} \rho(r,r')
+ \frac{1}{2m} \{ \hat{\Phi}(r) - \hat{\Phi}(r') \} + \frac{1}{2m} \{ \hat{\Phi}(r) + \hat{\Phi}(r') \} \rho(r,r') - \{ \hat{v}, \rho \} , \tag{36} \]

where the last term has been manipulated into the form of the generalised Poisson bracket defined previously in Eq. (18). Equation (36) may also be derived rather more directly by taking the expectation value (in the state \( |\psi(t)\rangle \)) of the Heisenberg equation of motion for the operator \( \rho(r,r') \) of Eq. (21). This generalised Poisson bracket may either be evaluated from a direct knowledge of \( \hat{\psi} = \hat{\psi}^\dagger \hat{\sigma}^\dagger_n \) or, more simply, from Eq. (17). We find, using Eq. (23),
\[
\{ \hat{v}, \rho \} = -i \frac{1}{\hbar} \int dx \int dy \, v(x-y) \langle [\hat{a}^+_x \hat{a}^+_y \hat{a}^+_z \hat{a}^+_r, \hat{a}_{x,x}^+, \hat{a}_{x,y}^+, \hat{a}_{y,z}^+, \hat{a}_{z,r}^+] \rangle
+ i \int dx \, [v(x-r) - v(x-r')] \langle \hat{a}^+_x \hat{a}^+_y \hat{a}^+_z \hat{a}^+_r \rangle , \tag{37} \]

where, in the last equality, we have used the basic commutation relations of Eq. (1).

The gauge transformation properties of the one-body density matrix are readily derived from Eqs. (27) and (29),
\[
\bar{\rho}(r,r') \rightarrow \rho(r,r') = e^{i[\Lambda(r) - \Lambda(r')] \rho(r,r')} . \tag{38} \]

We also see from Eqs. (31)-(33) that the functional \( \hat{V} = \hat{V}^\dagger \hat{C}_{\sigma_n}^\dagger_n \) is gauge-invariant, as is the diagonal term \( \bar{\rho}(r,r) \) in the one-body density matrix. Since we are interested in deriving the balance equations of fluid dynamics for the most important local physical field densities, it is convenient at this point to transform to relative and centre-of-mass coordinates defined respectively as,
\[
\bar{\xi} = \bar{r} - \bar{r}' ; \quad \bar{R} = \frac{1}{3} (\bar{r} + \bar{r}') , \tag{39} \]

and to expand the equation of motion (36) for \( \bar{\rho}(r,r') \equiv \rho(R|\xi) \) in powers of \( \xi \) up to a given order. The Taylor expansion,
\[
\bar{\rho}(R|\xi) = \bar{\rho}(R) + \bar{\rho}^a(R) \xi^a + \frac{1}{2} \gamma^{ab}(R) \xi^a \xi^b + \frac{1}{6} \delta^{abc}(R) \xi^a \xi^b \xi^c + O(\xi^4) , \tag{40} \]

is first made about \( \xi = 0 \), where \( \bar{\rho}(R) \equiv \bar{\rho}(R,R) \) is the number density at space-point \( R \). The roman superscripts \( a,b,c \cdots \) in Eq. (40) are Cartesian three-vector indices; and the tensor coefficients \( \gamma^{ab}, \delta^{abc} \cdots \) may be assumed, without loss of generality, to be completely symmetric in their indices. The Einstein summation convention for repeated indices is also adopted in Eq. (40) and henceforth.

