Fig. 4 shows the simulation results obtained when a 50 Hz sinusoidal signal is used as a reference for two values of k_r . The results show that, as expected, a lower value of k_r gives a slower closed-loop time response. Furthermore, in both cases the output tracks the reference almost perfectly in steady state and there is no delay in the beginning of the time response. This last characteristic is different in the conventional repetitive control systems where the high relative degree of the controller causes a significant delay in the response.

V. CONCLUSION

This work has proposed an internal model for repetitive control design. This internal model is a Discrete-Time Passive, or equivalently, Discrete-Time Positive Real. According to this, the proposed internal model can be connected in feedback form with a generic DTP system with the closed-loop stability being ensured.

Taking advantage of the passivity structure, a design procedure for FFDTPR systems has been proposed and analyzed. The design procedure allows to set the desired gain margin through the value of parameter k_r and filter $H(z)$. Furthermore, the proposed internal model allows to shape the open-loop frequency response in a more accurate way because it adds an additional degree of freedom to its structure.

Additionally, this passivity structure of the internal model establishes a simple way to deal with multiperiodic references and disturbances.

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Entropy Optimization Filtering for Fault Isolation of Nonlinear Non-Gaussian Stochastic Systems

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Abstract—In this paper, the fault isolation (FI) problem is investigated for nonlinear non-Gaussian systems with multiple faults (or abrupt changes of system parameters) in the presence of noises. By constructing a filter to estimate the states, the FI problem can be reduced to an entropy optimization problem subjected to the non-Gaussian estimation error systems. The design objective for the FI purpose is that the entropy of the estimation error is maximized in the presence of diagnosed fault and is minimized in the presence of the nuisance faults or noises. It is shown that the error dynamics is represented by a nonlinear non-Gaussian stochastic system, for which new relationships are applied to formulate the probability density functions (PDFs) of the stochastic error in terms of the PDFs of the noises and the faults. The Renyi's entropy has been used to simplify the computations in the filtering for the recursive design algorithms. It is noted that the output can be supposed to be immeasurable (but with known stochastic distributions), which is different from the existing results where the output is always measurable for feedback. Finally, simulations are given to demonstrate the effectiveness of the proposed data-driven FI filtering algorithms.

Index Terms—Entropy optimization, fault isolation, non-Gaussian systems, non-linear filtering, optimal control.

I. INTRODUCTION

Fault detection and isolation (FDI) for stochastic systems has drawn a considerable attention in the past decades, where many effective methodologies have been developed as seen from the survey works

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[2], [6], [10]. Two kinds of approaches have been developed for the related FDI problems of stochastic systems. One was originated from the statistic theory where the likelihood ratio and Bayesian methods are applied to estimate the fault (or abrupt change of parameters), by combining with some numeral computations such as Monto Carlo method or the particle filtering [1], [2], [10]. Another group of methodologies are the filter based method, where the minimax techniques are used such that the estimation errors satisfy various performance indexes [3].

In many previous filtering results on FDI problems, the linear systems with Gaussian variables are studied, for which the Kalman filters or the extended Kalman filters (EKFs) can be applied [3]. For nonlinear (stochastic) systems, some transformations or approximations are required such that the error equation satisfies some linear structures [4], [11].

However, even for the systems with Gaussian noises, nonlinearity can lead to a non-Gaussian output. For a non-Gaussian variable, especially for that with a non-symmetric PDF, the mean and variance are insufficient to characterize its stochastic property. Hence, FDI for non-Gaussian and nonlinear systems is a challenging subject because EKFs cannot be fully applied as it only gives a minimum variance estimation. In this regard, another general measure of randomness, namely the entropy, should be considered. Entropy has been widely used in information, thermodynamics and control theory [5], [12], [13]. In [9], entropy was used as a measure for the average information contained in the PDF of the estimation errors using the filters. Generally a PDF is a positive nonlinear function with an integral constraint, the main obstacle is to find the relationships between the PDFs of the input and the concerned errors, with which the entropy of the errors can be optimized [9]. In [15], B-spline expansions were used for the PDFs of the system output and the problem was transferred into the fault detection and diagnosis (FDD) problem for the weights dynamics governed by linear equations. The results in [7] addressed a solution for the FDD problems by using the measured output PDFs and provided a robust FDD scheme for the nonlinear weighting models. However, the results in [7], [15] hold only when the output PDFs can be approximated by B-spline expansions and the weighting-dynamics can be modeled between the control input and the weights on-line. Also, the FI problem was not studied there. Recently, some primary results have been reported in [9] for the minimum entropy filtering and applied to the fault detection in [8], where the neuro-fuzzy approximations for the PDFs are not required. It is noted that only fault detection was considered there and the design procedures involved heavy computations. Moreover, the outputs were supposed to be measured as in the classical filter based approaches [14], [16].

