Efficient evaluation of the J_2 -integral using the boundary element crack shape sensitivities

A Tafreshi

School of Mechanical, Aerospace and Civil Engineering

University of Manchester, M13 9PL, United Kingdom

azam.tafreshi@manchester.ac.uk

At present the most common method used in the industry for fracture analysis is the J_1 -integral in conjunction with the Finite Element (FE) or Boundary Element (BE) methods. The J_1 -integral requires time-consuming stress analysis at a series of internal points around the crack front. In addition, for fracture of in-plane mixed mode crack problems, the J_1 -integral is related to a combination of stress intensity factors (SIFs), due to the different fracture modes. Therefore, additional analysis must be carried out to separate SIFs. This paper presents the novel application of BE crack shape sensitivities for evaluation of the J_2 -integral. The J_2 -integral combined with the J_1 -integral or strain energy release rate (SERR), which was also previously determined using the same method, can be employed for efficient decoupling and evaluation of SIFs. Since the J_k -integrals are calculated by direct differentiation of the structural response without any additional computations or auxiliary equations, this method is more accurate and efficient. Its generality allows for the analysis of both curved and straight cracks. Three example problems with isotropic and homogeneous material properties are presented and the results are in excellent agreement with the corresponding analytical results. . For each crack shape and loading condition, the corresponding values of J_1 , J_2 and also the contribution to J_2 from the strain energy density discontinuity are presented.

Keywords

Boundary element method, J_k -integral, crack shape sensitivities, stress intensity factor, mixed mode fracture

1. INTRODUCTION

At present, the most common method used in industry and by academia for solving fracture mechanics problems is the J_1 -integral [1-2] in conjunction with the boundary element method (BEM) [3] or the finite element method (FEM) [4]. Obviously, the J_1 -integral with BEM reduces the computational time, but it would still require stress analysis at a series of internal points around the crack. The J_1 -integral is the Rice's path independent integral. This method was first developed by Rice to characterize fracture for two-dimensional structures with linear or nonlinear elastic material behavior.

For fracture of in-plane mixed mode crack problems, the J_1 -integral is related to a combination of the values of SIFs, due to the different fracture modes. An alternative to decouple the SIFs is to evaluate the J_2 -integral. However, the J_2 -integral not only involves the computation of stresses and strains at a series of internal points around the crack, but also the evaluation of highly singular integrals over the crack surfaces. There are many other methods currently employed for separating the SIFs. The decomposition method is the most popular technique for this purpose [4].

For elastic problems, the J_1 integral represents the energy release rate of a self-similar growing crack. In conjunction with the FEM or BEM, it is possible to employ shape sensitivity analysis to directly evaluate the sensitivities of the total strain energy in which the crack length is being treated as the shape variable. This approach has been used by a few other researchers [5] who have mainly studied the line cracks in homogeneous and isotropic materials and in which only J_1 has been evaluated. Reddy and Rao [6] used the continuum shape sensitivity analysis of a mixed-mode fracture in functionally graded materials using the FEM. Ref. [7] shows the evaluation of the J_k integrals in homogeneous and isotropic materials using the FE shape sensitivities in which only straight cracks were analysed.

Ref. [8] presents the novel application of the BEM for evaluation of the crack shape sensitivities of anisotropic structures. The crack of arbitrary geometric shape was treated as the shape design variable. The shape variable was associated with the coordinates of a series of nodes located at the crack surfaces. Thus, the relevant

velocity terms were applied together in the sensitivity analysis to directly determine the SERR or the J_1 -integral. Since fracture mechanics parameters were evaluated by direct differentiation of the structural response, the method is computationally more accurate and efficient than the J_1 -integral.

