Stochastic consensus control with finite frequency specification for Markov jump networks

Xiaoli Luan, Chaojie Zhou, Zhengtao Ding and Fei Liu

Abstract

A stochastic finite frequency consensus protocol for directed networks with Markov jump topologies and external disturbances is proposed in this paper. By introducing the frequency band information into consensus control design, the disagreement dynamics of interconnected networks asymptotically converge to zero with an improved level of disturbance attenuation in the specific frequency band. In addition, the new model transformation approach is presented by exploring certain features of Laplacian matrix in real Jordan form, which leads to more generality of the designed protocol. A numerical example validates the potential of the developed results.

Index Terms

Markov jump network topology, Finite frequency H_∞ control, Stochastic consensus control, Frequency and time domain inequalities

I. INTRODUCTION

In practical engineering phenomenon, the dynamic systems are often connected with each other via a sensing or communication network to achieve the specified control tasks, such as satellite formation flying, hazardous material handling, and industrial process controlling, etc [1], [2], [3], [4]. For such a type of network-connected systems, consensus control is an important and

Xiaoli Luan, Chaojie Zhou and Fei Liu are with the Key Laboratory of Advanced Control for Light Industry Processes, Ministry of Education, and the Institute of Automation, Jiangnan University, Wuxi 214122, China. E-mail: xlluan@jiangnan.edu.cn, chaojie_zhou@126.com, fliu@jiangnan.edu.cn

Zhengtao Ding is with the Control Systems Centre, School of Electrical and Electronic Engineering, University of Manchester, Sackville Street Building, Manchester M 13 9PL, UK. E-mail: Zhengtao.Ding@manchester.ac.uk

a practical problem to be considered. For example, an industrial heating furnace with multiple passes need to be controlled for all individual pass outlet temperature reaching the same value in order to achieve higher energy efficiency and obtain better profit. Consensus control aims to make the states of subsystems reach some kind of agreement using the information obtained from the network. Since the pioneering works on the consensus problems for distributed decision making and parallel computing, up to now, considerable amount of results have been reported in the literatures [5], [6], [7] with the assumption that the communication topology is fixed.

However, the changes of communication network structure may happen with time due to link failures, packet drops, stochastic noises, etc. The switching network topology, which may degrade the system performance has been considered in many existing literatures [8], [9], [10], [11]. Recently, Markov jump process is utilized to characterize the switching phenomenon stochastically happened in network topology. For example, the mean square consensus stability was studied in [12] for a network in Markov switching topology; Necessary and sufficient conditions of mean square consensus were presented in [13] by transforming the multi-agent systems into the Markov jump systems; Both continuous and discrete time consensus problems were investigated in [14], in which the effect of switching topologies on consensus is determined by the union of topologies associated with the positive recurrent states of the Markov process.

On the other hand, network-connected systems are often subjected to various disturbances such as random communication noises, measurement errors and external disturbances, etc. The existence of disturbances may deteriorate the convergence properties of the systems. To attenuate the effect of external disturbance signal on the behavior of the systems, robust H_∞ control approach is usually employed to achieve consensus. For example, the consensus problem of identical linear systems with directed topologies and external disturbances was discussed in [15] based on H_∞ control theory. The H_∞ consensus performance problem for linear multi-agent systems with undirected topologies was analyzed in [16], [17], [18]. Furthermore, the distributed H_∞ control for switching networks with linear coupling dynamics was solved in [19] using output feedback protocol. Due to the nonlinearities of practical systems, the consensus protocol with prescribed L_2 -gain performance was proposed in [20] for networks with nonlinear dynamics subject to intermittent communication. H_∞ global consensus problem for nonlinear multi-agent systems with strongly connected balanced communication graphs was investigated in [21].

Carding the existing literature research on networks with disturbances, including linear and

nonlinear dynamics, determined and stochastic topologies, state and output feedback protocols, almost all of the reported approaches as mentioned above have only considered the disturbances in the entire frequency domain. If the noise is within a finite frequency range, more accurately, low/middle/high/ frequency, the existing design methods for the entire frequency range will result in the pretty much conservative design. Therefore, it is necessary to develop a design algorithm for consensus control with finite frequency ranges for network-connected dynamic systems. To the best knowledge of the authors, very few research results have been found so far on finite frequency consensus problem, even less in network-connected dynamic systems with Markov jump topologies. This situation actually motivates the present work.

The main objective of this research is to identify a mode-dependent finite frequency consensus condition for directed networks with Markov jump topologies. Based on the generalized Kalman-Yakubovich-Popov (GKYP) lemma, the frequency domain inequalities are transformed into the linear matrix inequalities (LMIs). The attention is focused on developing methods to design a mode-dependent consensus protocol such that the disagreement dynamics of network-connected systems asymptotically converge to zero with an improved level of disturbance attenuation in the specific frequency range. The major contributions of this paper are threefold. Firstly, we consider the frequency information of disturbance in consensus design of network-connected systems. It is proven that the new method presents much better disturbance rejection performance for the considered frequency range. Secondly, different from other design methods of consensus controller for network topologies governed by Markov jump process, the proposed consensus protocol is determined not only by the jump mode, but also the jump rate. Lastly, some new features of Laplacian matrix are explored in real Jordan form to identify conditions for finite frequency consensus, which extends the results in [16], [17] only for undirected topologies.

II. PROBLEM DESCRIPTION

The dynamics of N subsystems take the following form:

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) + B_w w_i(t) \quad (1)$$

for $i = 1, \dots, N$, where $x_i(t) \in R^n$ and $u_i(t) \in R^m$ represent the state and control input of subsystem i , respectively. $w_i(t) \in L_2^p \left[\begin{array}{cc} \varpi_1 & \varpi_2 \end{array} \right]$ is the external disturbance, and A , B , B_w are constant matrices with (A, B) controllable.