In this paper, the main contribution is to find a novel filtering approach to isolate the target fault for the nonlinear non-Gaussian systems. Two cases will be considered. The first one is that the output is measured as studied in the classical methodologies. The second case is that the stochastic distributions of the output is supposed to be immeasurable. By constructing a filter to generalize a residual, we present an entropy optimization principle to design the filter such that the target fault can be isolated from the nuisance faults in the presence of disturbances. In order to simplify the computations, some auxiliary mappings are constructed to formulate the error PDFs in terms of the input PDFs and the Renyi's entropy is adopted in design algorithms, which are different from the previous results in [8], [9].

The remainder of the paper is organized as follows: In Section II, we describe the nonlinear difference models and the structure of the filter, then the filtering problem can be formulated for the FI problem. In Section III, the entropy optimization principle is established for the FI filter errors and the recursive design algorithms are given. Simulations

are provided in Section IV to demonstrate the effectiveness of the main results whilst a short conclusion is given in Section V.

II. PROBLEM FORMULATION

A. Plant Models and Fault Isolation Problems

Consider nonlinear stochastic discrete systems described by

$$\begin{cases} x_{k+1} = f(x_k, \delta_k, \eta_k, \omega_k) \\ y_k = h(x_k, \nu_k) \end{cases} \quad (1)$$

where x_k is the state sequence, y_k is the output sequence, ω_k and ν_k are the stochastic disturbances influencing on the state equation and the output equation respectively. δ_k , η_k are two faults where δ_k is the target fault to be isolated and η_k is the nuisance fault. It is noted that δ_k and η_k can also represent the abrupt changes of the model parameter. The model is actually a generalization of those which only have the additive fault inputs or the unexpected changes of model parameters. It is supposed that δ_k , η_k , ω_k and ν_k are bounded independent random variables rather than the Gaussian ones as concerned in the existing FI approaches based on the Kalman filter theory.

The following assumptions are required to simplify the filter design procedures, which could be satisfied by many practical cases:

Assumption 1: $f(\cdot)$ and $h(\cdot)$ are two known Borel measurable and smooth nonlinear functions of their arguments, $f(0, 0, 0) = 0$ holds.

Assumption 2: Exogenous inputs δ_k , η_k , ω_k and ν_k ($k = 0, 1, 2, \dots$) are bounded, mutually independent random variables with known PDFs $\gamma_\delta(\tau)$, $\gamma_\eta(\tau)$, $\gamma_\omega(\tau)$ and $\gamma_\nu(\tau)$. The initial value x_0 is also supposed to be a random variable with a known PDF $\gamma_{x_0}(\tau)$.

To provide a feasible FI approach with explicit design procedures, in this paper, we propose a novel approach for the FI filtering using the entropy optimization principle.

B. Fault Isolation Filter and Error Dynamics

For the nonlinear system given by (1), a filter can be given by

$$\begin{cases} \hat{x}_{k+1} = f(\hat{x}_k, 0, 0, 0) + U_k(y_k - \hat{y}_k) \\ \hat{y}_k = h(\hat{x}_k, 0) \end{cases} \quad (2)$$

where U_k is the filter gain to be determined. The resulting estimation error $e_k = x_k - \hat{x}_k$ satisfies

$$e_{k+1} = f(x_k, \delta_k, \eta_k, \omega_k) - f(\hat{x}_k, 0, 0, 0) - U_k y_k + U_k \hat{y}_k. \quad (3)$$

It is noted that for the filter-based fault diagnosis, the residual variable $\hat{e}_k = h(x_k, \nu_k) - h(\hat{x}_k, 0)$ should be considered rather than e_k . However, under Assumption 1, it can be verified that $h(x_k)$ can be written as $h(x_k, \nu_k) - h(\hat{x}_k, 0) = L_k e_k + \rho_k(x_k, \hat{x}_k)$, where L_k is a known constant and $\rho_k(x_k, \hat{x}_k)$ is an unknown function satisfying $M_1 \leq \rho_k(x_k, \hat{x}_k) \leq M_2$. Here M_1 and M_2 are given constants. Then the entropy of random variable $\hat{e}_k$ satisfies

$$H(\hat{e}_k) \leq H(e_k) + \ln |L_k| + \ln |M_2 - M_1|.$$

In this paper, the following optimization will be focused on $H(e_k)$ to save space.

The filter gain U_k will be designed such that e_{k+1} is affected primarily by the target fault δ_k and minimally by others including η_k , ω_k and ν_k . It is denoted that as the estimation error, $e_{k+1} \in [a, b]$, where a and b can be respectively chosen as $-\infty$ and $+\infty$. In this note, different from most previous works on filter or observer theory, two cases for the output y_k will be considered, respectively.

Case 1) y_k is measurable at every sample time k .

Case 2) y_k is immeasurable at every sample time k , but its PDF $\gamma_{y_k}(\tau)$ can be measured or estimated.

In the following sections, we will establish the relationships between the PDFs of inputs and outputs for two different cases.