Now, the tensor coefficients \( \gamma^{ab}, \delta^{abc} \cdots \) in the Taylor expansion (40) are
not gauge-invariant. For example, using Eq. (21), we find that the first-order coefficient $\beta^a(R)$ is simply proportional to the local canonical current density $j^a(R)$,

$$\beta^a(R) = -imj^a(R) \quad ,$$

which is itself certainly not gauge-invariant. Our first task is therefore to find suitable combinations of the various tensors at our disposal, one for each order, which are gauge-invariant. By making use of Eqs. (38)-(41), together with the gauge transformation of Eq. (26), one can readily show that the following first, second and third order combinations are all gauge-invariant,

$${\hat J}^a(R) = j^a = j^a - (\vec \rho/m)A^a \quad ,$$

$${\hat r}^{ab}(R) = t^{ab} = -(1/m^2)\gamma^{ab} - (1/\vec \rho)j^a j^b \quad ,$$

$${\Delta}^{abc}(R) = \Delta^{abc} = (1/2m^2)\delta^{abc} + (1/2m)(j^a \gamma^{bc} + j^b \gamma^{ca} + j^c \gamma^{ab})$$

$$+ (m/\vec \rho^2)j^a j^b j^c + (\vec \rho/24m^2)(\delta^{a\beta}c^\alpha A^a + \delta^{b\beta}c^\alpha A^b + \delta^{c\beta}c^\alpha A^c) ;$$

where we have used the notation $\delta^a V^b = a V^b / aR^a$ for an arbitrary vector field $V^a = V^a(R)$. (We recall at this point that all of the tensor coefficients in Eqs. (42a-c) depend both on space and time, although we continue to suppress explicit dependence on the time coordinate in accordance with our earlier notation). Equation (42a), for example, expresses the (invariant) true current density $\hat J$ in terms of the (non-invariant) canonical current density $\check J$.

Equation (36) may now be readily expanded in powers of $\xi^a$, using Eq. (40). The only term that requires some extra attention is the last term in Eq. (36) involving the generalised Poisson bracket. Using the representation in Eq. (37), we may readily expand this term in gauge-invariant quantities as follows,

$$[\tilde V, \rho] = i\xi^a - \rho^a_{\text{int}} + \xi^b \rho^c (A^b_{\text{int}} + A^b_{\text{int}} + mw^{ab}) + O(\xi^3) \quad ,$$

where the gauge-invariant tensors $\tilde F_{\text{int}}$ and $W_{ab}$ are defined as,

$$\tilde F_{\text{int}}(R) = - \rho^a \left[ \frac{1}{\rho(R)} \int dx \left[ \tilde V_R (R-x) \right] \right] a^a_{\hat R} \quad ,$$

$$W_{ab}(R) = - \rho^a \left[ \frac{1}{\rho(R)} \int dx \left[ \frac{\partial V(R-x)}{\partial R^a} \right] a^a_{\hat R} \right] + \frac{\partial V(R-x)}{\partial R^b} \left[ \frac{1}{\rho(R)} \int dx \left[ \tilde j^a_{\hat R}(R) a^a_{\hat R} \right] \right] .$$

The operator $\hat J(R)$ in Eq. (44b) is the gauge-invariant total current density operator, defined by analogy with Eqs. (21) and (42a) as,

$$\hat J(R) = - \frac{1}{2m} \left[ \hat a^a_{\hat R} (\tilde V^a_{\hat R a_{\hat R}}) - (\tilde V^a_{\hat R a_{\hat R}}) \right] - \frac{1}{m} \left[ \hat A(R) a^a_{\hat R} \right] .$$

The terms $\tilde F_{\text{int}}$ and $W_{ab}$ may also usefully be given a physical interpretation.
Thus, $F_{\text{int}}(R)$ represents the average force per particle due to the internal inter-particle forces. Also, since the gauge-invariant velocity field $\tilde{\mathbf{u}}(R)$ may be defined as,

$$\tilde{\mathbf{u}}(R) = \tilde{\mathbf{J}}(R)/\tilde{\rho}(R) \quad ,$$

it is clear that $W^{ab}(R)$ represents the symmetrised velocity-force correlation function. Its trace, $W^{aa}(R)$, thus represents twice the average rate of increase of the internal kinetic energy due to the interparticle forces.

