Computation of the J_k -integrals for bimaterial interface cracks using boundary element crack shape sensitivities

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This paper presents a new algorithm for the efficient evaluation of J_k -integrals for cracks between bonded homogeneous and isotropic materials using the boundary element crack shape sensitivities (BECSS). The flexibility of this novel method allows for analysis of both curved and straight interface cracks. In contrast to the available algorithms, the present method does not require stress analysis at a series of internal points around the crack or employment of an auxiliary equation. For an interface crack, the J_1 -integral is the strain energy release rate (SERR) or the derivative of the total potential energy with respect to the crack length extension. Although the J_2 -integral shows an oscillatory type behaviour and is non-existent at the crack tip, it can also be evaluated by direct differentiation of the structural response. It is well-known that a bimaterial interface crack induces both opening and shearing behaviour even for a single mode loading. Here it is shown that the computed J_k can be used to decouple and estimate the stress intensity factors (SIFs). Here, three example problems are analysed and their J_k values are presented which are in excellent agreement with the corresponding analytical results. Each case includes the contribution to J_2 by the jump of displacement derivatives across the interface and the strain energy density discontinuity on the crack surfaces and interface region.

Keywords

Bimaterial interface crack, J_k -integral, boundary element crack shape sensitivities, debonded dissimilar materials, mixed mode fracture, stress intensity factor

1. INTRODUCTION

Composite materials have endless applications in a variety of industries including aerospace, automobile, naval and electronics. In a composite structure consisting of two or more materials with various properties such as fibre reinforced laminated composites or multilayered electronic devices, failure is more likely to initiate at interfaces. Williams [1] was the first scientist to discover that the stress field along an interface crack between two dissimilar elastic materials is not only singular, but also has an oscillatory behaviour of type $r^{-\frac{1}{2}+i\varepsilon}$ where r is the radial distance from the crack tip and ε is a bimaterial constant. This shows that when r approaches zero, the displacements and stresses change sign indefinitely and there is an interpenetration of two crack faces near the crack tip which is not physically possible. As suggested by several researchers, for most practical crack sizes and materials, this zone of contact is extremely small and typically of the order of a few nanometers. In linear elastic interfacial fracture mechanics (LEIFM), the overlap between crack faces at the crack tip is usually ignored. However, an asymptotic field characterizing the stress and strain is usually employed in the vicinity of the crack tip.

William's work was followed by the studies carried out by Rice and Sih [2], Erdogan [3-4] and England [5]. Following, their pioneering research, a variety of algorithms have been developed based on LEIFM and in conjunction with the boundary element method (BEM), finite element method (FEM) or analytical method. These methods are based on the virtual crack extension, M-integral, interaction integral, complex variable, numerical manifold, element free Galerkin, extended finite element (XFEM) and analytical mode separation [6-15].

At present, the most common method used in industry and by academia for solving fracture mechanics of homogeneous structures is the J_1 -integral [16-17] in conjunction with BEM or FEM. The J_1 -integral is the Rice's path independent integral. This method was first developed by Rice to characterize fractures for two-dimensional structures with linear or nonlinear elastic material behaviour. Although the J_1 -integral

with BEM reduces computational time, it still requires time-consuming stress analysis at a series of internal points around the crack.

For elastic problems, the J_1 -integral is the SERR per unit of the crack extension. In conjunction with the FEM or BEM, it is possible to directly evaluate the sensitivities of the total strain energy where the crack length is being treated as the shape variable. Ref. [18] presents the novel application of the BECSS for the evaluation of the J_1 -integral in anisotropic materials where the crack of arbitrary geometric shape, straight or curved, was treated as the shape design variable. Since fracture mechanics parameters were evaluated by direct differentiation of the structural response, the BECSS method is computationally more accurate and efficient than the J_1 -integral method.

For fracture of in-plane mixed mode cracks in homogeneous structures, the J_1 -integral is related to a combination of SIF values, due to the different fracture modes. The decomposition method is the most popular technique for the evaluation of SIFs [19]. An alternative for decoupling the SIFs is to evaluate the J_2 -integral which not only involves the computation of stresses and strains at a series of internal points around the crack but also the evaluation of highly singular integrals over the crack surfaces. In Ref. [20] this deficiency is overcome by direct evaluation of J_2 using BECSS where a small region around the crack tip is treated as the shape design variable. It is shown that the derivative of the total potential energy with respect to the transverse direction of the crack is not the J_2 -integral. However, by addition of an integral, involving the strain energy density discontinuity, to this derivative the J_2 -integral can be efficiently evaluated. That study focused on isotropic and homogeneous materials. For the sake of validation the selected case studies with known analytical solutions were employed where for each crack shape and loading condition, the corresponding values of J_1 , J_2 and also the contribution to J_2 from the strain energy density discontinuity were presented.

In Ref. [21], using the BECSS of multi-region domains, coupled with an optimization algorithm and an automatic mesh generator, the crack kink angle and crack

propagation path in anisotropic and homogeneous elastic solids, based on the maximum SERR criterion, were predicted. In contrast to the J_1 -integral method, the computation of stresses and strains at a series of internal points during the automatic incremental crack procedure was not required. Therefore, the method was more accurate and efficient. The prediction of the crack propagation path of a central slant crack in a titanium plate subject to tension was in very good agreement with the corresponding experimental results published elsewhere. The findings confirmed the simplicity, accuracy and flexibility of the method which can be applied to both curved and straight cracks.

Here, the J_k -integrals for interface cracks between bonded homogeneous, isotropic and dissimilar materials are obtained using the BECSS. It is demonstrated how a good estimation of SIFs can be made using the computed J_k values. Three example problems with straight or curved interface cracks are analysed and their corresponding J_k values are presented. The results include the contribution to J_2 from the jump of displacement derivatives or strain across the interface and also the strain energy density discontinuity on the crack surfaces and interface region.