C. PDFs of the Error for Case 1

Under Case 1, in (3), the variable

$$\theta_{1k} := -f(\hat{x}_k, 0, 0, 0) - U_k y_k + U_k \hat{y}_k \quad (4)$$

can be regarded as the deterministic terms with unknown gain U_k at the sample time k . The main obstacle is that $f(x_k, \delta_k, \eta_k, \omega_k)$ is both nonlinear and stochastic at the sample time k .

We consider the formulation of γ_{x_k} and γ_{e_k} recursively. Suppose that the initial conditions are given by $x_0, \hat{x}_0$ and $U_0 = 0$. It is noted that (1) reduces to $x_1 = f(x_0, \delta_0, \eta_0, \omega_0)$. Using the PDFs of x_0, δ_0, η_0 and ω_0 and under Assumption 2, we can formulate the joint PDF of $(x_0, \delta_0, \eta_0, \omega_0)$ as

$$\gamma_{x_0 \delta \eta \omega}(\tau_1, \tau_2, \tau_3, \tau_4) = \gamma_{x_0}(\tau_1) \gamma_{\delta}(\tau_2) \gamma_{\eta}(\tau_3) \gamma_{\omega}(\tau_4).$$

As such, the PDF of x_1 can be given in terms of the multiple curve integrals based on the transformation theory of probability distributions. Recursively, we denote the PDF of x_k and e_k as $\gamma_{x_k}(\tau)$ and $\gamma_{e_k}(\tau)$, then we can have the following results.

Lemma 1: For Case 1, at the sample time k ($k = 1, 2, \dots$), $\gamma_{x_{k+1}}(\tau)$ can be formulated by

$$\frac{\partial \left[\iiint_{\Omega_k} \gamma_{x_k \delta \eta \omega}(\tau_1, \tau_2, \tau_3, \tau_4) d\tau_1 d\tau_2 d\tau_3 d\tau_4 \right]}{\partial \tau} \quad (5)$$

where for a given τ , the integrated domain is

$$\Omega_k := \{(\tau_1, \tau_2, \tau_3, \tau_4) | f(\tau_1, \tau_2, \tau_3, \tau_4) \leq \tau\}.$$

Thus, we have $\gamma_{e_k}(\tau) = \gamma_{x_k}(\tau - \theta_{1k})$, where θ_{1k} is denoted by (4).

Proof: At the sample time $k+1$, x_{k+1} is a function with four arguments x_k, δ_k, η_k and ω_k which are the stochastic variables with known PDFs. The proof can be easily provided by the classic probability theory, see e. g. (6–35) of [12]. Q.E.D

D. PDFs of the Error for Case 2

Under Case 2, correspondingly to (4), we denote

$$\theta_{2k} := -f(\hat{x}_k, 0, 0, 0) + U_k \hat{y}_k \quad (6)$$

which can be regarded as the deterministic terms with the unknown gain U_k . The left two terms of (3) are both nonlinear and stochastic. This means that the statistics of y_k has to be used for the formulation of $\gamma_{e_k}(\tau)$ for the FI purpose rather than y_k itself. In this case, initially, (1) reduces to

$$x_1 = f(x_0, \delta_0, \eta_0, \omega_0), \quad y_0 = h(x_0, \nu_0).$$

To determine the statistics of x_1 and e_1 , not only the statistics of x_0, δ_0, η_0 and ω_0 , but also that of y_0 should be used. Recursively, at every sample time k , θ_{1k} is unavailable to formulate $\gamma_{e_1}(\tau)$, which is different from Case 1. As such, we denote

$$d_{k+1} = f(x_k, \delta_k, \eta_k, \omega_k) - U_k y_k \quad (7)$$

as a sum of two random variables.

Lemma 2: For Case 2, at the sample time k , the PDF of d_{k+1} can be given by

$$\gamma_{d_{k+1}}(\tau) = \int_a^b \gamma_{x_{k+1}}(\tau + U_k \rho) \gamma_{y_k}(\rho) d\rho \quad (8)$$

where $\gamma_{x_{k+1}}$ can also be formulated by (5). Thus $\gamma_{e_k}(\tau) = \gamma_{d_k}(\tau - \theta_{2k})$.

Proof: In this case, $e_{k+1} = d_{k+1} + \theta_{2k}$, while d_{k+1} denoted in (7) possesses a sum form of two random variables. Since γ_{y_k} is supposed to be known, the sum operation can be used to represent $\gamma_{d_{k+1}}(\tau)$ as (8). The rest part of proof is similar to that for Lemma 1. Q.E.D

At this stage, $\gamma_{e_k}(\tau)$ can be represented by $\gamma_{\delta}(\tau)$, $\gamma_{\eta}(\tau)$, $\gamma_{\omega}(\tau)$, $\gamma_{\nu}(\tau)$ and $\gamma_{x_{k-1}}(\tau)$ as well as the under-determined gain U_k . However, it is noted that in the above results, the formulation of the PDFs were reduced to the differentials of some multiple integrals which depend on their integral domains.