Insertion of the expansions (40) and (43) into the equation of motion (36) for the (non-local) one-body density matrix, then finally yields the following local evolution equations, by comparing respectively the coefficients of the zeroth, first and second-order terms in $\xi^a$,

$$\frac{\partial \rho}{\partial t} = - a^a j^a + \frac{1}{m} \tilde{\rho} (a^a A^a) + \frac{1}{m} A^a (a^a \tilde{\rho}) \quad ,$$

$$\frac{\partial j^a}{\partial t} = \frac{1}{m^2} \tilde{\rho} \delta^{ab} j^b + \frac{1}{m} \left[ (a^a A^a) j^b + A^b (a^b j^a) \right] - \frac{\tilde{\rho}}{m} \left[ a^a \phi + \frac{1}{m} (a^a A^a) A^b \right]$$

$$+ \frac{1}{m} j^a (a^b A^b) + \frac{\tilde{\rho}}{m} F_{\text{int}}^a ,$$

$$\frac{\partial \gamma_{ab}}{\partial t} = - i \frac{1}{m} a^c \delta^{abc} + \frac{1}{m} \left[ \gamma^{ac} (a^b A^c) + \gamma^{bc} (a^a A^c) + \gamma^{ab} (a^c A^c) + (a^c \gamma_{ab}) A^c \right]$$

$$+ j^a \left[ m a^b \phi + (a^b A^c) A^c \right] + j^b \left[ m a^a \phi + (a^a A^c) A^c \right] + \frac{1}{4m} \tilde{\rho} (a^a A^b) A^c$$

$$- \tilde{\rho} \left[ a^a A^b_{\text{int}} + A^b_{\text{int}} a^a + m W^{ab} \right] .$$

We show in Sec. 6 that Eqs. (47a-c) represent the local balance equations of fluid dynamics for the most important local quantities of physical interest, by rewriting them in terms of gauge-invariant quantities and analysing them further.

6. QUANTUM FLUID DYNAMICS: THE LOCAL BALANCE EQUATIONS

The most basic physical quantities that are used to describe a system in terms of fluid dynamics are the number, momentum and energy densities and the corresponding fluxes or currents that define their flows. We now show how Eqs. (47a-c) may be related to the continuity or balance equations for these respective parameters of the fluid-dynamical description.

6.1. Number Density Balance

By making use of the definition (42a) of the total current density $\tilde{\mathbf{J}}$, it is a trivial matter to rewrite Eq. (47a) wholly in terms of gauge-invariant quantities as,
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \quad (48a)
\]

This is just the familiar form of the **current continuity equation**, which is simply the equation of **local number density balance**. By integrating over the entire volume of the system, we simply get the **global conservation law for particle number**, \[ \dot{N} = \int \rho(R) \, dR, \] the correct imposition of which has been our primary motivation in introducing the external gauge fields.

It is of particular importance to realise that not only are the conservation laws of Eqs. (48a,b) valid in the exact (untruncated) ECCM formalism, as they must be, but they also remain true in most of the usual approximation schemes which are necessary for practical implementation of the method. For example, the most natural truncation scheme is the SUBn scheme already discussed in Sec. 4, in which the amplitudes \{\sigma_m, \tilde{\sigma}_m\} with \( m > n \) are set to zero. It should be clear by our method of derivation, which was essentially based on Eqs. (29) and (36), that the conservation laws of Eqs. (48a,b) also hold in every SUBn approximation.