In another study [9], using the BE crack shape sensitivities of multi-region domains coupled with an optimization algorithm and an automatic mesh generator, the crack kink angle and crack propagation path in anisotropic elastic solids, based on the maximum SERR criterion, were predicted. In contrast to the J_1 -integral method, the computation of stresses and strains at a series of internal points during the automatic incremental crack procedure was not required. Therefore, the method was more accurate and efficient. In order to determine the SERR in each increment, the length of each kinked crack was treated as the shape design variable. This shape variable was then associated with the coordinates of the boundary nodes located on the new crack surfaces. Thus, the relevant velocity terms were applied in the sensitivity analysis to determine SERR. In order to validate the application of this formulation, the results of two example problems with different anisotropic material properties were compared with the corresponding experimental and/or analytical results in which a good agreement was observed. The crack propagation path of a central slant crack in a titanium plate subject to tension was in very good agreement with the corresponding experimental results published elsewhere. The generality of the method allows for the analysis of curved cracks as well as straight cracks. However, for fracture analysis and calculation of the SIFs for the mixed mode loading cases, an auxiliary relation based on the ratio of relative displacements was still required to be used with J_1 to decouple SIFs [8]. Here, this deficiency has been overcome by direct evaluation of both J_1 and J_2 in one pass using BEM crack shape sensitivities. Therefore, the present method is accurate, reliable and efficient. This study focuses on isotropic and homogeneous materials and for the sake of validation the selected case studies with known analytical solutions are employed. For each crack shape and loading condition, the corresponding values of J_1 , J_2 and also the contribution to J_2 from the strain energy density discontinuity are presented.

Nomenclature

a	=	crack length or crack half chord length
a_1, a_2	=	local coordinates at the crack tip
$C_{jk}(P)$	=	limiting value of the surface integral of $T_{jk}(P,Q)$
E	=	Young's modulus
E_P	=	Potential energy
$I_{S_{PC}}$	=	Strain energy density discontinuity term
J_k	=	($k=1,2$), J-Integral
K_I, K_{II}	=	modes I and II stress intensity factors, respectively
n_1, n_2	=	direction cosines of the unit outward normal vector to the surface of the elastic body
P	=	load point at the surface of the elastic domain
Q	=	field point at the surface of the elastic domain
r	=	Radius of the arc crack
S	=	domain boundary
$T_{jk}(P,Q)$	=	j th component of the traction vector at point Q due to a unit point load in the k th direction at P
t_j	=	traction vector
U_{jk}	=	j th component of the displacement vector at point Q due to a unit point load in the k th direction at P
u_j	=	displacement vector
W	=	strain energy density
x_i	=	rectangular Cartesian coordinates (Global)
α	=	half crack angle
β	=	slant crack orientation angle

ν = Poisson's ratio

σ_{ij} = stress tensor

σ_{x0}, σ_{y0} , = Normal stress components in the x- and y-directions, respectively

η = Biaxiality ratio

τ_0 = Shear stress component

ζ_1, ζ_2 = Coordinates of load point

2. REVIEW OF THE BOUNDARY ELEMENT CRACK SHAPE SENSITIVITY ANALYSIS

The BEM is based on the unit load solutions in an infinite body, known as the fundamental solutions; used with the reciprocal work theorem and appropriate limit operations. The Boundary Integral Equation (BIE) of the BEM for homogeneous and isotropic materials is an integral constraint equation relating boundary tractions (t_j) and boundary displacements (u_j) and it may be written as [3]

$$C_{ij}u_j(P) + \int T_{ij}(P,Q)u_j(Q)ds(Q) = \int U_{ij}(P,Q)t_j(Q)ds(Q) \quad i, j = 1, 2 \quad (1)$$

$P(\zeta_1, \zeta_2)$ and $Q(x_1, x_2)$ are the load and field points, respectively. Following the numerical integration, BIE can then be reduced to a set of simultaneous linear equations and be solved. The constant C_{ij} depends on the local geometry of the boundary at P, whether it is smooth or sharp. An elastic body with several cracks may be divided into several subregions in which the crack faces coincide with the boundaries of the subregions [10]. The BIE can then be employed for each subregion (L), in turn. Then appropriate continuity and equilibrium conditions are applied at the common sub-region interface before the linear algebraic equations are solved.