III. ENTROPY OPTIMIZATION FILTERING

A. Entropy Optimization Principle

The entropy of e_k can be given by

$$H(e_k) = - \int_a^b \gamma_{e_k}(\tau) \ln(\gamma_{e_k}(\tau)) d\tau \quad \gamma_{e_k}(\tau) > 0 \quad (9)$$

which is also a functional of $\gamma_{\delta}(\tau)$, $\gamma_{\omega}(\tau)$, $\gamma_{\nu}(\tau)$ and $\gamma_{x_k}(\tau)$ as well as the under determined gain U_k . It should be pointed that $H(e_k)$ is actually a conditional entropy which can be further represented by $H(e_k | x_k, \delta_{k-1}, \eta_{k-1}, \omega_{k-1}, \nu_{k-1}, U_{k-1})$.

A desired FI approach can make the error be affected maximally by δ_k and minimally by η_k, ω_k and ν_k . For this purpose, it is expected that the entropy from δ_k to the error should be maximized while that from the other random vectors including η_k, ω_k and ν_k be minimized. As such, e_k should be formulated in two different situations. One responds to the case in the presence of δ_k but in the absence of η_k, ω_k and ν_k , where the error is denoted as e_k^1 . Concretely, in Case 1, the error dynamics is driven by

$$e_{k+1}^1 = f(x_k^1, \delta_k, 0, 0) + \theta_{1k} \quad (10)$$

where θ_{1k} is defined by (4) satisfying $x_{k+1}^1 = f(x_k^1, \delta_k, 0, 0)$. Another situation corresponds to the case in the presence of η_k, ω_k and ν_k but in the absence of δ_k , where the error denoted as e_k^2 is driven by

$$e_{k+1}^2 = f(x_k^2, 0, \eta_k, \omega_k) + \theta_{1k} \quad (11)$$

where $x_{k+1}^2 = f(x_k^2, 0, \eta_k, \omega_k)$.

The design objective is to find U_k at each step such that the entropy resulting from the first situation is maximized whilst the entropy resulting from the second situation is minimized.

Thus the following two entropies have to be considered:

$$H(e_{k+1}^1) = H(e_{k+1}^1 | x_k, \delta_k, 0, 0, U_k) \quad (12)$$

$$H(e_{k+1}^2) = H(e_{k+1}^2 | x_k, 0, \eta_k, \omega_k, \nu_k, U_k) \quad (13)$$

where 0 still means that the corresponding argument is set to be the deterministic constant 0 (i.e., the corresponding variables do no contributions on the resultant error).

For a given τ , the following two sets as the integrated domains are denoted:

$$\begin{aligned}\Pi_k^1 &:= \{(\tau_1, \tau_2) | f(\tau_1, \tau_2, 0, 0) \leq \tau\}, \\ \Pi_k^2 &:= \{(\tau_1, \tau_2, \tau_3) | f(\tau_1, 0, \tau_2, \tau_3) \leq \tau\}.\end{aligned}$$

Similarly to lemma 1, for Case 1, the formulations for $\gamma_{e_k^i}$ ($i = 1, 2$) can be reduced as follows:

Theorem 1: For Case 1, at the sample time k ($k = 1, 2, \dots$), $\gamma_{x_{k+1}^i}$ ($i = 1, 2$) can be formulated by

$$\begin{aligned}\gamma_{x_{k+1}^1}(\tau) &= \frac{\partial \left[\int \int_{\Pi_k^1} \gamma_{x_k^1} \delta_k(\tau_1, \tau_2) d\tau_1 d\tau_2 \right]}{\partial \tau}, \\ \gamma_{x_{k+1}^2}(\tau) &= \frac{\partial \left[\int \int_{\Pi_k^2} \gamma_{x_k^2} \eta_k \omega_k(\tau_1, \tau_2, \tau_3) d\tau_1 d\tau_2 d\tau_3 \right]}{\partial \tau}.\end{aligned}$$

Thus, we have $\gamma_{e_k^i}(\tau) = \gamma_{x_{k+1}^i}(\tau - \theta_{1k})$, where θ_{1k} is denoted by (4).

Proof: This result can be proved similarly to lemma 1. Q.E.D. In Case 2, the corresponding error dynamics are given by

$$e_{k+1}^1 = f_1(x_k^1, \delta_k) + \theta_{2k}, \quad e_{k+1}^2 = f_2(x_k^2, \eta_k, \omega_k) + \theta_{2k} \quad (14)$$

where $f_1(x_k^1, \delta_k) = f(x_k^1, \delta_k, 0, 0) - U_k y_k$, $f_2(x_k^2, \eta_k, \omega_k) = f(x_k^2, \delta_k, 0, 0) - U_k y_k$, θ_{2k} is denoted by (6) and the corresponding states satisfy $x_{k+1}^1 = f(x_k^1, \delta_k, 0, 0)$, $x_{k+1}^2 = f(x_k^2, 0, \eta_k, \omega_k)$. The results for Case 2 can be obtained in the similar way following Lemma 2 and Theorem 1.