6.2. Momentum Density Balance

Equation (47b) may similarly be re-expressed in terms of the gauge-invariant tensors \( J^a \) and \( \tau^{ab} \) of Eqs. (42a,b). By making use also of the lower-order continuity equation (48a), we find that Eq. (47b) reduces to the form,

\[
\frac{\partial J^a}{\partial t} + \frac{\partial}{\partial t} \left( \mathbf{j} \cdot \mathbf{j} \right) = \left( \frac{\rho}{m} \right) F^a \quad , \tag{49}
\]

where \( F^a = F^a(R) \) is composed of three pieces,

\[
F^a(R) = F^a_{\text{int}}(R) + F^a_{\text{ext}}(R) + F^a_{\text{kin}}(R) \quad , \tag{50}
\]

and has the interpretation of the total average force per particle in the many-body condensed system. The three separate components are themselves defined and interpreted as follows. Firstly, \( F^a_{\text{int}} \) has already been defined in Eq. (44a), and has been explained to represent the average force per particle due to the internal interaction forces between the particles, which have been parametrised here by the two-body potential, \( \mathbf{v} = \mathbf{v}(x-y) \). The second term \( F^a_{\text{ext}} \) is defined as,

\[
F^a_{\text{ext}} = \mathbf{E} + \mathbf{v} \times \mathbf{B} \quad ; \quad \mathbf{B} = \mathbf{\nabla} \times \mathbf{A} \quad , \quad \mathbf{E} = -\mathbf{\nabla} \phi - \mathbf{A}/\mathbf{\partial t} \quad , \tag{51}
\]

in terms of the velocity field \( \mathbf{v} = \vec{U}(R) \) of Eq. (46). It is seen to have the familiar Lorentz form in terms of the gauge-invariant "magnetic" and "electric" fields, \( \mathbf{B} \) and \( \mathbf{E} \) respectively, where the nomenclature is by analogy with electrodynamics. The term \( F^a_{\text{ext}} \) thus represents the external force per particle due to the coupling of the system to the externally applied gauge fields. Finally, the term \( F^a_{\text{kin}} \) is defined as,
and is thus proportional to the divergence of the gauge-invariant kinetic stress tensor, \( \mathbf{\tau}^{ab} = \mathbf{\tau}^{ab}(R) \), which is itself readily seen from Eqs. (40) and (42) to be proportional to the second-order cumulant in the expansion for the one-body density matrix. Thus, \( \mathbf{\tau}^{\text{kin}} \) is the kinetic component of the average force per particle in the system. Equation (49) is clearly the equation of motion for the current density \( \mathbf{J}(R) \) or, equivalently the local balance equation for the momentum density, \( \mathbf{\pi}(R) = m\mathbf{J}(R) \).

By integrating over the entire volume of the system, we get the corresponding global form of the Newtonian equation for the total momentum, \( \mathbf{\Pi} = \int \mathbf{\pi}(R) dR \),

\[
\frac{d\mathbf{\Pi}}{dt} = \mathbf{\mathcal{F}}_{\text{ext}} ; \quad \mathbf{\mathcal{F}}_{\text{ext}} \equiv \int \rho(R) \mathbf{\mathcal{F}}_{\text{ext}}(R) dR ,
\]

where \( \mathbf{\mathcal{F}}_{\text{ext}} \) is the total (external) force acting on the system. The total momentum is thus conserved in the absence of the external gauge fields, as expected. We may also express the local equation (49) in Newtonian form per particle. This is best done by rewriting Eq. (49) in the form,

\[
m\left(1/\rho\right) \frac{D\mathbf{U}}{Dt} + \mathbf{\hat{u}}(\mathbf{\nabla} \cdot \mathbf{\hat{u}}) = \mathbf{\mathcal{F}} ,
\]

where we have made use of Eq. (46) and have introduced the usual definition of the convective time-derivative \( D/Dt \) as,

\[
D/Dt \equiv \frac{\partial}{\partial t} + \mathbf{\hat{u}} \cdot \nabla .
\]

A final combination of Eq.(54) with the lower-order continuity equation (48a) then yields the simple Newtonian form of the local momentum balance equation,

\[
m\mathbf{\hat{u}}(R)/Dt = \mathbf{\mathcal{F}}(R) .
\]