The connections between the subsystems are specified by a directed graph Z which consists of a set of vertices denoted by Λ and a set of edges denoted by Υ . A vertex represents a subsystem,

and each edge represents a connection. Associated with the graph, its adjacency matrix Q with elements q_{ij} denotes the connections such that $q_{ij} = 1$ if there is a connection from subsystem j to subsystem i , and $q_{ij} = 0$, otherwise. The Laplacian matrix $L = \{l_{ij}\}$ is commonly defined as $l_{ii} = \sum_{j=1, j \neq i}^N q_{ij}$, $l_{ij} = -q_{ij}$ for $i \neq j$. For a directed graph, some properties about the Laplacian matrix L will be given in the following lemma.

Assumption 2.1: The eigenvalues of the Laplacian matrix are distinct.

Note that this assumption is made for the convenience of presentation. The cases with multiple eigenvalues can be dealt in a similar way as shown in [22]. Nevertheless, it is common and prevailing for the Laplacian matrices to have distinct eigenvalues [23].

Lemma 2.1: ([24]) Zero is an eigenvalue of L with $\mathbf{1} = \begin{bmatrix} 1, & \dots, & 1 \end{bmatrix}^T \in R^N$ as the corresponding right eigenvector and all the non-zero eigenvalues have positive real parts. Furthermore, zero is a single eigenvalue of L if and only if the graph contains a directed spanning tree.

In this work, the network connection topology is switching, and the switching process $\{r_t, t \geq 0\}$ is governed by a homogeneous Markov chain. Let $(\Omega, E, \mathcal{O})$ be complete probability space, where Ω is the sample space, E is the algebra of events and $\mathcal{O}$ is the probability measure defined on E . Let the random form process $\{r_t, t \geq 0\}$ be the Markov stochastic process taking values in a finite set $S = \{1, 2, \dots, s\}$ with transition rate matrix $\Pi = \{\pi_{rl}\}$, $r, l \in S$ and define the following transition probability from mode r at time t to mode l at time $t + \Delta t$ as

$$\mathcal{O}_{rl} = \mathcal{O}_r \{r_{t+\Delta t} = l | r_t = r\} = \begin{cases} \pi_{rl}\Delta t + o(\Delta t), & r \neq l \\ 1 + \pi_{rr}\Delta t + o(\Delta t), & r = l \end{cases}$$

with transition probability rates $\pi_{rl} \geq 0$ for $r, l \in S$, $r \neq l$ and $\sum_{l=1, l \neq r}^s \pi_{rl} = -\pi_{rr}$ where $\Delta t > 0$ and $\lim_{\Delta t \rightarrow 0} o(\Delta t)/\Delta t \rightarrow 0$.

Under the influence of external disturbances $w_i(t)$, the accurate consensus may be hard to be achieved. In order to quantitatively analyze the effect of disturbances to the consensus, the controlled output function $z_i(t)$ is defined to measure the state disagreement variable as follows

$$z_i = x_i - \sum_{j=1}^N [\kappa(r_t)]_j x_j \quad (2)$$

where $\kappa(r_t) \in R^N$ is defined as the left eigenvector of jump Laplacian $L(r_t)$ associated with the eigenvalue 0 and satisfies $\kappa(r_t)^T \mathbf{1} = 1$.

Denoting $z = \begin{bmatrix} z_1^T & z_2^T & \cdots & z_N^T \end{bmatrix}^T$, equality (2) is rewritten as follows:

$$\begin{aligned} z &= x - \left(\left(\mathbf{1}\kappa(r_t)^T \right) \otimes I_n \right) x \\ &= M(r_t) \otimes I_n x \end{aligned} \quad (3)$$

where $M(r_t) = I_N - \mathbf{1}\kappa(r_t)^T$. Note that the condition $z = 0$ implies $x_1 = x_2 = \cdots = x_N$, which means the consensus control is achieved as $z = 0$.

Since $z = 0$ implies that the network-connected system (1) achieves consensus, the attenuating ability of system (1) on consensus against external disturbances with spectral energies concentrating at narrow frequency bands can be quantitatively measured by the H_∞ norm of the transfer function $G(j\omega)$ from the disturbance inputs $w = \begin{bmatrix} w_1^T & w_2^T & \cdots & w_N^T \end{bmatrix}^T$ to the controlled output z as follows:

$$\sup_{\varpi_1 < \omega < \varpi_2} \|G(j\omega)\|_\infty < \gamma \quad (4)$$

where $\gamma > 0$ is a given H_∞ performance index, and ϖ_1, ϖ_2 represent the upper and lower bounds of the concerned frequency. To formulate the problem to be addressed in this paper, let us recall the following lemmas.

Lemma 2.2: ([22]) For a Laplacian matrix that satisfies Assumption 2.1, there exists a similarity transformation $T(r_t)$ with the last column of $T(r_t) = \mathbf{1}$ and the last row of $T(r_t)^{-1} = \kappa(r_t)^T$ such that

$$T(r_t)^{-1} L(r_t) T(r_t) = J(r_t) \quad (5)$$

where $J(r_t)$ is a block-diagonal matrix with the structure

$$J(r_t) = \begin{bmatrix} \lambda_1(r_t) & & & & & \\ & \ddots & & & & \\ & & \lambda_{n_\lambda}(r_t) & & & \\ & & & \mu_1(r_t) & & \\ & & & & \ddots & \\ & & & & & \mu_{n_\mu}(r_t) \\ & & & & & & 0 \end{bmatrix}$$

where $\lambda_i(r_t) \in R$ for $i = 1, \dots, n_\lambda$ and $\mu_i(r_t) = \begin{bmatrix} \alpha_i(r_t) & \beta_i(r_t) \\ -\beta_i(r_t) & \alpha_i(r_t) \end{bmatrix} \in R^{2 \times 2}$ for $i = 1, \dots, n_\mu$. In above expressions, $\lambda_i(r_t)$, $\alpha_i(r_t)$ and $\beta_i(r_t)$ are positive real numbers with $\lambda_i(r_t)$ denoting real eigenvalues of $L(r_t)$ and $\alpha_i(r_t) \pm j\beta_i(r_t)$ denoting complex conjugate eigenvalues of $L(r_t)$ respectively. Obviously we have $1 + n_\lambda + 2n_\mu = N$.