Nomenclature

a	=	half crack length or crack half chord length
a_1, a_2	=	local coordinates at the crack tip
$C_{jk}(P)$	=	limiting value of the surface integral of $T_{jk}(P,Q)$
E	=	Young's modulus
F_i	=	shape functions for the continuous quadratic elements
f_i	=	shape functions for the discontinuous quadratic elements
E_p	=	potential energy
G	=	shear modulus
I_{sc}	=	integral involving strain energy density and strain discontinuities
J_k	=	($k=1,2$), J-Integral
K_1, K_2	=	modes 1 and 2 stress intensity factors, respectively
N_i ($i=1,3$)	=	geometrical nodes
N_{ci} ($i=1,3$)	=	collocation nodes

n_1, n_2	=	direction cosines of the unit outward normal vector to the surface of the elastic body
P	=	load point at the surface of the elastic domain
Q	=	field point at the surface of the elastic domain
r	=	small radial distance from the crack tip
R	=	radius of the circular inclusion
S	=	domain boundary
$T_{jk}(P,Q)$	=	jth component of the traction vector at point Q due to a unit point load in the kth direction at P
t_j	=	traction vector
U_{jk}	=	jth component of the displacement vector at point Q due to a unit point load in the kth direction at P
u_j	=	displacement vector
W	=	strain energy density
x_i	=	rectangular Cartesian coordinates (Global)
ε	=	bimaterial constant
ν	=	Poisson's ratio
θ_0	=	half the central crack angle
σ_{ij}	=	stress tensor
σ_{x0}, σ_{y0}	=	normal stress components in the x- and y-directions, respectively
η	=	ratio E_1/E_2
τ_0	=	shear stress component
ψ_i	=	$\frac{3-\nu_i}{1+\nu_i}$ Plane stress , $3-4\nu_i$ Plane strain
ζ_1, ζ_2	=	coordinates of load point
ζ	=	natural coordinate

2. REVIEW OF THE BOUNDARY ELEMENT CRACK SHAPE SENSITIVITY ANALYSIS

The BEM is based on the unit load solutions in an infinite body, known as the fundamental solutions; used with the reciprocal work theorem and appropriate limit

operations. The Boundary Integral Equation (BIE) of the BEM for homogeneous and isotropic materials is an integral constraint equation relating boundary tractions (t_j) and boundary displacements (u_j) and it may be written as [22]

$$C_{ij}u_j(P) + \int T_{ij}(P,Q)u_j(Q)ds(Q) = \int U_{ij}(P,Q)t_j(Q)ds(Q) \quad i, j = 1, 2 \quad (1)$$

$P(\zeta_1, \zeta_2)$ and $Q(x_1, x_2)$ are the load and field points, respectively. Following the numerical integration, BIE can then be reduced to a set of simultaneous linear equations and be solved. The constant C_{ij} depends on the local geometry of the boundary at P , whether it is smooth or sharp. For a general crack problem involving a homogeneous body with mixed mode deformation or an interface crack between bonded dissimilar materials, the domain may be divided into several subregions in which the crack faces coincide with the boundaries of the subregions [23]. The BIE can then be employed for each subregion (L), in turn. Then appropriate continuity and equilibrium conditions are applied at the common sub-region interface before the linear algebraic equations are solved.

Shape sensitivity analysis (SSA) is the calculation of quantitative information on how the response of a structure is affected by changes in the variables that define its shape. SSA is the fundamental requirement for shape optimization. The BEM, being a surface oriented technique, is well suited for shape and topology optimization problems, in particular for SSA [24-26]. In order to obtain sensitivities of the structural response with respect to a design variable, both sides of equation (1) can be directly differentiated. For calculation of the sensitivities of a multiregion domain, direct differentiation of the BIE for each region with respect to a typical design variable (x_h) can be performed.

$$\begin{aligned} C_{jk}^{(L)} \frac{\partial u_j^{(L)}(P)}{\partial x_h} + \frac{\partial C_{jk}^{(L)}}{\partial x_h} u_j^{(L)}(P) + \int_{s(L)} \left[\frac{\partial T_{jk}^{(L)}(P,Q)}{\partial x_h} u_j^{(L)}(Q) + T_{jk}^{(L)}(P,Q) \frac{\partial u_j^{(L)}(Q)}{\partial x_h} \right] ds^{(L)}(Q) + \\ \int_{s(L)} \left[T_{jk}^{(L)}(P,Q) u_j^{(L)}(Q) \right] \frac{\partial [ds^{(L)}(Q)]}{\partial x_h} = \int_{s(L)} \left[\frac{\partial U_{jk}^{(L)}(P,Q)}{\partial x_h} t_j^{(L)}(Q) + U_{jk}^{(L)}(P,Q) \frac{\partial t_j^{(L)}(Q)}{\partial x_h} \right] ds^{(L)}(Q) \\ \int_s U_{jk}^{(L)}(P,Q) t_j^{(L)}(Q) \frac{\partial [ds^{(L)}(Q)]}{\partial x_h} \quad j, k = 1, 2 \end{aligned} \quad (2)$$

This is followed by differentiation of the boundary constraints at the common subregion interfaces [27].

In Ref. [28], the effect of material properties on the optimum positioning of voids or cutouts of arbitrary shapes in anisotropic and homogeneous structures was investigated. The direct differentiation of the BIE with respect to the boundary nodes was initially carried out. Then, by the application of design velocity fields, the shape variables, such as void positions and/or orientations were related to the relevant boundary nodes. Based on this principle, in Ref. [20] the BECSS analysis of the J_k -integrals for straight or curved cracks in isotropic and homogeneous materials was performed. Here using a similar method, the J_k -integrals for an interface crack between bonded dissimilar, isotropic and homogeneous materials are evaluated.

3. CRACK MODELLING

In the present study, straight semi-discontinuous quadratic quarter point elements [29] are used for the crack tips and standard quadratic elements are used for the rest of the boundary.