Shape sensitivity analysis (SSA) is the calculation of quantitative information on how the response of a structure is affected by changes in the variables that define its shape. SSA is the fundamental requirement for shape optimization. The BEM, being a surface oriented technique, is well suited for shape and topology optimization

problems, in particular for SSA [11-13]. In order to obtain sensitivities of the structural response with respect to a design variable, both sides of equation (1) can be directly differentiated [14]. For calculation of the sensitivities of a multiregion domain, direct differentiation of the BIE for each region with respect to a typical design variable (x_h) can be performed.

$$\begin{aligned}
 C_{jk}^{(L)} \frac{\partial u_j^{(L)}(P)}{\partial x_h} + \frac{\partial C_{jk}^{(L)}}{\partial x_h} u_j^{(L)}(P) + \int_{s(L)} \left[\frac{\partial T_{jk}^{(L)}(P, Q)}{\partial x_h} u_j^{(L)}(Q) + T_{jk}^{(L)}(P, Q) \frac{\partial u_j^{(L)}(Q)}{\partial x_h} \right] ds^{(L)}(Q) + \\
 \int_{s(L)} \left[T_{jk}^{(L)}(P, Q) u_j^{(L)}(Q) \right] \frac{\partial [ds^{(L)}(Q)]}{\partial x_h} = \int_{s(L)} \left[\frac{\partial U_{jk}^{(L)}(P, Q)}{\partial x_h} t_j^{(L)}(Q) + U_{jk}^{(L)}(P, Q) \frac{\partial t_j^{(L)}(Q)}{\partial x_h} \right] ds^{(L)}(Q) \\
 \int_s U_{jk}^{(L)}(P, Q) t_j^{(L)}(Q) \frac{\partial [ds^{(L)}(Q)]}{\partial x_h} \quad j, k = 1, 2
 \end{aligned} \tag{2}$$

This will be followed by differentiation of the boundary constraints at the common subregion interfaces [14].

The design velocity field describes the movements of the material points due to a change in the shape of the structural domain. If a relationship exists between the coordinates of a boundary point in the structure and the geometric/shape variable, then the design velocity term will be the derivative of the point position with respect to that variable. In Ref. [15], the effect of material properties on the optimum positioning of voids or cutouts of arbitrary shapes in anisotropic structures was investigated. The direct differentiation of the BIE with respect to the boundary nodes was initially carried out. Then, by the application of design velocity fields, the shape variables, such as void positions and/or orientations were related to the relevant boundary nodes. The same principle will be used in the current study for the evaluation of the J_2 integrals.

As mentioned earlier, although in Ref. [8] the J_1 -integral was directly evaluated, for mixed mode loading, an auxiliary relationship in terms of displacements was still required to be used with J_1 to separate SIFs. This deficiency has been overcome here by evaluation of both J_1 and J_2 simultaneously using the BE crack shape sensitivities.

3. EVALUTION OF THE J_k -INTEGRALS USING THE CRACK SHAPE SENSITIVITIES

Fig. 1 shows a crack of arbitrary shape within a continuum. S_1 is a contour surrounding the crack tip which includes $(S_{PC})^+$, $(S_{PC})^-$ and S_p . $(S_{PC})^+$, $(S_{PC})^-$ are the upper and lower crack surfaces near the crack tip while S_p is a circular curve starting from the lower crack surface and ending at the upper crack surface. For problems containing a crack tip in linear elasticity, the mixed-mode singular near-tip stress field can be characterized by a pair of SIFs (K_I and K_{II}). SIFs characterize the stress fields in the vicinity of the crack tip with respect to a local Cartesian system (a_1 , a_2) that has its origin (o) at the tip of the crack. a_1 and a_2 are the axes in the tangent and transverse directions of the crack, respectively, as shown in Fig. 1. (x_1, x_2) is the global coordinate system which is fixed in space. For such a material of unit thickness containing a traction free crack, the J_k -integral in the absence of body forces is defined as [4].

$$J_k \equiv \lim_{S_1 \rightarrow 0} \oint_{S_1} (W n_k - t_j u_{j,k}) ds \quad (3)$$

Where $j, k=1, 2$. W is the strain energy density, n_k are the components of the unit outward normal to the contour path, t_i and u_i are the traction and displacement components along the path.

For traction free straight cracks, the J_1 -integral vanishes along the crack surfaces therefore,

$$J_1 = \int_{S_p} (W n_1 - t_j u_{j,1}) ds \quad (4)$$

This integral would require the computation of stresses and strains at a series of internal points around the path S_p .