As was the case for the current continuity equation (48), so the momentum density balance equation (49) or (56) also remains true at an arbitrary SUBn level of truncation. Furthermore, and perhaps even more importantly, the momentum balance equations -- which, by contrast with the lower-order continuity equation (48), now also depend explicitly on the potential term \( \hat{V} \) -- continue to remain valid when even more severe sub-approximations are made to the basic SUBn scheme. Such sub-approximations involve the further neglect of some specific class of terms (or diagrams) that rightfully contribute to the expectation value \( \langle \hat{V} \rangle \) of the potential, at the SUBn level. In such cases, since each term contributing to \( \hat{V} \) transforms under the gauge transformations in exactly the same way, the symmetry is thus obeyed by each term separately, and the conservation laws therefore continue to hold even after the neglect of some arbitrary terms which contribute to \( \hat{V} \). A typical example is provided by systems with strongly
repulsive short-range interparticle forces of the hard-core variety, for which the
SUBn scheme is not properly defined in either the NCCM or ECCM formalisms. Further
details of the necessary sub-approximation scheme in this case are provided in Ref.
[11] in the context of the NCCM. A similar hard-core truncation scheme is possible
within the present ECCM formalism.

We also note that the well-known \( f \)-sum rule may be derived from the present
momentum balance equations (49)-(52) by specialising to the linear response regime
close to the static equilibrium point, after perturbing the system with an infinitesim-
ial impulsive scalar potential, \( \delta \phi(r,t) = \psi(r)\delta(t) \). We may readily show that the
Corresponding infinitesimal change in number density, \( \delta \rho(r,t) \), is continuous, where-
as its time-derivative is discontinuous at \( t = 0 \) by an amount that can be derived
from Eqs. (49)-(52). By relating the change \( \delta \rho \) to the change \( \delta \phi \) that caused it,
via the well-known linear response formalism which thereby introduces the usual
density-density response function, we can show that this linear response of the system
obeys the \( f \)-sum rule. Again, it should be clear that the \( f \)-sum rule will continue to
be obeyed in the various approximation schemes discussed above. We may thus regard
our momentum balance equation (49) as a generalisation of the \( f \)-sum rule away from
the linear response regime.

6.3. Energy Density Balance

The second-order (in \( \xi^a \)) equation (47c) may finally also be expressed in terms
of the gauge-invariant tensors of Eqs. (42a-c). It turns out to be particularly use-
ful to express it as a continuity equation not for the stress tensor \( \tau^{ab} \), but for a
related gauge-invariant kinetic stress tensor \( K^{ab} \) defined as,

\[
K^{ab} = K^{ab}(R) = \frac{1}{4\pi} \left[ \tau^{ab}(R) + \rho(R)u^a(R)u^b(R) \right], \tag{57}
\]

which is related to the kinetic energy density, as we shall see below. A straight-
forward but tedious calculation yields the local balance equation for the kinetic
stress density as,

\[
\frac{1}{\rho} \frac{dK^{ab}}{dt} + \nabla [\delta_{abc} + u^a \delta_{bc} + u^b \delta_{ac} + u^c \delta_{ab} - \mu u^a u^b u^c + (\rho/24\pi^2) (\epsilon_{bcd} a^d + \epsilon_{acd} b^d)] = \frac{1}{\rho} (w^{ab} + u^a w^b + u^b w^a) + (1/m) (\epsilon_{bcd} a^{bc} + \epsilon_{acd} b^{bc})b^d, \tag{58}
\]

where \( \vec{W} \) and \( \vec{B} \) are the external force fields defined previously in Eq.(51), and
\( \epsilon^{abc} \) is the usual Levi-Civita symbol or antisymmetric unit tensor of third rank.
The derivation of Eq.(58) also employs the lower-order balance equations (48a) and
(49). Rather than trying to put a physical interpretation on Eq.(58) in its general-
ity, we shall consider only its trace. We show below how the equation for the trace
\( K^{aa} \) relates to the local balance equation for the energy density.