Lemma 2.3: (GKYP Lemma [25]) Consider the linear system $\begin{pmatrix} \bar{A} & \bar{B} & \bar{C} & \bar{D} \end{pmatrix}$. Given a symmetric matrix Φ , the following statements are equivalent.

1) The finite frequency inequality

$$\begin{pmatrix} G(j\omega) \\ I \end{pmatrix}^T \Phi \begin{pmatrix} G(j\omega) \\ I \end{pmatrix} < 0, \varpi_1 \leq \omega \leq \varpi_2 \quad (6)$$

2) There exist symmetric matrices X and Y satisfying $Y > 0$ and

$$\begin{pmatrix} \bar{A} & \bar{B} \\ I & 0 \end{pmatrix}^T \Xi \begin{pmatrix} \bar{A} & \bar{B} \\ I & 0 \end{pmatrix} + \begin{pmatrix} \bar{C} & \bar{D} \\ 0 & I \end{pmatrix}^T \Phi \begin{pmatrix} \bar{C} & \bar{D} \\ 0 & I \end{pmatrix} < 0 \quad (7)$$

where

$$\begin{aligned} G(j\omega) &= \bar{C}(j\omega I - \bar{A})^{-1}\bar{B} + \bar{D} \\ \Xi &= \begin{pmatrix} -Y & X + j\varpi_c Y \\ X - j\varpi_c Y & -\varpi_1 \varpi_2 Y \end{pmatrix} \\ \varpi_c &= (\varpi_1 + \varpi_2)/2 \end{aligned}$$

Lemma 2.4: (Projection Lemma [26]) For given Γ, H, Θ , there exists a matrix F satisfying

$$\Gamma F H + (\Gamma F H)^T + \Theta < 0 \quad (8)$$

if and only if the following two conditions hold

$$\Gamma^\perp \Theta \Gamma^\perp{}^T < 0 \quad (9)$$

$$H^{T^\perp} \Theta H^{T^\perp{}^T} < 0 \quad (10)$$

Lemma 2.5: (Reciprocal Projection Lemma [26]) For any given positive definite matrix U , the inequality $\Psi + S + S^T < 0$ is equivalent to the LMI problem

$$\begin{bmatrix} \Psi + U - (W + W^T) & S^T + W \\ * & -U \end{bmatrix} < 0 \quad (11)$$

The purpose of this research is to design a controller to make the disagreement dynamics of interconnected networks converge to zero asymptotically, with a prescribed H_∞ disturbance attenuation level over finite frequency ranges in the presence of Markov jump network topologies and external disturbance with spectral energies concentrating at the narrow frequency bands.

III. CONTROLLER DESIGN

In this section, sufficient conditions in term of LMIs are identified to ensure the consensus performance with a desired H_∞ disturbance attention level over finite frequency ranges. To achieve finite frequency H_∞ consensus, the following protocol is adopted:

$$u_i = K(r_t) \sum_{j=1}^N q_{ij}(r_t) (x_j - x_i) \quad (12)$$

where $q_{ij}(r_t)$ are adjacency elements of the stochastic jump graph $G(r_t)$, and $K(r_t) \in R^{m \times n}$ is the mode-dependent control gain to be designed.

Now we can write the protocol (12) in an alternative format as follows:

$$u_i = -K(r_t) \sum_{j=1}^N l_{ij}(r_t) x_j \quad (13)$$

With protocol (13) and defining $x = \begin{bmatrix} x_1^T & x_2^T & \dots & x_N^T \end{bmatrix}^T$, the network dynamics can be expressed as

$$\begin{cases} \dot{x} = (I_N \otimes A - L(r_t) \otimes BK(r_t))x + (I_N \otimes B_w)w \\ z = (M(r_t) \otimes I_n)x \end{cases} \quad (14)$$

where $L(r_t)$ is the Laplacian matrix associated with jump graph $Z(r_t)$, and $\otimes$ denotes the Kronecker product of matrices.

To exploit the structure of $L(r_t)$, let $\xi = T(r_t)^{-1} \otimes I_n x$, $\tilde{z} = T(r_t)^{-1} \otimes I_n z$, $\tilde{w} = (T(r_t)^{-1} \otimes I_p)w$, then the network dynamics (14) can be transformed to the following system

$$\begin{cases} \dot{\xi} = (I_N \otimes A - J(r_t) \otimes BK(r_t))\xi + (I_N \otimes B_w)\tilde{w} \\ \tilde{z} = (T(r_t)^{-1} \otimes I_n)(M(r_t) \otimes I_n)(T(r_t) \otimes I_n)\xi \end{cases} \quad (15)$$

Performing the matrix calculation to equation (15), one has

$$\begin{aligned}
\tilde{z} &= (T(r_t)^{-1} \otimes I_n) (M(r_t) \otimes I_n) (T(r_t) \otimes I_n) \xi \\
&= T(r_t)^{-1} M(r_t) T(r_t) \otimes I_n \xi \\
&= \left(T(r_t)^{-1} I_N T(r_t) - T(r_t)^{-1} (\mathbf{1} \cdot \kappa(r_t)^T) T(r_t) \right) \otimes I_n \xi \\
&= \left(I_N - (T(r_t)^{-1} \mathbf{1}) \cdot \left(\kappa(r_t)^T T(r_t) \right) \right) \otimes I_n \xi
\end{aligned} \tag{16}$$

After calculation, equation (16) can be restated as

$$\tilde{z} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \otimes I_n \xi \tag{17}$$

For the notational convenience, let us partition the state variable ξ into ξ_i for $i = 1, \dots, N$.