The J_2 -integral can be written as

$$J_2 = \int_{S_p} (W n_2 - t_j u_{j,2}) ds + \int_{(S_{PC})^+} W n_2 ds + \int_{(S_{PC})^-} W n_2 ds \quad (5)$$

This not only involves evaluation of stresses and strains at a series of internal points but also integration of highly singular integrals over the crack surfaces.

For a linear elastic, isotropic and homogeneous material in mixed mode fracture, the SIFs and J_k -integrals are related by the following relations [16].

$$J_1 = \frac{K_I^2 + K_{II}^2}{E} \quad (6)$$

$$J_2 = -\frac{2K_I K_{II}}{E} \quad (7)$$

As shown, the J_1 -integral is related to a combination of SIFs. This means for decoupling of SIFs, the computation of both J_1 and J_2 is required. The following shows how the sensitivities of the total potential energy of a cracked structure with respect to its crack shape variation in the a_k direction are related to the J_k -integrals. Due to the generic nature of the present method, this algorithm can be applied to curved as well as straight cracks.

For a continuum, free of body forces and cracks, with the domain A and boundary S the total potential energy (E_p) is

$$(E_p) = \int_A W dA - \int_{S_t} t_i u_i dS \quad (8)$$

S_t is the portion of the boundary with prescribed boundary conditions. The strain energy density (W) can be written in terms of the stress (σ_{ij}) and strain (ε_{ij}) tensors,

$$W = \int \sigma_{ij} d\varepsilon_{ij} \quad (9)$$

where

$$t_i = \sigma_{ij} n_j \quad (10)$$

Due to equilibrium

$$\sigma_{ij,j} = 0. \quad (11)$$

Now assume the continuum shown in Fig. 1 is divided into two subregions, 1 and 2. The boundary of region 1 with the area A_1 is S_1 which surrounds the crack tip. The boundary of region 2 with the area A_2 , includes S_O , S_P , $(S_{OC})^+$ and $(S_{OC})^-$. S_O is the outer boundary while $(S_{OC})^+$ and $(S_{OC})^-$ are the portions of the upper and lower crack surfaces, respectively. S_P is the interface between the two regions. Since the continuum is in static equilibrium, then the tractions on the interface boundary must be equal and in opposite directions.

The total potential energy (E_p) of the continuum is equal to the summation of the potential energies of regions 1 and 2.

$$E_p = (E_p)_1 + (E_p)_2 \quad (12)$$

For region 1 the potential energy is

$$(E_p)_1 = \int_{A_1} W dA - \int_{S_1} t_i u_i dS \quad (13)$$

Now assume the region 1 has a small translation in the a_k direction without any shape variation [15]. Therefore, the derivative of the potential energy of this region can be written as

$$\frac{\partial (E_p)_1}{\partial a_k} = \int_{A_1} \frac{\partial W}{\partial a_k} dA - \int_{S_1} \left(\frac{\partial t_i}{\partial a_k} u_i + t_i \frac{\partial u_i}{\partial a_k} \right) dS \quad (14)$$

Differentiating the strain energy density we have [1] ,

$$\frac{\partial W}{\partial a_k} = \sigma_{ij} \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial a_k} \right) = \frac{\partial}{\partial x_j} \left(\sigma_{ij} \frac{\partial u_i}{\partial a_k} \right) \quad (15)$$

After integrating the above equation over the region 1 and application of the Green's theory [17] we have,

$$\int_{A_1} \frac{\partial W}{\partial a_k} dA = \int_{A_1} \frac{\partial}{\partial x_j} \left(\sigma_{ij} \frac{\partial u_i}{\partial a_k} \right) dA = \int_{S_1} \left(\sigma_{ij} n_j \frac{\partial u_i}{\partial a_k} \right) dS = \int_{S_1} \left(t_i \frac{\partial u_i}{\partial a_k} \right) dS \quad (16)$$

Substituting equation (16) in equation (14), the derivative of the potential energy of region 1 can be written as

$$\frac{\partial (E_p)_1}{\partial a_k} = - \int_{S_1} \left(\frac{\partial t_i}{\partial a_k} u_i \right) dS \quad (17)$$