Fig. 1a shows a discontinuous or non-conforming quadratic element where the two extreme collocation nodes are shifted towards the inside of the element [29]. N_i and N_{c_i} ($i=1,3$) are the geometrical and collocation nodes, respectively. The geometry of the semi-discontinuous quadratic element is represented using the same shape functions (F_i) as the standard continuous elements where

$$F_1 = \frac{1}{2}\zeta(\zeta - 1) \quad , \quad F_2 = (1 - \zeta^2) \quad , \quad F_3 = \frac{1}{2}\zeta(\zeta + 1) \quad (3)$$

The boundary variables such as tractions and displacements are represented in terms of their values at the collocation nodes (N_{c_i}) and the shape functions (f_i) where

$$f_1 = \frac{(\zeta - \zeta_2)(\zeta - \zeta_3)}{(\zeta_1 - \zeta_2)(\zeta_1 - \zeta_3)} \quad , \quad f_2 = \frac{(\zeta - \zeta_1)(\zeta - \zeta_3)}{(\zeta_2 - \zeta_1)(\zeta_2 - \zeta_3)} \quad , \quad f_3 = \frac{(\zeta - \zeta_1)(\zeta - \zeta_2)}{(\zeta_3 - \zeta_1)(\zeta_3 - \zeta_2)} \quad (4)$$

Fig. 1b shows a straight discontinuous quarter point quadratic element with the mid node located at one quarter of the element length. This element is able to represent $r^{1/2}$ singularity for the displacement variation along the crack [29].

Here, considering that the rest of the domain boundary is discretized to standard continuous quadratic elements, each crack tip is modelled using a straight semi-discontinuous quadratic quarter point element, as shown in Fig. 1c. For this element only the collocation node at the crack tip is shifted towards inside of the element. The collocation nodes for the crack tip elements are either located at $\left(-\frac{3}{4}, 0., 1.0\right)$ or $\left(-1.0, 0., \frac{3}{4}\right)$.

For the curved cracks, a very small straight semi-discontinuous quadratic quarter point element is used at each side of the crack tip and curved quadratic elements are used for the rest of the boundary [30]. For more information the reader may refer to Refs. [29-30]

In the vicinity of the interface crack tips, similar to homogeneous materials, the singularities of the displacements and tractions are proportional to $r^{1/2}$ and $r^{-1/2}$, respectively. However, using the BEM with quadratic quarter point crack tip elements, both displacement and traction variations have $r^{1/2}$ singularity. In order to achieve the $r^{-1/2}$ singularity for the traction variation, the shape functions can be multiplied by $\sqrt{r/L}$ where L is the length of the crack tip element [23].

4. J_k-INTEGRALS FOR BIMATERIAL INTERFACE CRACKS

Fig. 2 shows a bimaterial interface crack between two bonded isotropic and homogeneous elastic materials. ε is the bimaterial constant which is defined as

$$\varepsilon = \frac{1}{2\pi} \ln \left(\frac{\psi_1 G_2 + G_1}{\psi_2 G_1 + G_2} \right) \quad (5)$$

G_i, ν_i ($i = 1, 2$) are the shear moduli and Poisson's ratios of the materials, respectively [2]. The parameter ψ_i is defined as

$$\psi_i = \begin{cases} \frac{3-\nu_i}{1+\nu_i} & \text{Plane stress} \\ 3-4\nu_i & \text{Plane strain} \end{cases} \quad (6)$$

The local coordinate system (a_1, a_2) , as shown in Fig. 2, has its origin (o) at the tip of the crack. a_1 and a_2 are the axes in the tangent and transverse directions of the crack, respectively. (x_1, x_2) is the global coordinate system which is fixed in space.

Now assume material 1 is divided into the subregions 1a and 1b while material 2 is divided into the subregions 2a and 2b. S_{1a} is a contour in material 1 surrounding the crack tip which includes $S_p^+, S_{p_c}^+, S_r^+$, and $S_{U_c}^+$. S_{2a} is a contour in material 2 surrounding the crack tip which includes $S_p^-, S_{p_c}^-, S_r^-$ and $S_{U_c}^-$. S_{1a} and S_{2a} exclude the crack tip and r is a small radial distance from the crack tip. $S_{p_c}^+$ and $S_{p_c}^-$ are the upper and lower crack surfaces, respectively. $S_{U_c}^+$ and $S_{U_c}^-$ are the upper and lower interface surfaces near the crack tip, respectively. The boundary of region 1b includes $S_p^+, S_{I_c}^+, S_1$ and $S_{O_c}^+$. The boundary of region 2b includes $S_p^-, S_{O_c}^-, S_2$ and $S_{I_c}^-$. $S_{O_c}^+$ and $S_{O_c}^-$ are the portions of the upper and lower crack surfaces, respectively. $S_{I_c}^+$ and $S_{I_c}^-$ are the portions of the upper and lower interface surfaces, respectively. S_p^+ is the interface between the regions 1a and 1b. Therefore, the tractions on this interface are equal and in opposite directions. Similarly, S_p^- is the interface between the regions 2a and 2b and the tractions on this interface are also equal and in opposite directions.

J_k -integrals are the line integrals evaluated along a counterclockwise contour enclosing and shrinking onto the crack tip where

$$J_k \equiv \lim_{r \rightarrow 0} \oint_S (W n_k - t_j u_{j,k}) ds \quad (7)$$

where $j, k=1, 2$. W is the strain energy density, n_k are the components of the unit outward normal to the contour path, t_i and u_i are the traction and displacement components along the path.

For a traction free straight crack, the J_1 -integral vanishes along the crack surfaces and considering that the tractions on the interface (S_{Uc}) are equal and in opposite directions, therefore [7],

$$J_1 = \int_{S_p} (Wn_1 - t_j u_{j,1}) ds \quad (8)$$

where $S_p \equiv S_p^+ + S_p^-$. This integral would require the computation of stresses and strains at a series of internal points around the path S_p . The J_2 -integral can be written as

$$J_2 = \int_{S_p + S_{Pc} + S_{Uc}} (Wn_2 - t_j u_{j,2}) ds \quad (9)$$

where $S_{Pc} \equiv S_{Pc}^+ + S_{Pc}^-$, $S_{Uc} \equiv S_{Uc}^+ + S_{Uc}^-$. The crack surfaces are traction free and there is continuity of tractions across the interface. Although, the displacement field across the interface is continuous, its first derivative or the strain field is discontinuous. Therefore,

$$J_2 = \int_{S_p} (Wn_2 - t_j u_{j,2}) ds + \int_{S_{Pc} + S_{Uc}} \langle \langle W \rangle \rangle n_2 ds - \int_{S_{Uc}} t_i \langle \langle \frac{\partial u_i}{\partial a_2} \rangle \rangle ds \quad (10)$$

where each double angle bracketed term denotes the jump of the corresponding function across the crack surface and/or the interface. The above equation not only involves evaluation of stresses and strains at a series of internal points but also integration of highly singular integrals over the crack surfaces and the interface region.