Then for $i = 1, \dots, n_\lambda$, we have

$$\begin{cases} \dot{\xi}_i = (A - \lambda_i(r_t)BK(r_t))\xi_i + B_w \tilde{w}_i \\ \tilde{z}_i = \xi_i \end{cases} \tag{18}$$

and for $i = n_\lambda + 1, \dots, N - 1$

$$\begin{cases} \dot{\xi}_{i_1} = (A - \alpha_i(r_t)BK(r_t))\xi_{i_1} - \beta_i(r_t)BK(r_t)\xi_{i_2} + B_w \tilde{w}_{i_1} \\ \tilde{z}_{i_1} = \xi_{i_1} \end{cases} \tag{19}$$

$$\begin{cases} \dot{\xi}_{i_2} = (A - \alpha_i(r_t)BK(r_t))\xi_{i_2} + \beta_i(r_t)BK(r_t)\xi_{i_1} + B_w \tilde{w}_{i_2} \\ \tilde{z}_{i_2} = \xi_{i_2} \end{cases} \tag{20}$$

and for $i = N$

$$\begin{cases} \dot{\xi}_N = A\xi_N + B_w \tilde{w}_N \\ \tilde{z}_N = 0 \end{cases} \tag{21}$$

where $i_1 = 1 + n_\lambda + 2k - 1$, $i_2 = 1 + n_\lambda + 2k$, $k \in 1, \dots, n_\mu$. For notational simplicity, $r_t = r$, $r \in S$, $K(r_t)$, $\lambda_i(r_t)$, $\alpha_i(r_t)$ and $\beta_i(r_t)$ are denoted as K_r , $\lambda_{i,r}$, $\alpha_{i,r}$ and $\beta_{i,r}$, respectively. It is needed to design the mode-dependent control gain matrix K_r to ensure the states of system (1)

reach consensus performance with a constrained H_∞ disturbance attention level over the fixed frequency band $\varpi_1 \leq \omega \leq \varpi_2$. By using Lemmas 2.3-2.5, we have the following theorem.

Theorem 3.1: For the network-connected dynamic system (1) and a scalar $\gamma > 0$, if there exist symmetric positive definite matrices $\bar{P}_r, \bar{Y}$, symmetric matrix $\bar{X}$, and general matrix $\bar{F}$, such that the following LMIs for each $r \in S$:

$$\begin{pmatrix} -[\bar{F}]_z & \bar{F}^T A^T - \sigma \bar{K}^T B^T + \bar{P}_r & \bar{M} & \sqrt{\pi_{rr} + 1} \bar{F}^T \\ * & -\bar{P}_r & 0 & 0 \\ * & * & -\tilde{P}_r & 0 \\ * & * & * & -\bar{P}_r \end{pmatrix} < 0 \quad (22)$$

$$\begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2^T & \Sigma_1 \end{bmatrix} < 0 \quad (23)$$

where

$$\Sigma_1 = \begin{bmatrix} -\bar{Y} & \bar{X} - \bar{F} & 0 & 0 \\ \bar{X}^T - \bar{F}^T & -\varpi_1 \varpi_2 \bar{Y} + [A\bar{F} - \sigma B\bar{K}]_z & B_w & \bar{F}^T \\ 0 & B_w^T & -\gamma^2 I & 0 \\ 0 & \bar{F} & 0 & -I \end{bmatrix}$$

$$\Sigma_2 = \begin{bmatrix} 0 & \varpi_c \bar{Y} & 0 & 0 \\ -\varpi_c \bar{Y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{M} = \begin{pmatrix} \sqrt{\pi_{r1}} \bar{F}^T & \sqrt{\pi_{r2}} \bar{F}^T & \cdots & \sqrt{\pi_{r(r-1)}} \bar{F}^T & \sqrt{\pi_{r(r+1)}} \bar{F}^T & \cdots & \sqrt{\pi_{rs}} \bar{F}^T \end{pmatrix}$$

$$\tilde{P}_r = \text{diag}\{\bar{P}_1, \bar{P}_2, \dots, \bar{P}_{r-1}, \bar{P}_{r+1}, \dots, \bar{P}_s\}, [\bar{F}]_z = \bar{F}^T + \bar{F}$$

$$\bar{P}_r = P_r^{-1}, \bar{K} = K_r \bar{F} = K_r F^{-1}, \bar{Y} = (F^{-1})^T Y F^{-1}$$

$$\bar{X} = (F^{-1})^T X F^{-1}, \varpi_c = (\varpi_1 + \varpi_2)/2$$

$$\sigma = \min \{ \lambda_{1,r}, \dots, \lambda_{n_\lambda,r}, \alpha_{1,r}, \dots, \alpha_{n_\mu,r} \}$$

are satisfied simultaneously, the mode-dependent controller in the form of (12) can be obtained as $K_r = \bar{K} \bar{F}^{-1}$, and the closed-loop system (15) is asymptotically stable with $\sup_{\varpi_1 < \omega < \varpi_2} \|G(j\omega)\|_\infty < \gamma$.