The tractions on the crack surfaces are equal to zero, therefore, the above equation can be written as,

$$\frac{\partial (E_p)_1}{\partial a_k} = - \int_{S_p} \left(\frac{\partial t_i}{\partial a_k} u_i \right) dS \quad (18)$$

Similarly, the potential energy of region 2 can be written as

$$(E_p)_2 = \int_{A_2} W dA - \int_{S_2} t_i u_i dS \quad (19)$$

and its derivative as

$$\frac{\partial (E_p)_2}{\partial a_k} = \int_{S_2} \frac{\partial W}{\partial a_k} dA - \int_{S_2} \left(\frac{\partial t_i}{\partial a_k} u_i + t_i \frac{\partial u_i}{\partial a_k} \right) dS \quad (20)$$

Using the Green's theory, the above equation can be written as

$$\frac{\partial(E_p)_2}{\partial a_k} = \int_{S_2} W n_k dS - \int_{S_2} \left(\frac{\partial t_i}{\partial a_k} u_i + t_i \frac{\partial u_i}{\partial a_k} \right) dS \quad (21)$$

The variation of the potential energy of region 2 is due to the translation of region 1 while its outer boundary and crack surfaces remain unchanged. Therefore,

$$\frac{\partial(E_p)_2}{\partial a_k} = \int_{S_p} W n_k dS - \int_{S_p} \left(\frac{\partial t_i}{\partial a_k} u_i + t_i \frac{\partial u_i}{\partial a_k} \right) dS \quad (22)$$

By summation of equations (22) and (18) and considering that the tractions (t_i) on the interface region (S_p) in regions 1 and 2 have equal values but being in opposite directions then

$$\frac{\partial(E_p)}{\partial a_k} = \int_{S_p} \left(W n_k - t_i \frac{\partial u_i}{\partial a_k} \right) ds \quad (23)$$

For a straight crack, if the derivative is with respect to the crack direction (a_1) then the above integral is the J_1 -integral.

$$\frac{\partial(E_p)}{\partial a_1} = J_1 \quad (24)$$

If the derivative is with respect to the transverse direction of the crack (a_2), then equation 23 is not the J_2 -integral but by addition of the integral $I_{S_{PC}}$ to it, J_2 can be easily obtained.

$$\frac{\partial(E_p)}{\partial a_2} = J_2 + I_{S_{PC}} \quad (25)$$

where

$$I_{S_{PC}} = \int_{(S_{PC})^+} W n_2 ds + \int_{(S_{PC})^-} W n_2 ds \quad (26)$$

As shown, $I_{S_{PC}}$ includes the integrals involving the crack tip singularity. Here, the line integrals in equation 26 were only calculated for the singular quarter point crack tip elements. The same integration method as for the non-singular elements but with lower number of integration points was employed. A relatively finer mesh near each crack tip was also required. Here, it is shown that using the BEM crack shape sensitivities, J_2 can be obtained directly with excellent accuracy and then SIFs can be decoupled efficiently without any additional computation or auxiliary equation in terms of displacements.

It should be noted that using equation 23, the contribution of crack face integrals to J_1 vanishes when straight crack faces are considered, where $n_1 = 0$ and $t_j = 0$. If curved crack faces are considered, e.g. a crack as shown in Fig. 1, then $n_1 \neq 0$ and $t_j = 0$. In this case the jump of the strain energy density along crack faces must be considered for the calculation of J_1 in the same manner as for J_2 to provide the path-independence [18].

In reference [8], by direct differentiation of the structural response, the strain energy release rate at the crack tip for anisotropic materials was obtained where the length of the curved or straight crack was treated as the shape design variable. Using this method J_1 is directly obtained and the integration over the curved cracks is not required. This method was successfully applied in reference [9] for prediction of the crack propagation path in anisotropic solids where the strain energy release rates at the existing crack tip and new cracks for the period of crack growth were determined.