It should be noted that using equation 8, the contribution of crack face integrals to J_1 vanishes when straight crack faces are considered where $n_1 = 0$. and $t_j = 0$. If curved crack faces are considered, e.g. a crack as shown in Fig. 2, then $n_1 \neq 0$. and $t_j = 0$. [20].

For a linear elastic, isotropic and homogeneous material in mixed mode fracture, the SIFs and J_k -integrals are related by the following relations,

$$J_1 = \frac{K_1^2 + K_2^2}{E} \quad (11)$$

$$J_2 = -\frac{2K_1 K_2}{E}$$

As shown, J_1 -integral is related to a combination of SIFs. This means for decoupling of SIFs, the computation of both J_1 and J_2 is required. In Ref. [20] by direct differentiation of the structural response, the sensitivities of the total potential energy of a cracked structure with respect to its crack shape variation in the a_k -direction were related to the J_k -integrals and the SIFs were directly evaluated without any auxiliary equations. Due to the generic nature of the method, the algorithm can be applied to both curved and straight cracks. Here the algorithm is expanded and used for evaluation of the J_k -integrals for a bimaterial interface crack.

5. RELATIONS BETWEEN THE J_k -INTEGRALS AND SIFS OF A BIMATERIAL INTERFACE CRACK

The following shows how the SIFs of a bimaterial interface crack are related to the corresponding J_k -integrals.

The complex forms of SIFs and J can be written as

$$K = K_1 - iK_2 \quad (12)$$

$$J = J_1 + iJ_2 \quad (13)$$

For bimaterial interface cracks between two dissimilar homogeneous and isotropic materials, Khandelwal and Chadra Kishen [10] showed that J can be expressed in terms of the SIFs as

$$J = \frac{(K_1^2 + K_2^2)}{B_\varepsilon} + i \lim_{r \rightarrow 0} \left[(K_1^2 - K_2^2) S_\varepsilon + 2K_1 K_2 C_\varepsilon \right] \quad (14)$$

where

$$\frac{1}{B_\varepsilon} = \left[\frac{1 + \psi_1}{16G_1} + \frac{1 + \psi_2}{16G_2} \right] \quad (15)$$

$$D_\varepsilon = -\frac{1}{32\pi\varepsilon} \left[\frac{1 + \psi_1}{G_1} (1 - e^{-2\pi\varepsilon}) + \frac{1 + \psi_2}{G_2} (e^{2\pi\varepsilon} - 1) \right] \quad (16)$$

$$S_\varepsilon = D_\varepsilon \sin(2\varepsilon \ln r) \text{ and } C_\varepsilon = D_\varepsilon \cos(2\varepsilon \ln r).$$

From equations 13 and 14, it can be concluded that

$$J_1 = \frac{(K_1^2 + K_2^2)}{B_\varepsilon} \quad (17)$$

Equations 13-16 show that J_2 has an oscillatory type behaviour because of the terms $\sin(2\varepsilon \ln r)$ and $\cos(2\varepsilon \ln r)$ and does not exist when $r \rightarrow 0$. However, Khandelwal and Chadra Kishen [10] in conjunction with the FEM evaluated J_{2r} around a contour path of small radius from the crack tip within the singularity dominated zone where

$$J_{2r} = (K_1^2 - K_2^2)S_\varepsilon + 2K_1K_2C_\varepsilon \quad (18)$$

For a cracked homogeneous material, where $\varepsilon=0$, equations 17 and 18 will be identical to equations 11.

For a bimaterial interface crack once J_1 and J_{2r} are calculated, K_1 and K_2 can be determined using the following equations,

$$K_1 = \pm \sqrt{\frac{[J_1 B_\varepsilon (S_\varepsilon^2 + C_\varepsilon^2) + J_{2r} S_\varepsilon] \pm \sqrt{[J_1 B_\varepsilon (S_\varepsilon^2 + C_\varepsilon^2) + J_{2r} S_\varepsilon]^2 - (S_\varepsilon^2 + C_\varepsilon^2)(J_{2r} + J_1 B_\varepsilon S_\varepsilon)^2}}{2(S_\varepsilon^2 + C_\varepsilon^2)}} \quad (19)$$

$$K_2 = \pm \sqrt{\frac{[J_1 B_\varepsilon (S_\varepsilon^2 + C_\varepsilon^2) + J_{2r} S_\varepsilon] \pm (-1) \sqrt{[J_1 B_\varepsilon (S_\varepsilon^2 + C_\varepsilon^2) - J_{2r} S_\varepsilon]^2 - (S_\varepsilon^2 + C_\varepsilon^2)(J_{2r} - J_1 B_\varepsilon S_\varepsilon)^2}}{2(S_\varepsilon^2 + C_\varepsilon^2)}}$$

The above equations are obtained by combining equations 17 and 18 and then rearranging them with respect to K_1 and K_2 . The signs of SIFs can be determined by checking the values of the crack opening displacements near the crack tip which are readily available by BEM.