Proof: Consider a stochastic Lyapunov function candidate as

$$V(\xi_i(t), r_t = r) = V(\xi_i, r) = \xi_i^T P_r \xi_i \quad (24)$$

where P_r is mode-dependent positive definite symmetric matrix for each r . For $i = 1, \dots, n_\lambda$, along the trajectories of system (18), the corresponding time derivative of $V(\xi_i, r)$ is given by

$$\begin{aligned} & \frac{d}{dt} E \{V(\xi_i, r)\} \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} [E \{V(\xi_i(t + \Delta t), r_{t+\Delta t}, t + \Delta t) | \xi_i(t) = \xi_i, r_t = r\} - V(\xi_i(t), r, t)] \\ &\leq \xi_i^T \left((A - \lambda_{i,r} B K_r)^T P_r + P_r (A - \lambda_{i,r} B K_r) + \sum_{l=1}^s \pi_{rl} P_l \right) \xi_i + 2\xi_i^T P_r B_w w_i \end{aligned} \quad (25)$$

For $i = n_\lambda + 1, \dots, N - 1$, we consider $V(\xi_i, r)$ in pairs for the complex conjugate eigenvalues. It can be obtained by the same procedure for the real eigenvalue case that

$$\begin{aligned} & \frac{d}{dt} E \{V(\xi_{i_1}, r)\} + \frac{d}{dt} E \{V(\xi_{i_2}, r)\} \\ &\leq \xi_{i_1}^T \left((A - \alpha_{i,r} B K_r)^T P_r + P_r (A - \alpha_{i,r} B K_r) + \sum_{l=1}^s \pi_{rl} P_l \right) \xi_{i_1} \\ &+ \xi_{i_2}^T \left((A - \alpha_{i,r} B K_r)^T P_r + P_r (A - \alpha_{i,r} B K_r) + \sum_{l=1}^s \pi_{rl} P_l \right) \xi_{i_2} \\ &+ 2\xi_{i_1}^T P_r B_w w_{i_1} + 2\xi_{i_2}^T P_r B_w w_{i_2} \end{aligned} \quad (26)$$

Firstly, we discuss the stability problem of system (15) without external disturbance, i.e. $\tilde{w} = 0$.

Letting $V(\xi(t), r) = \sum_{i=1}^{N-1} V(\xi_i, r)$, it follows that

$$\begin{aligned} & \frac{d}{dt} E \{V(\xi(t), r)\} \\ &\leq \sum_{i=1}^{n_\lambda} \xi_i^T \left((A - \lambda_{i,r} B K_r)^T P_r + P_r (A - \lambda_{i,r} B K_r) + \sum_{l=1}^s \pi_{rl} P_l \right) \xi_i \\ &+ \sum_{i=n_\lambda+1}^{N-1} \left[\xi_{i_1}^T \left((A - \alpha_{i,r} B K_r)^T P_r + P_r (A - \alpha_{i,r} B K_r) + \sum_{l=1}^s \pi_{rl} P_l \right) \xi_{i_1} \right. \\ &\left. + \xi_{i_2}^T \left((A - \alpha_{i,r} B K_r)^T P_r + P_r (A - \alpha_{i,r} B K_r) + \sum_{l=1}^s \pi_{rl} P_l \right) \xi_{i_2} \right] \end{aligned} \quad (27)$$

As long as $(A - \sigma BK_r)^T P_r + P_r (A - \sigma BK_r) + \sum_{l=1}^s \pi_{rl} P_l < 0$ holds, $\frac{d}{dt} E \{V(\xi(t), r)\} < 0$. Performing the congruence transformation to inequality (22) by $\text{diag}\{F, I, \dots, I\}$, we can transform inequality (22) into the following form:

$$\begin{pmatrix} -(F + F^T) & \bar{A}^T + F\bar{P}_r & \bar{N} & \sqrt{\pi_{rr} + 1} \\ * & -\bar{P}_r & 0 & 0 \\ * & * & -\bar{P}_r & 0 \\ * & * & * & -\bar{P}_r \end{pmatrix} < 0 \quad (28)$$

where

$$\bar{N} = \begin{pmatrix} \sqrt{\pi_{r1}} & \sqrt{\pi_{r2}} & \cdots & \sqrt{\pi_{r(r-1)}} & \sqrt{\pi_{r(r+1)}} & \cdots & \sqrt{\pi_{rs}} \end{pmatrix}$$

$$\bar{A} = A - \sigma BK_r$$

Employing Schur complement, and performing the congruence transformation to inequality (28) by $\text{diag}\{I, P_r\}$, inequality (28) is transformed into

$$\begin{pmatrix} -(F + F^T) + P_r + \sum_{l=1}^s \pi_{rl} P_l & \bar{A}^T P_r + F^T \\ * & -P_r \end{pmatrix} < 0 \quad (29)$$

By using Lemma 2.5, and letting $S^T = \bar{A}^T P_r$, $U = P_r$, inequality (29) is equivalent to

$$\bar{A}^T P_r + P_r \bar{A} + \sum_{l=1}^s \pi_{rl} P_l < 0 \quad (30)$$

Therefore, system (15) is asymptotically stable when $\tilde{w} = 0$, i.e. system (1) achieves consensus.