4. RESULTS AND DISCUSSIONS

The method presented here has been applied to three mixed mode crack problems and the results have been compared with the corresponding analytical solutions. The material properties of $E=100$ GPa and $\nu=0.3$ are employed throughout the paper. E and ν are Young's modulus and Poisson's ratio, respectively. The examples include a thin flat plate having different crack shapes and loading conditions as follows: a central straight crack subject to biaxial and shear loads (Fig. 2), a central slant crack subject to biaxial loads (Fig. 3) and a central arc crack subject to biaxial loads (Fig.4). Although, the solutions for the corresponding infinite cracked plates have been used as reference solutions, the ratio of $H/W=1.5$ and $a/W=0.1$ gave a very good agreement for all the cases. H and W are the plate's width and height, respectively. a is the half crack length for the straight cracks and half the chord length for the arc cracks. For the sake of comparison, $a=10$ mm was kept the same for all the models. Here, the results for the right crack tips are presented. In example 3, the left and right crack tips have the same K_I and J_1 values while K_{II} and J_2 values are equal in magnitudes but opposite in sign.

The first example is a thin flat plate with a central crack of $2a$ subject to biaxial and shear stresses σ_{x0} , σ_{y0} and τ_0 , as shown in Fig. 2. This is a very good example to assess the accuracy in calculation of J_2 and also the contribution to J_2 from the strain energy density discontinuity because for this case an exact analytical relation for the stresses and displacements is already available [19]. The stress intensity factors of this plate are

$$\begin{aligned} K_I &= \sigma_{y0} \sqrt{\pi a} \\ K_{II} &= \tau_0 \sqrt{\pi a} \end{aligned} \quad (27)$$

If W^+ and W^- are the magnitudes of the strain energy density on the top and bottom crack surfaces, respectively, the term $(W^+ - W^-)$ represents a discontinuity or jump in the strain energy density across the crack opening. This term is given by Herrman and Herrman [20] as

$$(W^+ - W^-) = \frac{-4\tau_0(\sigma_{x0} - \sigma_{y0})}{E} \times \frac{a_1}{(a^2 - a_1^2)^{1/2}} \quad (28)$$

Eischen [19] obtained the contribution to J_2 from the strain energy density discontinuity exactly by integrating the above equation along half the crack length which is

$$I_{S_{pc}} = \frac{4K_{II}(\sigma_{x0} - \sigma_{y0})\sqrt{a}}{E\sqrt{\pi}} \quad (29)$$

Here, an analysis was performed for the following stress magnitudes $\sigma_{y0} = 4\sigma_{x0} = 8\tau_0 = 200 \text{ MPa}$. The numerical results using the BEM crack shape

sensitivities for J_1 , J_2 , $I_{S_{pc}}$ and $\frac{dE_p}{da_2}$ are presented in Table (1) and then compared with

the corresponding analytical results obtained using equations (6), (7), (27) and (29). It

should be noted that the analytical value of $\frac{dE_p}{da_2}$ is obtained by subtracting the

analytical value of $I_{S_{pc}}$ from J_2 . As shown, a very good agreement between the numerical and analytical results can be observed.

Next, the effect of the size of the crack tip element on the accuracy of $I_{S_{pc}}$, $\frac{dE_p}{da_2}$, J_1

and J_2 was investigated. The above example was meshed with 90 isoparametric quadratic elements. The crack size and loading condition remained the same while

varying the element bias factor (EBF) towards the crack tips. The EBF is the ratio between the lengths of two adjacent elements. There were 8 elements on each half crack length. Since there are two crack tips, the line of crack was treated as two geometric lines. This allowed the elements to be graded from small (at one crack tip) to larger and back down to small again at the other crack tip. The numerical values of $I_{\text{Spc}}, \frac{dE_p}{da_2}, J_1$ and J_2 were initially normalized with respect to their corresponding analytical values, as presented in Table 1, and then plotted vs the EBF, as shown in Fig. 5. It can be seen that for EBF greater than 1.4, a good agreement between the analytical and the corresponding numerical results of J_2 can be observed. The accuracy of the numerical values is almost independent of EBF when EBF is greater than 1.6. The accuracy of the derivatives $\frac{dE_p}{da_1} \equiv J_1$ and $\frac{dE_p}{da_2}$ were generally very good even with a relatively coarse mesh and a low EBF.