6. EVALUATION OF THE J_k -INTEGRALS USING THE BECSS

For a homogeneous elastic solid, free of body forces with the domain A and boundary S the total potential energy (E_p) is

$$(E_p) = \int_A W dA - \int_{S_t} t_i u_i dS \quad (20)$$

S_t is the portion of the boundary with prescribed boundary conditions. The strain energy density (W) can be written in terms of the stress (σ_{ij}) and strain (ε_{ij}) tensors,

$$W = \int \sigma_{ij} d\varepsilon_{ij} \quad (21)$$

where

$$t_i = \sigma_{ij} n_j \quad (22)$$

Due to equilibrium

$$\sigma_{ij,j} = 0. \quad (23)$$

The total potential energy (E_p) of the bimaterial domain shown in Fig. 2 is equal to the summation of the potential energies of materials 1 and 2.

$$E_p = (E_p)_1 + (E_p)_2 \quad (24)$$

Considering that each material is divided into two subregions, therefore,

$$E_p = (E_p)_{1a} + (E_p)_{1b} + (E_p)_{2a} + (E_p)_{2b} \quad (25)$$

For regions 1a and 2a, the potential energies are

$$(E_p)_{1a} = \int_{A_{1a}} W dA - \int_{S_{1a}} t_i u_i dS \quad (26)$$

$$(E_p)_{2a} = \int_{A_{2a}} W dA - \int_{S_{2a}} t_i u_i dS$$

Now assume that the interface along S_{uc} is intact and the region 1a+2a has a small translation in the a_k direction without any shape variation. Therefore, the derivative of the potential energy of this region can be written as

$$\frac{\partial (E_p)_{1a+2a}}{\partial a_k} = \int_{A_{1a}} \frac{\partial W}{\partial a_k} dA - \int_{S_{1a}} \left(\frac{\partial t_i}{\partial a_k} u_i + t_i \frac{\partial u_i}{\partial a_k} \right) dS + \int_{A_{2a}} \frac{\partial W}{\partial a_k} dA - \int_{S_{2a}} \left(\frac{\partial t_i}{\partial a_k} u_i + t_i \frac{\partial u_i}{\partial a_k} \right) dS \quad (27)$$

Differentiating the strain energy density we have [16] ,

$$\frac{\partial W}{\partial a_k} = \sigma_{ij} \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial a_k} \right) = \frac{\partial}{\partial x_j} \left(\sigma_{ij} \frac{\partial u_i}{\partial a_k} \right) \quad (28)$$

After integrating the above equation over the region 1a+2a and application of the Green's theory [31] we have,

$$\int_{A_{1a}} \frac{\partial W}{\partial a_k} dA + \int_{A_{2a}} \frac{\partial W}{\partial a_k} dA = \int_{S_{1a}} \left(t_i \frac{\partial u_i}{\partial a_k} \right) dS + \int_{S_{2a}} \left(t_i \frac{\partial u_i}{\partial a_k} \right) dS \quad (29)$$

Substituting equation (29) in equation (27), the derivative of the potential energy of region 1a+2a can be written as

$$\frac{\partial (E_p)_{1a+2a}}{\partial a_k} = - \int_{S_{1a}} \left(\frac{\partial t_i}{\partial a_k} u_i \right) dS - \int_{S_{2a}} \left(\frac{\partial t_i}{\partial a_k} u_i \right) dS \quad (30)$$

The tractions on the crack surfaces are equal to zero, therefore, the above equation can be written as,

$$\frac{\partial(E_p)_{1a+2a}}{\partial a_k} = - \int_{S_{p^+}} \left(\frac{\partial t_i}{\partial a_k} u_i \right) dS - \int_{S_{p^-}} \left(\frac{\partial t_i}{\partial a_k} u_i \right) dS \quad (31)$$

Similarly, the potential energy of region 1b+2b can be written as

$$(E_p)_{1b+2b} = (E_p)_{1b} + (E_p)_{2b} = \int_{A_{1b}} W dA - \int_{S_{1b}} t_i u_i dS + \int_{A_{2b}} W dA - \int_{S_{2b}} t_i u_i dS \quad (32)$$

and its derivative as

$$\frac{\partial(E_p)_{1b+2b}}{\partial a_k} = \int_{S_{1b}} \frac{\partial W}{\partial a_k} dA - \int_{S_{1b}} \left(\frac{\partial t_i}{\partial a_k} u_i + t_i \frac{\partial u_i}{\partial a_k} \right) dS + \int_{S_{2b}} \frac{\partial W}{\partial a_k} dA - \int_{S_{2b}} \left(\frac{\partial t_i}{\partial a_k} u_i + t_i \frac{\partial u_i}{\partial a_k} \right) dS \quad (33)$$

Using the Green's theory, the above equation can be written as

$$\frac{\partial(E_p)_{1b+2b}}{\partial a_k} = \int_{S_{1b}} W n_k dS - \int_{S_{1b}} \left(\frac{\partial t_i}{\partial a_k} u_i + t_i \frac{\partial u_i}{\partial a_k} \right) dS + \int_{S_{2b}} W n_k dS - \int_{S_{2b}} \left(\frac{\partial t_i}{\partial a_k} u_i + t_i \frac{\partial u_i}{\partial a_k} \right) dS \quad (34)$$

The variation of the potential energy of the region 1b+2b is due to the translation of region 1a+2a while its outer boundary and crack surfaces remain unchanged. Therefore,

$$\frac{\partial(E_p)_{1b+2b}}{\partial a_k} = \int_{S_{p^+}} W n_k dS - \int_{S_{p^+}} \left(\frac{\partial t_i}{\partial a_k} u_i + t_i \frac{\partial u_i}{\partial a_k} \right) dS + \int_{S_{p^-}} W n_k dS - \int_{S_{p^-}} \left(\frac{\partial t_i}{\partial a_k} u_i + t_i \frac{\partial u_i}{\partial a_k} \right) dS \quad (35)$$

By summation of equations (35) and (31) and considering that the tractions (t_i) on the interface regions S_{p^+} (between regions 1a and 1b) and S_{p^-} (between regions 2a and 2b) have equal values but are in opposite directions then

$$\frac{\partial(E_p)}{\partial a_k} = \int_{S_p} \left(W n_k - t_i \frac{\partial u_i}{\partial a_k} \right) ds \quad (36)$$

For a straight crack, if the derivative is with respect to the crack direction (a_1) then the above integral is the J_1 -integral.

$$\frac{\partial(E_p)}{\partial a_1} = J_1 \quad (37)$$

If the derivative is with respect to the transverse direction of the crack (a_2), then equation 36 is not the J_2 -integral but by addition of the integral I_{Sc} to it, J_2 can be easily obtained.

$$J_2 = \frac{\partial(E_p)}{\partial a_2} + I_{Sc} \quad (38)$$

where

$$I_{Sc} = \int_{S_{P_c}^+ + S_{P_c}^- + S_{U_c}^+ + S_{U_c}^-} \left(W n_k - t_i \frac{\partial u_i}{\partial a_2} \right) ds \quad (39)$$

By comparing equations 39 and 10, it can be concluded that,

$$I_{Sc} = \int_{S_{P_c} + S_{U_c}} \langle \langle W \rangle \rangle n_2 ds - \int_{S_{U_c}} t_i \langle \langle \frac{\partial u_i}{\partial a_2} \rangle \rangle ds \quad (40)$$

As shown, I_{Sc} includes two integrals containing the crack tip singularity. The first integral involves the strain energy density discontinuities on the crack surfaces and interface region. The second integral involves the jump of the displacement derivative or strain in the interface region.