In order to derive an expression for the energy density we begin with the
Hamiltonian of Eq. (23), which we write as the sum, $H = H + V$, of a one-body part
1 and a two-body part $V$ due to the interparticle interactions. The one-body
piece gives a contribution $H$ to the energy expectation value $\tilde{H}$ which we write as
the volume integral of a local energy density $\varepsilon_1(R)$ as,

$$\tilde{H}_1 = \int dR \varepsilon_1(R) \ . \quad (59)$$

By making use of Eqs. (39), (40), (42a, b) and (46), it is straightforward to derive the
explicit expression,

$$\varepsilon_1(R) = \frac{1}{1/8\pi} \rho_1^2(R) + \frac{1}{2} m^2 \varepsilon_1(R) \rho_1^2(R) + \frac{1}{2} m^2 \rho_1^2(R) + \rho_1^2(R) \left[ \phi(R) - \mu \right] \ . \quad (60)$$

The first two terms in Eq. (60) represent the kinetic energy density in the local rest
frame. The first term is strictly speaking redundant since, as a perfect divergence,
its contribution to $\tilde{H}_1$ vanishes. Nevertheless its inclusion guarantees the posi-
tivity of the kinetic energy density. The third term in Eq. (60) represents the fluid-
dynamical kinetic energy density of the translational motion due to the flow; and the
second and third terms together are just the trace of the kinetic stress tensor, $\rho_1^2$,
introduced in Eq. (57). We also note that each of the individual terms in Eq. (60) is
gauge-invariant except only the potential energy term, $\rho_1^2$, due to the coupling to
the external scalar potential.

It turns out that although it is not possible to express the two-body energy
expectation value $\tilde{V}$ similarly as the volume integral of a local energy density
$\varepsilon_2(R)$, it is possible so to express its time-derivative, $\partial V / \partial t$: Making use of the
Heisenberg equation of motion for an arbitrary operator, we first calculate $\partial D(r, r')/\partial t$
from Eq. (31) as,

$$\partial D(r, r')/\partial t = - i \langle a_{r}^+ a_{r'}, a_{r}, a_{r'} H \rangle$$

$$= - \frac{v}{2} \cdot \langle a_{r}^+, J(r) a_{r}, - \frac{v}{2} \cdot \langle a_{r}^+, J(r') a_{r} \rangle \ . \quad (61)$$

In this way we obtain the result,

$$\partial V / \partial t = \frac{1}{2} \int dr \int dr' v(r-r') \partial D(r, r') / \partial t \equiv (\partial / \partial t) \int dr \varepsilon_2(r) \ , \quad (62)$$

with the following expression for $\partial \varepsilon_2(r) / \partial t$,

$$\partial \varepsilon_2(r) / \partial t = - \int dr' v(r-r') \langle a_{r}^+, J(r) a_{r}, >$$

$$= - \frac{1}{2} \rho_1^2(r) \omega_1^2(r) - \frac{v}{2} \cdot \rho_1^2(r) \ , \quad (63)$$

where in the second equality in Eq. (63) we have integrated by parts, made use of the
definition (44b), and have introduced the new definition,

$$\dot{P}(r) \equiv \int dr' v(r-r') \langle a_{r}^+, J(r) a_{r} \rangle \ . \quad (64)$$
The new local vector field $\vec{P}(r)$ clearly has the physical interpretation of an energy flux due to the internal forces. It is defined in such a way that the interaction energy of a given pair of particles is associated wholly with just one of them. Equations (59) and (62) may now be combined to give,

$$\frac{a\vec{\xi}}{at} = \int dr \left[ a_{1}(r)\frac{at}{at} + a_{2}(r)\frac{at}{at} \right] = (\frac{a}{at}) \int dr \xi(r) \quad . \tag{65}$$