Theorem 2: For Case 2, at the sample time k ($k = 1, 2, \dots$), the PDF of $\varrho_{k+1}^1 = f_1(x_k^1, \delta_k)$ and $\varrho_{k+1}^2 = f_2(x_k^2, \eta_k, \omega_k)$ can be given by

$$\gamma_{\varrho_{k+1}^i}(\tau) = \int_a^b \gamma_{x_{k+1}^i}(\tau + U_k \rho) \gamma_{y_k}(\rho) d\rho \quad (i = 1, 2) \quad (15)$$

where

$$\begin{aligned}\gamma_{x_{k+1}^1}(\tau) &= \frac{\partial \left[\int \int_{\Pi_k^1} \gamma_{x_k^1} \delta_k(\tau_1, \tau_2) d\tau_1 d\tau_2 \right]}{\partial \tau}, \\ \gamma_{x_{k+1}^2}(\tau) &= \frac{\partial \left[\int \int \int_{\Pi_k^2} \gamma_{x_k^2} \eta_k \omega_k(\tau_1, \tau_2, \tau_3) d\tau_1 d\tau_2 d\tau_3 \right]}{\partial \tau}.\end{aligned}$$

Thus, we have $\gamma_{e_k^i}(\tau) = \gamma_{\varrho_{k+1}^i}(\tau - \theta_{2k})$.

Proof: $\gamma_{x_{k+1}^i}(\tau)$ ($i = 1, 2$) can be formulated similarly to Lemma 1 based on the classical probability theory. Besides, as shown in the definition, ϱ_k^i ($i = 1, 2$) can be obtained by subtracting $U_k y_k$ from x_{k+1}^i ($i = 1, 2$). Thus, we have $\gamma_{e_k^i}(\tau) = \gamma_{\varrho_{k+1}^i}(\tau - \theta_{2k})$. Q.E.D.

To isolate the fault and attenuate the disturbances, we propose the following performance index called as entropy optimization principle (EOP)

$$J_N = \sum_{k=1}^N \left[-H(e_k^1) + R_1 H(e_k^2) + \frac{1}{2} R_2 u_k^2 \right] \quad (16)$$

where R_1 and R_2 can be selected as weights.

Remark 1: Entropy optimization is equivalent to the variance optimization for Gaussian signals. It can be verified that the entropy optimization principle is consistent with the results for linear Gaussian systems [3].

If there exists a filter such that J_k is minimized for each sample time, then it is called to be an entropy optimization (EO) filter (EOF).

B. Simplified Calculation of the Error PDFs

It is noted that surface integrals are involved in the formulations of the error PDFs. To simplify the design methods, we introduce the following assumption. It is noted that the more general cases can be dealt with and are omitted due to space limitation.

Assumption 3: $(\partial f(x_k, \delta_k, \eta_k, \omega_k) / \partial x_k) \neq 0$ holds for any $x_k \in [a, b]$.

Under Assumption 3, given a constant τ , for functions $f(\tau_1, \tau_2, 0, 0)$, $f(\tau_1, 0, \tau_2, \tau_3)$, $f_1(\tau_1, \tau_2)$ and $f_2(\tau_1, \tau_2, \tau_3)$ denoted, there exist the following inverse functions

$$\begin{aligned}\kappa_1 &= f^{-1}(\tau, \tau_1, 0, 0), \quad \varpi_1 = f^{-1}(\tau, 0, \tau_1, \tau_2), \\ \kappa_2 &= f_1^{-1}(\tau, \tau_1), \quad \varpi_2 = f_2^{-1}(\tau, \tau_1, \tau_2)\end{aligned} \quad (17)$$

satisfying $\tau = f(\kappa_1, \tau_1, 0, 0)$, $\tau = f(\varpi_1, 0, \tau_1, \tau_2)$, $\tau = f_1(\kappa_2, \tau_1)$, $\tau = f_2(\varpi_2, \tau_1, \tau_2)$ respectively. Construct the auxiliary multivariate mapping as

$$\begin{cases} \sigma_{k+1}^1 = \begin{bmatrix} x_{k+1}^1 \\ \delta_{k+1} \end{bmatrix} = \begin{bmatrix} f(x_k^1, \delta_k, 0, 0) \\ \delta_{k+1} \end{bmatrix}, \\ \sigma_{k+1}^2 = \begin{bmatrix} x_{k+1}^2 \\ \eta_{k+1} \end{bmatrix} = \begin{bmatrix} f(x_k^2, 0, \eta_k, \omega_k) \\ \eta_{k+1} \end{bmatrix}, \\ \xi_{k+1}^1 = \begin{bmatrix} \varrho_{k+1}^1 \\ \delta_{k+1} \end{bmatrix} = \begin{bmatrix} f_1(x_k^1, \delta_k) \\ \delta_{k+1} \end{bmatrix}, \\ \xi_{k+1}^2 = \begin{bmatrix} \varrho_{k+1}^2 \\ \eta_{k+1} \end{bmatrix} = \begin{bmatrix} f_2(x_k^2, \eta_k, \omega_k) \\ \eta_{k+1} \end{bmatrix}.\end{cases} \quad (18)$$

With (18), the dynamics of estimation errors and the states in the presence of δ_k and in the absence of η_k and ω_k and vice versa can be represented respectively using the above augmented one-to-one multivariate functions. Different from the Lemmas 1, 2 and Theorems 1, 2, simplified algorithms can be further obtained to calculate $\gamma_{e_k^i}(\tau)$ ($i = 1, 2$) by constructing the above auxiliary multivariate functions. The following results can be provided for Case 1.