Here for calculation of the J_2 -integral, the line integrals in equation 40 were only evaluated for the semi-discontinuous quarter point crack tip elements [20]. On the other hand, the J_1 -integral or the strain energy release rate is obtained by direct differentiation of the structural response where the length of the curved or straight crack is treated as the shape design variable. Therefore, for evaluation of J_1 the integration over the curved crack surfaces is not required. This method was successfully applied in Ref. [21] for prediction of the crack propagation path in anisotropic solids where the strain energy release rates at the existing crack tip and new cracks for the period of crack growth were determined.

7. RESULTS AND DISCUSSIONS

The method presented here has been applied to three different interface crack problems and the results are compared with the corresponding analytical solutions or existing published results. The straight or curved cracks are located at the interface of two bonded dissimilar homogeneous and isotropic materials. Throughout the paper,

the properties of $E_1 = 100GPa$, $\nu_1 = 0.3$ and $E_2 = E_1 / \eta$, $\nu_2 = 0.3$ are employed for the materials 1 and 2, respectively. η is a constant parameter.

The presented examples include a bimaterial thin flat plate with a central straight crack subject to biaxial and shear loads (Fig. 3), an arc crack along the interface of a circular inclusion embedded in a matrix subject to tension (Fig. 4) and a side crack in a rectangular bimaterial plate subject to tension. Although, for the first two examples, the analytical solutions for the corresponding infinite bimaterial cracked plates have been used as reference solutions, the ratios $\frac{H}{W} \geq 1.5$ and $\frac{a}{W} \leq 0.1$ gave a very good agreement. H and W are the plate's width and height, respectively. a is the half crack length for the central straight cracks, half the chord length for the arc cracks and the crack length for the side cracks. For the arc crack, as shown in Fig. 4, $a = R \sin \theta_0$.

The first example is a thin bimaterial plate with a central straight crack of $2a$ subject to biaxial and shear stresses, as shown in Fig. 3. Here, a series of analyses were performed for the following stress magnitudes $\sigma_{y0} = 4(\sigma_{x0})_1 = 8\tau_0 = 200MPa$ where $a=10mm$. The ratio η varied from 1.0 to 10.0, with an increment of 1.0. To ensure strain continuity along the bimaterial interface, a stress jump across the interface has been considered [2] where

$$(\sigma_{x0})_2 = \omega(\sigma_{x0})_1 + \frac{\omega(e^{2\pi\epsilon} - 3) + (3e^{2\pi\epsilon} - 1)}{(1 + e^{2\pi\epsilon})}(\sigma_{y0}) \quad (41)$$

where $\omega = \frac{G_2(\psi_1 + 1)}{G_1(\psi_2 + 1)}$. The present numerical results for J_1 and J_{2r} with respect to η

are shown in Fig. 6. Table 1 shows the derivative $\frac{dE_p}{da_2}$ and also the contribution to J_2

from the integral I_{Sc} for each value of η . I_{Sc} is calculated using equation 40 and as mentioned earlier, it includes the jump of displacement derivatives or strain across the interface and also the strain energy density discontinuity on the crack surfaces and interface region.

Since $v_1=v_2$, the case $\eta=1$ is a central crack in a homogeneous plate. This case was also studied in Ref. [20] in which the integral I_{sc} calculated using BECSS was in good agreement with the corresponding analytical result obtained by Eischen [32]. For a homogeneous plate the integral I_{sc} only involves the contribution to J_2 due to the strain energy density discontinuity [20]. Based on the theoretical solution given by Herrman and Herrman [33], Eischen [32] analytically calculated I_{sc} for a homogeneous plate ($\eta=1, v_1=v_2$) with a central crack subject to biaxial and shear stresses (Fig. 3).

For the cases when $\eta \geq 1$, the analytical solutions for the SIFs at the tip of the crack of this example can be obtained using the following equations, [2]

$$\begin{aligned} K_1 &= \frac{\sqrt{\pi a}}{\cosh(\pi \varepsilon)} \{ \sigma_{y0} [\cos(\varepsilon \ln 2a) + 2\varepsilon \sin(\varepsilon \ln 2a)] + \tau_0 [\sin(\varepsilon \ln 2a) - 2\varepsilon \cos(\varepsilon \ln 2a)] \} \\ K_2 &= \frac{\sqrt{\pi a}}{\cosh(\pi \varepsilon)} \{ \tau_0 [\cos(\varepsilon \ln 2a) + 2\varepsilon \sin(\varepsilon \ln 2a)] - \sigma_{y0} [\sin(\varepsilon \ln 2a) - 2\varepsilon \cos(\varepsilon \ln 2a)] \} \end{aligned} \quad (42)$$

It can be observed that when $\varepsilon \neq 0$, even for a symmetric loading condition ($\tau_0 = 0$), the plate is still subject to mixed mode fracture and therefore both SIFs exist.

Here the analytical values of J_1 for this example are calculated using equations 17 and 42 and presented in Fig. 6 which are in very good agreement with the corresponding numerical results obtained in this study. The analytical values of J_{2r} are also calculated using equations 18 and 42 for different values of r and illustrated in Fig. 6 where $2 \times 10^{-3} \leq r/a \leq 1 \times 10^{-5}$. It can be observed that following a convergence test the present numerical results for J_{2r} are very close to the corresponding analytical values of J_{2r} when $r/a = 1 \times 10^{-5}$.