It is evident that the above analytical solution [19] is invalid for certain magnitudes of the remotely applied loads when the crack surfaces come into contact.

The next example is a thin flat plate with a central slant crack of length $2a$ oriented at an angle of β measured clockwise to the direction of the load. The plate is subject to a biaxial state of stress where $\sigma_{y0} = \sigma_0 = 100\text{MPa}$ and $\sigma_{x0} = \eta\sigma_0$, as shown in Fig. 3. η is the stress biaxiality ratio. The analytical solutions for SIFs for a traction free slant crack in an infinite plate subject to a similar loading condition are [21]

$$\begin{aligned} K_I &= \frac{\sigma_0 \sqrt{\pi a}}{2} [(1+\eta) - (1-\eta)\cos 2\beta] \\ K_{II} &= \frac{\sigma_0 \sqrt{\pi a}}{2} [(1-\eta)\sin 2\beta] \end{aligned} \quad (30)$$

Here using the present method, the numerical values of the J_k -integrals are calculated for different slant angles $15^\circ \leq \beta \leq 90^\circ$ and biaxiality ratios of $\eta = 0, 0.5, 1.0$. The corresponding analytical results are obtained using the equations 6, 7 and 30. Fig. 6 shows the variation of the analytical and numerical values of the J_k -integrals with respect to the slant angle which are in very good agreement. Table 2 shows the

derivative $\frac{dE_p}{da_2}$ and also the contribution to J_2 from the strain energy density discontinuity for each crack orientation and loading condition.

The final example is a thin flat plate with a central arc crack subject to biaxial tensile stresses, as shown in Fig. 4, where $\sigma_{y0} = \sigma_0 = 100MPa$ and $\sigma_{x0} = \eta\sigma_0$. α is the half crack central angle and a is the chord half length. The exact solution of Muskhelishvili [21] for a circular arc crack has been widely used [18, 22] to study curved crack behavior in an infinite homogeneous and isotropic elastic material. This analytical solution is not valid for certain orientations and magnitudes of the applied stresses due to contact of the crack surfaces. In a recent study, Ritz and Pollard [23] have shown that by addition of a complementary algorithm to a BEM program, the local displacements, stresses and SIFs for circular arc cracks that undergo partial closure can be accurately calculated.

From the stress function given by Muskhelishvili [21], Cotterel and Rice [24] formulated the SIFs for a traction free arc crack in an infinite plate loaded by remote stresses in terms of α and a . Based on their solution, the SIFs for an infinite plate subject to the biaxial tensile loads of σ_{y0} and σ_{x0} are

$$K_I = \sqrt{\pi a} \left\{ \left[\left(\frac{\sigma_{y0} + \sigma_{x0}}{2} \right) - \left(\frac{\sigma_{y0} - \sigma_{x0}}{2} \right) \sin^2 \frac{\alpha}{2} \cos^2 \frac{\alpha}{2} \right] \frac{\cos \frac{\alpha}{2}}{\left(1 + \sin^2 \frac{\alpha}{2} \right)} + \left(\frac{\sigma_{y0} - \sigma_{x0}}{2} \right) \cos \frac{3\alpha}{2} \right\}$$

$$K_{II} = \sqrt{\pi a} \left\{ \left[\left(\frac{\sigma_{y0} + \sigma_{x0}}{2} \right) - \left(\frac{\sigma_{y0} - \sigma_{x0}}{2} \right) \sin^2 \frac{\alpha}{2} \cos^2 \frac{\alpha}{2} \right] \frac{\sin \frac{\alpha}{2}}{\left(1 + \sin^2 \frac{\alpha}{2} \right)} + \left(\frac{\sigma_{y0} - \sigma_{x0}}{2} \right) \sin \frac{3\alpha}{2} \right\}$$

(31)

Here using the present method, the numerical values of the J_k -integrals are calculated for different arc angles $0 \leq 2\alpha \leq 80^\circ$ and loading conditions. Here, a is kept constant and the radius (r) varies according to $\frac{a}{\sin \alpha}$. As α goes to zero, the arc length goes to $2a$ which is a straight crack. The corresponding analytical results are obtained using

the equations 6, 7 and 31. Fig. 7 shows the variation of the analytical and numerical values of the J_k -integrals with respect to the slant angle for the biaxiality ratios of $\eta=0,0.5,1.0$, which are in very good agreement. Table 3 shows the derivative $\frac{dE_p}{da_2}$ and also the contribution to J_2 from the strain energy density discontinuity for each crack orientation and loading condition.