Now we discuss the finite frequency H_∞ performance of closed-loop system (15) with nonzero disturbance $\tilde{w}$ over fixed frequency band $\varpi_1 \leq \omega \leq \varpi_2$. Applying the approaches in [27] to inequality (23) and transforming it into another LMI with a smaller dimension in complex variables, one has

$$\begin{pmatrix} -\bar{Y} & \bar{X} + j\varpi_c \bar{Y} - \bar{F} & 0 & 0 \\ * & -\varpi_1 \varpi_2 \bar{Y} + [A\bar{F} - \sigma B\bar{K}]_z & B_w & \bar{F}^T \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -I \end{pmatrix} < 0 \quad (31)$$

Performing the congruence transformation to inequality (31) by $\text{diag}\{F, F, I, I\}$, it follows that

$$\begin{pmatrix} -Y & X + j\varpi_c Y - F & 0 & 0 \\ * & -\varpi_1 \varpi_2 Y + [F^T \bar{A}]_z & F^T B_w & I \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -I \end{pmatrix} < 0 \quad (32)$$

Rewrite inequality (32) as

$$W\Xi W^T + V\Phi V^T + [\Gamma F H]_z < 0 \quad (33)$$

where

$$\begin{aligned} W &= \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \end{pmatrix}^T, \Xi = \begin{pmatrix} -Y & X + j\varpi_c Y \\ * & -\varpi_1 \varpi_2 Y \end{pmatrix}, \Phi = \begin{pmatrix} I & 0 \\ 0 & -\gamma^2 I \end{pmatrix} \\ V &= \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}^T, \Gamma = \begin{pmatrix} -I & \bar{A} & B_w \end{pmatrix}^T, H = \begin{pmatrix} 0 & I & 0 \end{pmatrix} \end{aligned}$$

Then, according to Lemma 2.4, inequality (33) holds if and only if

$$\begin{aligned} \Gamma^\perp (W\Xi W^T + V\Phi V^T) \Gamma^{\perp T} &< 0 \\ H^{T\perp} (W\Xi W^T + V\Phi V^T) H^{\perp T} &< 0 \end{aligned} \quad (34)$$

where

$$\Gamma^\perp = \begin{bmatrix} \bar{A}^T & I & 0 \\ B_w^T & 0 & I \end{bmatrix}, H^\perp = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \end{pmatrix}^T$$

Note that inequality (34) can be transformed to the following form

$$\begin{pmatrix} \bar{A} & B_w \\ I & 0 \end{pmatrix}^T \Xi \begin{pmatrix} \bar{A} & B_w \\ I & 0 \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}^T \Phi \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} < 0 \quad (35)$$

By applying Lemma 2.3, we can obtain that

$$\begin{pmatrix} G(j\omega) \\ I \end{pmatrix}^T \Phi \begin{pmatrix} G(j\omega) \\ I \end{pmatrix} < 0, \varpi_1 \leq \omega \leq \varpi_2 \quad (36)$$

which is exactly the finite frequency H_∞ performance index inequality in inequality (4), which completes the proof.

Remark 3.1: The objective of Theorem 3.1 is to guarantee the asymptotical consensus of system (1) and transform the H_∞ performance index of system (1) from time domain to frequency domain. With the frequency band information of external disturbance, the results obtained in Theorem 3.1 are less conservative compared with the entire frequency H_∞ consensus controller.

Remark 3.2: It can be seen from Theorem 3.1 that the finite frequency H_∞ consensus of the system under consideration is influenced by not only the eigenvalues of Laplacian matrix, but also the transition rates of communication topology. Furthermore, to obtain an optimized finite frequency consensus controller, the following minimization problem can be used:

$$\begin{aligned} \min_{\bar{P}_r, \bar{X}, \bar{Y}, \bar{F}} \quad & \gamma \\ \text{s. t.} \quad & \text{LMIs (12 - 23)} \end{aligned} \quad (37)$$

Remark 3.3: Theorem 3.1 puts forward a new approach to design a consensus controller with a prescribed level of disturbance attenuation over finite frequency ranges. In order to verify the advantages of finite frequency consensus controller in attenuating disturbance, we will present the following entire frequency consensus controller design result for comparison:

$$\begin{pmatrix} (A\bar{P}_r - \sigma BY_r)^T + (A_i\bar{P}_r - BY_r) + \pi_{ii}\bar{P}_r & B_w & \bar{P}_r^T & (\bar{P}_r E)^T \\ B_w^T & -\gamma^2 I & 0 & 0 \\ \bar{P}_r & 0 & -I & 0 \\ E\bar{P}_r & 0 & 0 & -X \end{pmatrix} < 0 \quad (38)$$

where

$$E^T = \begin{pmatrix} \sqrt{\pi_{i1}} & \cdots & \sqrt{\pi_{i(i-1)}} & \sqrt{\pi_{i(i+1)}} & \cdots & \sqrt{\pi_{is}} \end{pmatrix}$$

$$X = \text{diag} \{ \bar{P}_1, \bar{P}_2, \cdots, \bar{P}_{r-1}, \bar{P}_{r+1}, \cdots, \bar{P}_s \}$$

$$Y_r = K_r \bar{P}_r$$

By solving the feasibility problem of LMIs shown in Theorem 3.1 and the inequality (38), we can eventually obtain consensus controllers for finite frequency and entire frequency domain. The comparison of them will be given in the simulation part of Section IV, and the advantages of the finite frequency band consensus controller will be presented.

IV. ILLUSTRATIVE EXAMPLE

In this section, we will study an industrial material heating unit to demonstrate the engineering application and advantages of finite frequency consensus controller over entire frequency consensus controller in rejecting the external disturbances. In the refineries or oil processing plants, the crude oil needs to be heated to the specific range of temperature before feeding to the distillation unit where the different products can be separated. The heating furnaces generally have multiple passes and a group of fuel burners. The heating balance of each pass is critical in order to achieve higher thermal efficiency and the consistent operation conditions in the downstream process. For example, the coil outlet temperature (COT) and tube skin temperature (TST) in different passes should be controlled as uniform as possible in order to increase the product yield and avoid cracking caused by underheating and/or overheating. The consensus control can be achieved by adjusting the feed flowrate of each pass and fuel supply to each burner in the furnace so that COT and TST converge to their required values.