The next stage is to evaluate the SIFs from the computed values of J_1 and J_{2r} . In a homogeneous material ($\varepsilon=0$), once J_k -integrals are calculated, using equations 11, the corresponding SIFs can be decoupled and calculated. However, in order to decouple and calculate SIFs for an interface crack ($\varepsilon \neq 0$) using the equations 17-19, the

correct value of r is required in advance. The following demonstrates how a good estimation of SIFs can be made using the computed J_k -integrals. For example, for the cases $\eta = 2.0$ and $\eta = 4.0$, the present numerical values of J_k -integrals are given in Table 2. Using this numerical data with three different values of r/a ($2.0E-4$, $1.0E-4$ and $1.0E-5$) and in conjunction with equations 19, the SIFs are calculated and presented in Table 2. Using equations 42, the corresponding analytical SIFs are also obtained and added to Table 2. It can be seen that the average of the present K_1 or K_2 values in each column is in good agreement with its corresponding analytical value [2].

The next example involves a debonded crack at the interface between a circular inclusion and a rectangular matrix which is subjected to remote uniaxial tension of $\sigma_0 = 100MPa$. The crack spans at an angle $2\theta_0$ as shown in Fig. 4 and is symmetric about the y -axis. The analytical solution for the complex SIF of an infinite plate with a central circular inclusion given by Perlman and Sih [34] is

$$K_1 - iK_2 = \sigma_0 \sqrt{\pi a} \frac{(1 - 2i\varepsilon)\beta}{1 + \alpha} e^{\varepsilon(\pi + \theta_0) - i\left[\frac{\theta_0}{2} - \varepsilon \ln(2a)\right]} \times \left[\frac{(M_1^2 - M_2) - (1 + \gamma)}{N_1 e^{2\varepsilon\theta_0} - \alpha(1 + 2\gamma)} + e^{-(2\varepsilon + i)\theta_0} \right] \quad (43)$$

where the parameters $\alpha, \beta, \gamma, M_1, M_2, N_1$ are defined in Appendix A.

Here, the J_k values for two different half crack angles (θ_0), 20° and 30° , are calculated and compared with the corresponding analytical results. The ratio η varies from 1 to 10, with an increment of 1.0. Here the inclusion and matrix are considered as materials 1 and 2, respectively. The radius of the inclusion is $R=1.0$ and $a = \sin \theta_0$.

Figs. 7 and 8 show the variation of the numerical and analytical values of J_1 vs η for $\theta_0 = 20^\circ$ and 30° , respectively. Table 3 shows the corresponding values of the derivative dE_p/da_2 and integral I_{Sc} for each value of η . The analytical values of J_1 are calculated using equations 17 and 43 which are almost identical to the present results. The analytical values of J_{2r} vs η for $\theta_0 = 20^\circ$ and 30° are calculated using equations 18 and 43 for different values of r and also illustrated in Figs. 6-7 where

$2 \times 10^{-3} \leq r/R \leq 1 \times 10^{-5}$. It can be observed that following a convergence test the present numerical results for J_{2r} are very close to the corresponding analytical values of J_{2r} when $r/R = 1 \times 10^{-4}$.

The next example is an edge crack with length a in a bimaterial plate subject to the remote uniaxial tension of $\sigma_0 = 100 \text{ MPa}$, as shown in Fig. 5. The analyses were carried out for the crack length $\frac{a}{w} = 0.1$, $\frac{H}{w} = 1.5$. The ratio η varied between 1.0 and 10.0 with an increment of 1.0 and also between 10.0 and 100.0 with an increment of 10.0. Fig. 9 shows the present J_k values vs η . Table 4 shows the corresponding values of the derivative dE_p/da_2 and integral I_{Sc} for each value of η .

This example was also analysed in Refs [7-8] in which the SIFs were obtained either using the M_1 -integral [7] or the virtual crack extension [8] method in conjunction with BEM. Having their calculated SIF values [7] and using equations 17-18, here their corresponding J_k -integrals are calculated and presented in Fig. 9. For calculation of their J_{2r} , the parameter $\frac{r}{a}$ varies between 2×10^{-3} and 1×10^{-5} . It should be noted that the SIFs presented in Refs [7-8] are only for the ratios $\eta = 1.0, 2.0, 3.0, 4.0, 10.0$ and 100.0 . Fig. 9 shows that the present J_1 values are almost identical to their J_1 values [7]. It can be observed that following a convergence test the present numerical results for J_{2r} is very close to their corresponding analytical values of J_{2r} when $r/a = 1 \times 10^{-4}$.

For the models analysed there were about 6-10 elements on either half the crack length (examples 1 and 2) or the full crack length (example 3). For each case, the convergence test was performed based on the J_1 or SERR. For the sake of consistency, once the value of J_1 was considered satisfactory, for the same mesh the corresponding value of dE_p/da_2 was added to the I_{Sc} integral to calculate J_{2r} . I_{Sc} was only computed for the crack tip elements and the shape sensitivities for E_p were obtained with respect to the movement of the crack tip in the a_k ($k=1,2$) directions.

CONCLUSION

Following a brief review of the mathematical basis of the BIE and SSA, the BECSS for evaluation of the J_k -integrals for the interface crack between two bonded homogeneous and isotropic materials was discussed. The CSS were evaluated by direct differentiation of the structural response for multi-region domains using the BEM. For calculation of the J_1 -integral the crack length was selected as the shape design variable and the SERR was directly determined. For calculation of the J_2 -integral a small region around the crack tip was treated as the shape design variable. For an interface crack, J_2 depends on r , has an oscillatory behavior and is non-existent for $r \rightarrow 0$. For this reason, instead of J_2 , the notation of J_{2r} was used throughout this paper. Here, it was shown that J_{2r} is equal to the summation of the derivative of the total potential energy with respect to the transverse direction of the crack and two integrals involving the crack tip elements. These integrals were related to the jump of displacement derivatives or strain across the interface and also the strain energy density discontinuity on the crack surfaces and interface region. It was shown that using J_1 and J_{2r} with non-dimensional value of r (r/a or r/R) varying between 0.0001 and 0.00001, the SIF values can be well estimated. The results show that the method is simple, accurate, flexible and applicable to both curved and straight cracks. Due to generic nature, this algorithm can be expanded and applied to the fracture analysis of homogeneous or bimaterials with anisotropic material properties. It can also be employed for the prediction of the crack kink angle and propagation path of interface cracks for isotropic and/or anisotropic bimaterials.