Theorem 3: For Case 1, under Assumption 1-3, at the sample time k , the PDF of x_{k+1}^i ($i = 1, 2$) can be given by

$$\gamma_{x_{k+1}^1}(\tau) = \int_a^b \gamma_{x_k^1}(f^{-1}(\tau, \tau_1, 0, 0)) \gamma_{\delta}(\tau_1) \psi_1(\tau, \tau_1) d\tau_1 \quad (19)$$

$$\begin{aligned}\gamma_{x_{k+1}^2}(\tau) &= \int_a^b \gamma_{x_k^2}(f^{-1}(\tau, 0, \tau_1, \tau_2)) \gamma_{\eta}(\tau_1) \gamma_{\omega}(\tau_2) \\ &\quad \times \psi_2(\tau, \tau_1, \tau_2) d\tau_1 d\tau_2\end{aligned} \quad (20)$$

where $\psi_1(\tau, \tau_1) = |(\partial f(\tau, \tau_1, 0, 0) / \partial \tau)|^{-1}$, $\psi_2(\tau, \tau_1, \tau_2) = |(\partial f(\tau, 0, \tau_1, \tau_2) / \partial \tau)|^{-1}$. Then $\gamma_{e_{k+1}^i}(\tau) = \gamma_{x_{k+1}^i}(\tau - \theta_{1k})$.

Proof: Firstly, from (18), it can be obtained that $\gamma_{x_{k+1}^1}(\tau) = \int_a^b \gamma_{\sigma_{k+1}^1}(\tau, \tau_1) d\tau_1$. It is noted that it is an one-to-one mapping from (x_k^1, δ_k) to σ_{k+1}^1 under Assumption 3. Thus, the PDF of σ_{k+1}^1 can be formulated in terms of that of (x_k^1, δ_k) [12]. On the other hand, random variables x_k^1 and δ_k can be regarded as mutually independent ones under Assumption 2. As such, based on the special structure described by (18) and the inverse function denoted in (17), it can be verified that

$$\begin{aligned}\gamma_{\sigma_{k+1}^1}(\tau, \tau_1) &= \gamma_{x_k^1 \delta_k}(\kappa_2, \tau_1) \left| \frac{\partial f(\tau, \tau_1, 0, 0)}{\partial \tau} \right|^{-1} \\ &= \gamma_{x_k^1}(f^{-1}(\tau, \tau_1, 0, 0)) \gamma_{\delta_k}(\tau_1) \psi_1(\tau, \tau_1)\end{aligned} \quad (21)$$

which means that (19) holds. Similarly, we can prove (20) holds. Q.E.D.

This procedure can also be used to prove the following result for Case 2 as combined with Theorem 2.

Theorem 4: For Case 2, under Assumption 1–3, at the sample time k , the PDF of e_{k+1}^i ($i = 1, 2$) can be given by $\gamma_{e_k^i}(\tau) = \gamma_{\varrho_k^i}(\tau - \theta_{2k})$. $\gamma_{\varrho_k^i}(\tau)$ is calculated by (15), where

$$\begin{aligned}\gamma_{x_{k+1}^1}(\tau) &= \int_a^b \gamma_{x_k^1}(f^{-1}(\tau, \tau_1, 0, 0)) \gamma_{\delta}(\tau_1) \psi_1(\tau, \tau_1) d\tau_1 \\ \gamma_{x_{k+1}^2}(\tau) &= \int_a^b \gamma_{x_k^2}(f^{-1}(\tau, 0, \tau_1, \tau_2)) \gamma_{\eta}(\tau_1) \gamma_{\omega}(\tau_2) \\ &\quad \times \psi_2(\tau, \tau_1, \tau_2) d\tau_1 d\tau_2\end{aligned}\quad (22)$$

and $\psi_1(\tau, \tau_1) = |(\partial f(\tau, \tau_1, 0, 0)/\partial \tau)|^{-1}$, $\psi_2(\tau, \tau_1, \tau_2) = |(\partial f(\tau, 0, \tau_1, \tau_2)/\partial \tau)|^{-1}$.

Proof: $\gamma_{x_{k+1}^i}(\tau)$ ($i = 1, 2$) can be obtained similarly to Theorem 3.