Take a heating furnace with four passes for example. In order to build the practical useful model for control purposes, one must make a series of moves on each pass flowrate to determine the effects to its COT and TST. The moving should be made in both positive and negative directions from the current operating point to observe the process response. The data collected are fit to algorithms in order to generate a dynamic model that best represents the process. In this case, a finite impulse response is used to identify the step-response curves then convert them to the following transfer function matrix:

$$G_i(s) = \begin{bmatrix} -\frac{1}{1+5s} & \frac{0.8}{1.4s^2+4.35s+1} \\ -\frac{0.7}{1+8s} & \frac{1}{0.75s^2+3.25s+1} \end{bmatrix}$$

where $i = 1, 2, 3, 4$ is number of pipeline passes, COT and TST are the controlled variable, and the feed rate and fuel gas flow are the manipulated variables, respectively. The inert behaviour of heat transfer from combustion flame to the inside temperature of tubes generally possesses the second-order dynamic property. On the other hand, the material feed impacts the temperatures by quick mixing process and results in the first-order dynamic property.

Firstly, we convert the above transfer function models into the following four subsystems.

Each of them is described by the following state-space model with additional disturbances:

$$\begin{aligned} \dot{x}(t) = & \begin{bmatrix} -0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.125 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3.107 & -0.7143 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4.333 & -1.333 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x(t) \\ & + \begin{bmatrix} 0.5 & 0 \\ 0.25 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u(t) + \begin{bmatrix} 0.1 \\ -0.3 \\ -0.3 \\ 0.1 \\ -0.2 \\ 0.2 \end{bmatrix} \omega(t) \\ y(t) = & \begin{bmatrix} -0.4 & 0 & 0 & 0.5714 & 0 & 0 \\ 0 & -0.35 & 0 & 0 & 0 & 1.3333 \end{bmatrix} x(t) \end{aligned}$$

The interconnection topology jumps between Z_1 and Z_2 with adjacency matrix described as:

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, Q_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and the transition rate between Z_1 and Z_2 is

$$\Pi = \begin{bmatrix} -0.7 & 0.7 \\ 0.4 & -0.4 \end{bmatrix}$$

TABLE I: Minimum upper bound of disturbance rejection level γ^2 for different frequency bands

$[\varpi_1 \quad \varpi_2]$	γ^2
$\frac{9}{10}\pi \leq \omega \leq \frac{11}{10}\pi$	5.0219
$\frac{7}{10}\pi \leq \omega \leq \frac{13}{10}\pi$	5.0222
$\frac{6}{10}\pi \leq \omega \leq \frac{14}{10}\pi$	5.0247
$\frac{5}{10}\pi \leq \omega \leq \frac{15}{10}\pi$	5.0253

The external disturbance is supposed to be

$$\begin{aligned}
 w(t) &= \begin{bmatrix} w_1(t) & w_2(t) & w_3(t) & w_4(t) \end{bmatrix}^T \\
 &= \begin{bmatrix} 0.2\omega(t) & 0.5\omega(t) & 0.4\omega(t) & 0.2\omega(t) \end{bmatrix}^T
 \end{aligned}$$

with $\omega(t) = \sin(t)$.

Our purpose here is to design a finite frequency consensus controller in the form of (12) such that the COT and TST of each pass asymptotically converge to the same value with a prescribed H_∞ disturbance attenuation level over finite frequency ranges in the presence of Markov jump network topologies and external disturbance with spectral energies concentrating at narrow frequency bands. Assuming the frequency range of disturbances belong to frequency band $\frac{9}{10}\pi \leq \omega \leq \frac{11}{10}\pi$, and solving the optimization problem (37), the corresponding mode-dependent controller gains are computed as follows:

$$\begin{aligned}
 K_1 &= \begin{bmatrix} 5.7969 & 4.8247 & 0.0131 & 0.0233 & 0.0035 & -0.0256 \\ 0.1944 & -0.4020 & 3.8250 & 5.5875 & 2.9479 & 3.4506 \end{bmatrix} \\
 K_2 &= \begin{bmatrix} 5.8450 & 4.8629 & 0.0144 & 0.0246 & 0.0051 & -0.0248 \\ 0.1949 & -0.4048 & 3.8470 & 5.6055 & 2.9699 & 3.4644 \end{bmatrix}
 \end{aligned}$$

In order to investigate the effect of increasing frequency band $[\varpi_1 \quad \varpi_2]$ in the search of minimum upper bound of disturbance rejection level γ^2 obtained by the condition (37), the results of the optimization problem are summarized in Table I. As shown in Table I, the wider the frequency range, the higher the disturbance rejection level.

Fig. 1: Evolution of topology versus time

Fig. 2: COT response curves of four passes

To demonstrate the effectiveness of the design method, the COT and TST for four passes under control protocol (12) in Markov jump topologies $\{Z_1, Z_2\}$ of finite frequency approach are shown in Fig. 2 and Fig. 3 by generating a possible mode evolution of topology shown in Fig. 1. From Fig. 2 and Fig. 3, it can be seen that the COT and TST of each pass reach consensus in the presence of both disturbances and switching topologies.

Fig. 3: TST response curves of four passes

V. CONCLUSION

In this paper, the finite frequency performance synthesis conditions via consensus protocol for Markov jump networks have been addressed. By exploring certain features of Laplacian matrix and using relative state information of the system, a mode-dependent consensus protocol based on frequency and time domain inequalities is designed such that the disagreement dynamics of interconnected networks asymptotically converge to zero with the prescribed level of disturbance attenuation over finite frequency ranges. Future work will focus on the techniques to obtain the sufficient and necessary conditions of finite frequency consensus protocol to further reduce the conservativeness of the results.