The total energy density $\xi(r)$ from Eqs. (60) and (63) is readily shown to obey a continuity equation which follows from taking the trace of our earlier balance equation (58) for the kinetic stress density of Eq. (57). We thus easily find the local balance equation for the energy density in the form,

$$\frac{a\xi}{at} + \vec{\nabla} \cdot \vec{J}_{\xi} = S_{\xi} \quad , \tag{66}$$

where the energy flux vector $\vec{J}_{\xi}$ and energy source density $S_{\xi}$ are given respectively as,

$$\vec{J}_{\xi} = \vec{J}_{\xi}(r) = (1/8\pi)v^2J + [(m/2\rho)trT + \frac{1}{2}m\vec{u}^2 + \vec{t} - \mu] \vec{J}$$

$$+ (\Delta^{ab} + \mu \cdot \Delta^{ab}) r^{b} - (\rho/12m^{2})\vec{V} \times \vec{B}^{\prime} \tag{67a}$$

$$S_{\xi} = S_{\xi}(r) = \frac{\rho}{at} \vec{z} \cdot \vec{z} \vec{A}/at$$

$$= \frac{\rho}{at} \Phi \vec{z} \cdot \vec{z} - \vec{J} \cdot \vec{z} \vec{A}/at \quad . \tag{67b}$$

In Eq. (67a), $r^{b}$ is a unit vector in the $b$-direction, and in the second equality in Eq. (67b) we have used the convective derivative of Eq. (55). We note that the presence of the scalar potential in each of the expressions of Eqs. (60), (67a) and (67b) ensures that none of the quantities $\xi(r)$, $\vec{J}_{\xi}(r)$ and $S_{\xi}(r)$ is gauge-invariant. It is however a trivial matter to rewrite Eq. (66) as a gauge-invariant balance equation of the form,

$$(\vec{a}/at)(\xi - \Phi) + \vec{V} \cdot (\vec{J}_{\xi} - \Phi \vec{J}) = \vec{J} \cdot \vec{E} \quad , \tag{68}$$

by combining it with the lower-order balance equation (48a). In this form the subtracted energy density $(\xi - \Phi)$ and subtracted energy flux $(\vec{J}_{\xi} - \Phi \vec{J})$ are both gauge-invariant, as is the "Ohmic" source term $\vec{J} \cdot \vec{E}$.

An integration over the entire volume of the system of the local balance equation (66) again gives the corresponding global form of the energy balance. By making use of the general equation (16) and the basic ECCM equations of motion (15) the global form is readily derived directly as,

$$\frac{d\vec{H}}{dt} = \frac{a\vec{H}}{at}$$

$$= \int dr S_{\xi}(r) \quad , \tag{69}$$

where the second equality follows from the first by making use of the explicit form
of Eq. (23) for the Hamiltonian, and the definitions of Eqs. (42a) and (67b). Equation (69) is precisely the same as obtained by a volume integration of the local form in Eq. (66).

Finally, we also point out that the balance equations (58) and (66) are again valid not only in the exact untruncated formalism, but also in the same SUBn approximation scheme and the various sub-approximations to it which have been discussed previously in connection with the lower-order balance equations.

7. CONCLUDING REMARKS

We have shown how in principle the ECCM exactly generalises the Gross-Pitaevskii treatment of weakly-interacting condensed Bose systems, to which it degenerates at the lowest (SUB1) natural level of truncation. The ECCM characterises the system completely in terms of a set of amplitudes \( \{ \sigma_n, \tilde{\sigma}_n \} \), the lowest of which, \( \sigma_1(x) \) and \( \tilde{\sigma}_1(x) \), are simply the condensate wavefunctions \( \langle \alpha_1 \rangle \) and \( \langle \alpha_1^\dagger \rangle \) respectively, whose approximate equations of motion are given by Gross and Pitaevskii.

We have also shown how the ECCM gives a complete description of the fluid dynamics of an arbitrary condensed Bose fluid at zero temperature. The fact that the basic ECCM amplitudes all strictly obey the cluster property implies that the formalism is, in principle, perfectly capable of describing possible states of topological excitation or deformation, such as the vortex lines observed in liquid \(^4\)He. Such excitations are created by specific topological boundary conditions on the amplitudes \( \{ \sigma_n, \tilde{\sigma}_n \} \) which prevent their decay by any quasi-local processes. It is apparent that such boundary conditions may only be properly incorporated in a formalism like the ECCM, in which the cluster property is exactly obeyed by all relevant parameters.