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Appendix A

The parameters in equation 43 are defined by equations A-1 as follows: [34]

$$\begin{aligned}
 \alpha &= \frac{G_1 + G_2 \psi_1}{G_2 + G_1 \psi_2} \\
 \beta &= \frac{G_1(1 + \psi_2)}{G_2 + G_1 \psi_2} \\
 \gamma &= \frac{G_1(1 + \psi_2)}{G_2(1 + \psi_1)} \\
 M_1 &= \cos \theta_0 + 2\varepsilon \sin \theta_0 \\
 N_1 &= \cos \theta_0 - 2\varepsilon \sin \theta_0 \\
 M_2 &= \frac{1}{4} + \varepsilon^2 + \left(\frac{3}{4} - \varepsilon^2 \right) \cos 2\theta_0 + 2\varepsilon \sin (2\theta_0)
 \end{aligned} \tag{A-1}$$

Table 1 Numerical values of dE/da_2 and I_{Sc} for a rectangular plate with a central straight crack subject to biaxial and shear stresses

($\sigma_{y0} = 4\sigma_{x0} = 8\tau_0 = 200MPa$), (Fig. 3)

$\eta=E_1/E_2$	dE_p/da_2 (mJ/mm ²)	I_{Sc} (mJ/mm ²)
1	-1.612 [Ref.20]	-1.520 [Ref.20]
2	-4.234	-9.541
3	-6.538	-14.331
4	-8.705	-17.731
5	-10.759	-19.818
6	-12.761	-21.305
7	-14.674	-22.386
8	-16.620	-23.101
9	-18.512	-23.597
10	-20.406	-23.927

Table 2 SIFs for a rectangular plate with a central straight crack subject to biaxial and shear stresses

$(\sigma_{y0} = 4\sigma_{x0} = 8\tau_0 = 200MPa)$, (Fig. 3)

	E ₁ /E ₂ =2.0 J ₁ =18.988, J _{2i} =-13.775 mJ/mm ² (present)		E ₁ /E ₂ =4.0 J ₁ =31.061, J _{2i} =-26.436 mJ/mm ² (present)	
	K ₁	K ₂	K ₁	K ₂
r=0.0002	35.384	3.719	35.219	1.435
r=0.0001	35.469	2.802	35.247	0.222
r=0.00005	35.529	1.884	35.198	1.878
Average SIFs (Present)	35.461	2.802	35.221	1.179
Analytical SIFs [2]	35.477	2.692	35.225	1.286

Table 3 Numerical values of dE/da_2 and I_{sc} for the plate with a circular inclusion (Fig. 4)

E_1/E_2	$\theta_0=20^\circ$		$\theta_0=30^\circ$	
	dE_p/da_2 (mJ/mm ²)	I_{sc} (mJ/mm ²)	dE_p/da_2 (mJ/mm ²)	I_{sc} (mJ/mm ²)
1	-0.305	-0.303	-0.556	-0.559
2	-1.226	-0.856	-1.192	-1.411
3	-2.082	-1.409	-2.204	-2.065
4	-2.580	-1.904	-2.632	-2.669
5	-3.021	-2.364	-3.080	-3.188
6	-3.319	-2.805	-3.398	-3.679
7	-3.542	-3.235	-3.615	-4.139
8	-3.709	-3.646	-3.628	-4.594
9	-3.804	-4.066	-3.624	-5.034
10	-3.880	-4.469	-3.625	-5.457

Table 4 Numerical values of dE/da_2 and I_{Sc} for a rectangular plate with a side crack (Fig. 5)

E_1/E_2	$dEp/da_2(\text{mJ/mm}^2)$	$I_{Sc} (\text{mJ/mm}^2)$	E_1/E_2	$dEp/da_2(\text{mJ/mm}^2)$	$I_{Sc} (\text{mJ/mm}^2)$
1	-2.73E-03	-1.20E-04	20	-12.751	-23.347
2	-0.795	-3.267	30	-19.213	-29.850
3	-1.859	-5.235	40	-25.602	-37.185
4	-2.713	-6.634	50	-32.024	-43.897
5	-3.529	-7.493	60	-38.416	-50.439
6	-3.619	-10.827	70	-44.891	-60.798
7	-4.286	-12.076	80	-51.199	-63.618
8	-4.951	-13.203	90	-57.648	-70.444
9	-5.609	-14.236	100	-58.766	-120.835
10	-6.270	-15.204

Fig. 1 a) Discontinuous quadratic element
b) Discontinuous quadratic quarter point element
c) Semi-discontinuous quadratic quarter point element

Fig. 3 A plate with an interface crack subject to biaxial and shear stresses

Fig. 4 An arc crack with the central angle of $2\theta_0$ and chord length of $2a$ along the interface of a circular inclusion embedded in a plate subject to tension

**Fig. 5 Side crack in a rectangular bimaterial plate
subject to tension**

Fig. 6 Variation of the numerical and analytical values of J_1 and J_{2r} vs E_1/E_2 for the plate with the central crack subject to biaxial and shear stresses

Fig. 7 Variation of the numerical and analytical values of J_1 and J_{2r} vs E_1/E_2 for an arc crack ($\theta_0 = 20^\circ$) along the interface of a circular inclusion embedded in a plate subject to tension

Fig. 8 Variation of the numerical and analytical values of J_1 and J_{2r} vs E_1/E_2 for an arc crack ($\theta_0 = 30^\circ$) along the interface of a circular inclusion embedded in a plate subject to tension

Fig. 9 Variation of the numerical and analytical values of J_1 and J_{2r} vs E_1/E_2 for the plate with the side crack subject to tension