C. Optimal FD Filter Design Strategy

It is noted that the above calculation procedure for entropy will lead to a slow and computational inefficient algorithm. To solve this, the Renyi's entropy will be used in the following [13]:

$$H_{\alpha}(e_k) = \frac{1}{1-\alpha} \ln \int_a^b \gamma_{e_k}^{\alpha}(\tau) d\tau. \quad (23)$$

In (23), it can be selected as $\alpha = 2$. In this case, the minimization (maximization) of $H_2(e_k)$ reduces to the maximization (minimization) of $V(e_k) = \int_a^b \gamma_{e_k}^2(\tau) d\tau$, which is called the information potential. Thus, we only consider the following instantaneous cost function

$$\psi_k(U_k) = V(e_k^1) - R_1 V(e_k^2) + \frac{1}{2} R_2 U_k^2 \quad (24)$$

where the logarithm computation in (23) can be avoided.

The optimal filtering strategy can be obtained via $(\partial J_k / \partial U_k) = 0$, where an explicit function from other augments to U_k should be determined. For this purpose, we consider a recursive procedure in the following. Denote

$$U_k = U_{k-1} + \Delta U_k, \quad k = 1, 2, \dots, N, \quad (25)$$

$$J_k(U_k) = V(e_k^1) - R_1 V(e_k^2). \quad (26)$$

It can be approximated that

$$J_k(U_k) = J_{k0} + J_{k1} \Delta U_k + \frac{1}{2} J_{k2} \Delta U_k^2 + o(\Delta U_k^2) \quad (27)$$

where $J_{k0} = J_k(U_k)|_{U_k=U_{k-1}}$, $J_{k1} = (\partial J_k(U_k)/\partial U_k)|_{U_k=U_{k-1}}$, $J_{k2} = (\partial^2 J_k(U_k)/\partial U_k^2)|_{U_k=U_{k-1}}$.

Theorem 5: For Case 1, an EO filtering strategy for the FI purpose for J_k subject to nonlinear error model (3) is given by

$$\Delta U_k^* = -\frac{J_{k1} + R_2 U_{k-1}}{J_{k2} + R_2} \quad (28)$$

for a weight $R_2 > 0$ satisfying $J_{k2} + R_2 > 0$.

Proof: It can be seen that

$$R_2 U_k^2 = U_{k-1}^T R_2 U_{k-1} + 2 U_{k-1}^T R_2 \Delta U_k + R_2 \Delta U_k^2. \quad (29)$$

Substituting (27) and (29) into (24) yields recursive suboptimal control law (28) for all k . However, the above algorithm (28) results from a necessary condition for optimization. To guarantee sufficiency, the following second-order derivative should be satisfied

$$\frac{\partial^2 [J_k(U_k) + \frac{1}{2} R_2 U_k^2]}{\partial \Delta U_k^2} = J_{k2} + R_2 > 0$$

which will hold if R_2 is selected sufficiently large.

Q.E.D

Fig. 1. Signal for the disturbance input ω_k .

Fig. 2. Signal for the output disturbance ν_k .

Remark 2: For Case 1, the suboptimal EO filtering algorithm can be summarized as follows:

- Initialize $x_0, \hat{x}_0$ and U_0 ;
- At each sample time k , compute $\gamma_{e_{k+1}^i}(\tau)$, $\tau \in [a, b]$ and $V(e_{k+1}^i)$ ($i = 1, 2$);
- Calculate ΔU_k and U_k using (28) and (25);
- Increase k by 1 and go back to a).

The design procedures for Case 2 follows similarly.

The following simulation is given for Case 1. For Case 2, the simulation is similar and is omitted here for brevity.

IV. SIMULATION RESULTS

To demonstrate the FI algorithm, we consider a simple nonlinear and non-Gaussian model described as

$$\begin{cases} x_{k+1} = \arctan x_k + \delta_k + \eta_k + \sqrt{\omega_k} \\ y_k = 0.9x_k + 0.1 \sin x_k + \nu_k \end{cases}$$

where the faults δ_k , η_k and the disturbances ω_k , ν_k ($k = 0, 1, 2, \dots$) are assumed to be mutually independent. The asymmetric PDF of δ_k is defined by

$$\gamma_{\delta}(x) = \begin{cases} -12x^5 + 12x^3, & x \in [0, 1], \\ 0, & \text{else} \end{cases}$$

Fig. 3. Response of the target fault δ_k .

Fig. 5. System response in the presence of the target fault.

Fig. 4. Response of the nuisance fault η_k .

Fig. 6. System response in the presence of the nuisance fault.

the PDF of η_k is given by

$$\gamma_\eta(x) = \begin{cases} -48(x^2 - x + \frac{3}{16}), & x \in [0.25, 0.75], \\ 0, & \text{else} \end{cases}$$

while the PDF of ω_k is defined by

$$\gamma_\omega(x) = \begin{cases} -\frac{3000}{4}(x^2 - 0.01), & x \in [-0.1, 0.1], \\ 0, & \text{else} \end{cases}$$

and ν_k is a Gaussian variable with the distribution $N(0, 1)$.