We have also seen in I how the ECCM maps an arbitrary quantum many-body problem defined in terms of a Schrödinger dynamics, into a non-local classical field theory for the basic n-body linked-cluster amplitudes \( \{ \sigma_n, \tilde{\sigma}_n \} \), \( n = 1, 2, \ldots \). These amplitudes play the role of generalised classical mean fields or generalised quasi-local order parameters. Their equations of motion have the usual classical canonical form. We saw further how the ECCM phase space is endowed with a set of generalised Poisson brackets which are mapped from the original quantum-mechanical commutators. The existence of the Poisson brackets in turn endows the complex differentiable ECCM phase space with a symplectic structure. The geometrical or topological properties of this space are clearly immensely more complicated than in the usual classical mechanics of continuous media. The full complexity of the original quantum-mechanical Hilbert space is, loosely speaking, mirrored by the neighbourhood of an arbitrary point in the ECCM phase space.

In the present paper we have focussed attention on the properties of a single trajectory in the ECCM phase space, where a representative point in the space is clearly characterised by the complete set of parameters \( \{ \sigma_n, \tilde{\sigma}_n \} \). We have demonstrated that these trajectories comply fully with the gauge symmetries and with the
local conservation laws or balance equations of fluid dynamics for such physical
local densities as particle number, momentum and energy. Further, we have shown how
the balance equations hold true even in the physically relevant subspaces which cor-
respond to the approximation schemes necessary to implement the method in practice.

It will be of considerable future interest to extend the present treatment of
single trajectories in phase space to incorporate the properties of the totality of
allowable trajectories -- namely, the entire phase portrait. For example, the modern
view of the statistical mechanics of a many-body system \([12,13]\) is that, as an alter-
native to the ensemble approach of Gibbs which is based on the statistical density
operator, a proper qualitative description of the system may be based on the phase
portrait. The present fluid dynamics treatment may thus be viewed as an initial step
in a complete investigation of the geometrical or topological properties of the ECCM
phase space, which are needed for this phase portrait approach to statistical mech-
nics. It is clearly vital for a proper description of such large-scale or global
properties that all of the parameters which characterise the system, behave properly
at large interparticle separations. We have seen in I how the ECCM is ideally posi-
tioned for such a task, since all of the basic amplitudes obey the cluster property.
By contrast, neither the normal coupled cluster approach nor the configuration-
interaction method shares this feature.

We have seen both in I and in the present paper that although the ECCM has its
foundations in perturbation theory -- like its predecessors, the normal coupled
cluster method and the configuration-interaction method -- its applicability now far
transcends these origins. We believe that the new capability of expressing all physi-
cal observables of the many-body quantum mechanics in terms of the geometrical prop-
ties of the equivalent ECCM phase space, opens up a whole new array of possible
approaches to many-body systems. For example, one might hope to be able to tailor
new approximation schemes to this geometrical approach, rather than having to rely on
such (admittedly powerful) approximations as the SUBn scheme, which are still firmly
rooted in perturbation theory. Further, the method holds the promise of being able
to describe more global properties of the physical system than the inherently local
descriptions which most fundamental formalisms have provided to date. Such global
descriptions are not restricted to the large-scale geometrical properties of the
system in ordinary coordinate space, but might also include such physically important
quantities as the potential energy surfaces. The success of the present application
to quantum fluid dynamics has reinforced our belief that, for all these reasons, the
ECCM has a promising future for applications to systems of interest in quantum
chemistry.

ACKNOWLEDGEMENT

One of us (RFB) wishes to acknowledge the support of a research grant from the
Science and Engineering Research Council of Great Britain.
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