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REFERENCES

- [1] J. A. Fax and R. M. Murray. Information flow and cooperative control of vehicle formation. *IEEE Transactions on Automatic Control*, 49:1465–1476, 2004.
- [2] J. Huang, S. M. Farritor, A. Qadi, and S. Goddard. Localization and follow-the-leader control of a heterogeneous group of mobile robots. *IEEE/ASME Transactions on Mechatronics*, 11(2):205–215, 2006.

- [3] W. J. Dong and J. A. Farrell. Cooperative control of multiple nonholonomic mobile agents. *IEEE Transactions on Automatic Control*, 53(6):262–268, 2009.
- [4] Y. Z. Feng, S. Y. Xu, and B. Y. Zhang. Group consensus control for double-integrator dynamic multiagent systems with fixed communication topology. *International Journal of Robust and Nonlinear Control*, 24(3):532–547, 2014.
- [5] L. Scardovi and R. Sepulchre. Synchronization in networks of identical linear systems. *Automatica*, 45(11):2557–2562, 2009.
- [6] Z. Li, Z. Duan, G. Chen, and L. Huang. Consensus of multi-agent systems and synchronization of complex networks, a unified viewpoint. *IEEE Transactions on Circuits and Systems I*, 57(1):213–224, 2010.
- [7] Z. Ding. Consensus output regulation of a class of heterogeneous nonlinear systems. *IEEE Transactions on Automatic Control*, 58:2648–2653, 2013.
- [8] Y. Hatano and M. Mesbahi. Agreement over random networks. *IEEE Transactions on Automatic Control*, 50:1867–1872, 2005.
- [9] M. Porfiri and D. J. Stilwell. Consensus seeking over random weighted directed graphs. *IEEE Transactions on Automatic Control*, 52:1767–1773, 2007.
- [10] A. Salehi and A. Jadbabaie. A necessary and sufficient condition for consensus over random networks. *IEEE Transactions on Automatic Control*, 53(3):791–795, 2008.
- [11] Z. Wang, J. X. Xi, Z. C. Yao, and G. B. Liu. Guaranteed cost consensus for multi-agent systems with switching topologies. *International Journal of Robust and Nonlinear Control*, DOI: 10.1002/rnc.325, 2014.
- [12] Y. Zhang and Y. P. Tian. Consentability and protocol design of multi-agent systems with stochastic switching topology. *Automatica*, 45:1195–1201, 2009.
- [13] G. Y. Miao, S. Y. Xu, and Y. Zou. Necessary and sufficient conditions for mean square consensus under Markov switching topologies. *International Journal of System Science*, 44(1):178–186, 2013.
- [14] K. Y. You, Z. K. Li, and L. H. Xie. Consensus condition for linear multi-agent systems over randomly switching topologies. *Automatica*, 49(10):3125–3132, 2013.
- [15] P. Lin, Y. Jia, and L. Li. Distributed robust H_∞ consensus control in directed networks of agents with time-delay. *Systems and Control Letters*, 57(8):643–653, 2008.
- [16] P. Massioni and M. Verhaegen. Distributed control for identical dynamically coupled systems: a decomposition approach. *IEEE Transactions on Automatic Control*, 55(1):124–135, 2009.
- [17] P. Lin and Y. Jia. Robust H_∞ consensus analysis of a class of second-order multi-agent systems with uncertainty. *IET Control Theory and applications*, 4(3):487–498, 2010.
- [18] Z. Li, Z. Duan, and G. Chen. On H_∞ and H_2 performance regions of multi-agent systems. *Automatica*, 47(4):797–803, 2011.
- [19] Y. Zhao, Z. Duan, G. Wen, and G. Chen. Distributed H_∞ consensus of multi-agent systems: a performance region-based approach. *International Journal of Control*, 85(3):332–341, 2012.
- [20] G. Wen, Z. Duan, Z. Li, and G. Chen. Consensus and its L_2 -gain performance of multi-agent systems with intermittent information transmissions. *International Journal of Control*, 85(4):384–396, 2012.
- [21] Z. Li, X. Liu, M. Fu, and L. Xie. Global H_∞ consensus of multiagent systems with Lipschitz nonlinear dynamics. *IET Control Theory and applications*, 6(13):2041–2048, 2012.
- [22] Z. Ding. Consensus control of a class of Lipschitz nonlinear systems. *International Journal of Control*, 87(11):2372–2382, 2014.
- [23] N. Cai, J. W. Cao, and M. J. Khan. Almost decouplability of any directed weighted network topology. *Physica A: Statistical Mechanics and its Applications*, 436:637–645, 2015.

- [24] W. W. Yu, G. R. Chen, and M. Gao. Consensus in directed networks of agents with nonlinear dynamics. *IEEE Transactions on Automatic Control*, 56(6):1436–1441, 2011.
- [25] T. Iwasaki and S. Hara. Generalized KYP lemma: unified frequency domain inequality with design applications. *IEEE Transactions on Automatic Control*, 50(1):41–59, 2005.
- [26] P. Apkarian, H. D. Tuan, and J. Bernussou. Continuous-time analysis, eigenstructure assignment, and H_2 synthesis with enhanced linear matrix inequalities (LMI) characterizations. *IEEE Transactions on Automatic Control*, 42(12):1941–1946, 2001.
- [27] P. Gahinet, A. Nemirovskii, A. J. Laub, and M. Chilali. *LMI control toolbox users guide*. MA: The Math. Works Inc., 1995.