According to (2), the FI filter can be constructed by

$$\begin{cases} \hat{x}_{k+1} = \arctan \hat{x}_k + U_k(y_k - \hat{y}_k), \\ \hat{y}_k = 0.9\hat{x}_k + 0.1 \sin \hat{x}_k. \end{cases}$$

In simulations, the weights are set to be $R_1 = 1$ and $R_2 = 20$. The Initial conditions are $U_0 = 0$, $\hat{x}_0 = 0$ and x_0 is under uniform distribution on the interval $[0, 1]$. Fig. 1 and Fig. 2 show the signals for the disturbance input ω_k and the output disturbance ν_k respectively. Fig. 3 displays the signals for the target fault emerging at the 20th second and Fig. 4 displays the signals for the nuisance fault emerging at the 15th second. Fig. 5 is provided to demonstrate the system response in the

presence of the target fault δ_k and in the absence of the nuisance fault η_k while Fig. 6 demonstrates the system response in the presence of the nuisance fault η_k and in the absence of the target fault δ_k . It is shown that when the target fault occurs the error jumps to a high level in a short time, whereas it converges in the presence of the nuisance fault.

V. CONCLUSION

In this paper, we present a novel FI approach for the dynamic nonlinear non-Gaussian systems with faults and the noises. By constructing a filter to estimate the states, the FI problem can be reduced to an entropy optimization problem subjected to the estimation error systems, which is also represented by a nonlinear non-Gaussian system. After new relationships are applied to describe the PDFs of the stochastic error in terms of PDFs of the stochastic noises and faults, we propose recursive approaches to design real-time suboptimal FI filters. The design objective is that the entropy of the stochastic estimation error is maximized with respect to the faults and is minimized with respect to other faults and the stochastic noises. Another feature of the proposed approach is that the output can be supposed to be immeasurable. With the measured PDF of the output, similar approaches can be provided, which is different from the existing results. Finally, simulations

are provided to demonstrate the effectiveness of the novel Filtering algorithms. Indeed, this paper has provided a new optimal approach for FI problems of stochastic systems with non-Gaussian variables.

The results in this paper can also be generalized to multivariate systems where $\gamma_\delta(x)$, $\gamma_\omega(x)$ and $\gamma_\nu(x)$ are joint PDFs under the assumption of vector-value Borel measurable functions for $f(x_k, \delta_k, \omega_k)$ and $h(x_k, \nu_k)$. See [9] for the filtering problem.

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Interval Stability and Stabilization of Linear Stochastic Systems

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Abstract—This technical note applies the spectrum technique to deal with the interval stability and stabilization of linear stochastic time-invariant systems. First, the notion of $(-\beta, -\alpha)$ -stability of stochastic systems is defined and some relationships among the $(-\beta, -\alpha)$ -stability, the decay rate of system state response, and the second-order moment Lyapunov exponent are revealed. We then address the $(-\beta, -\alpha)$ -stabilization problem and give a sufficient condition via a linear matrix inequality approach.

Index Terms—Asymptotic mean square stability, $(-\beta, -\alpha)$ -stability, $(-\beta, -\alpha)$ -stabilization, second-order moment Lyapunov exponent, spectrum assignment.

I. INTRODUCTION

This technical note applies the spectrum technique developed in [17] to study the stability and stabilization of linear stochastic time-invariant systems. For a long time, the study on stability analysis and stabilization for stochastic systems is mainly dominated by the Lyapunov function-based approach and the Lyapunov exponents; see monographs [1], [10], [11] and a recent book [14]. In particular, for linear stochastic time-invariant systems, many criteria have been presented for testing the asymptotic mean square stability in terms of linear matrix inequalities (LMIs) and algebraic Riccati equations, which are derived from the application of the Lyapunov function approach. As is well known, the time-invariant deterministic system

$$\dot{x} = Ax, \quad x(0) = x_0 \in \mathcal{R}^n \quad (1)$$

is asymptotically stable if and only if (iff) all eigenvalues of A belong to the open left-half complex plane, which is the so-called "eigenvalue criterion" or "spectrum criterion" for asymptotic stability. One of the advantages of spectrum characterization is its direct linkage with the system dynamic behavior. Intuitively speaking, the further left the spectrum set of $\sigma(A)$ is located to, the faster the system decay rate is. Secondly, the spectrum characterization leads to an important control design technique named as pole assignment, which plays an important role in linear systems theory, and has been studied extensively by many researchers; see [9], [13]. Likewise, for linear time-invariant systems governed by Itô-type stochastic differential equations, the spectrum technique has been respectively introduced in [7], [8] and [17] recently, and three different spectrum criteria for mean square stabilization [8], [17] and stability [7] were also given. In particular, by means of the spectrum technique, [17] presented a necessary and sufficient condition called "PBH Criterion" for stochastic observability,

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