Throughout this study the convergence test was performed for each model. The number of isoparametric quadratic boundary elements varied between 70-100.

It should be noted that once J_1 and J_2 have been calculated, K_I and K_{II} can be determined using the following equations [19].

$$\begin{aligned} K_I &= \pm \sqrt{\frac{EJ_1}{2} \left[1 \pm \sqrt{1 - \left(\frac{J_2}{J_1} \right)^2} \right]} \\ K_{II} &= \pm \sqrt{\frac{EJ_1}{2} \left[1 \pm (-1) \sqrt{1 - \left(\frac{J_2}{J_1} \right)^2} \right]} \end{aligned} \quad (32)$$

The above equations are obtained by combining equations 6 and 7 and then re-arranging them with respect to K_I and K_{II} . The signs of SIFs can be determined by checking the values of the crack opening displacements near the crack tip [19] which are readily available by BEM.

CONCLUSION

Following a brief review of the mathematical basis of the BIE and shape sensitivity analysis, the BE crack shape sensitivities for evaluation of the J_2 -integral was discussed. A small region around the crack tip was treated as the shape design variable to evaluate the J_2 -integral. It was shown that the derivative of the total potential energy with respect to the transverse direction of the crack is not the J_2 -integral. However, by addition of an integral involving the crack tip elements to this derivative the J_2 -integral can be efficiently evaluated. The J_2 -integral combined with

the J_1 -integral, which was also previously obtained using the BE crack shape sensitivities, can be employed for efficient and accurate separation and calculation of the SIFs. The present algorithm was validated using test cases with known analytical solutions. For each crack shape and loading condition, the value of the J_2 -integral as well as the contribution to J_2 due to the strain energy density discontinuity were presented. Three example problems with isotropic and homogeneous material properties are analysed and the results are presented. The findings show the simplicity, accuracy and flexibility of the method which is applicable to curved and straight cracks. Due to the generic nature of the method, this algorithm can be expanded and applied to the fracture analysis of bimetals with isotropic and/or anisotropic material properties.

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Table 1 Analytical and numerical values of J_1 , J_2 , $\frac{dE_p}{da_2}$ and $I_{S_{pc}}$ for a rectangular plate with a central straight crack subject to biaxial and shear stresses ($\sigma_{y0} = 4\sigma_{x0} = 8\tau_0 = 200MPa$), (Fig. 2)

$J_1 \equiv \frac{dE_p}{da_1}$ (mJ/mm ²)		$J_2 \equiv \frac{dE_p}{da_2} + I_{S_{pc}}$ (mJ/mm ²)		$\frac{dE_p}{da_2}$ (mJ/mm ²)		$I_{S_{pc}}$ (mJ/mm ²)	
Analytical	Numerical	Analytical	Numerical	Analytical	Numerical	Analytical	Numerical
12.756	12.749	-3.140	-3.132	-1.64	-1.612	-1.500	-1.520

Table 2 Numerical values of $\frac{dE_P}{da_2}$ and $I_{S_{pc}}$ for different crack slant angles (β) and biaxiality ratios (η) (Fig. 3)

Crack angle (β) Degrees	$\eta=0$		$\eta=0.5$		$\eta=1.0$	
	$\frac{dE_P}{da_2}$ (mJ/mm ²)	$I_{S_{pc}}$ (mJ/mm ²)	$\frac{dE_P}{da_2}$ (mJ/mm ²)	$I_{S_{pc}}$ (mJ/mm ²)	$\frac{dE_P}{da_2}$ (mJ/mm ²)	$I_{S_{pc}}$ (mJ/mm ²)
15	-0.073	-0.058	-0.258	-0.074	-0.029	-0.008
30	-0.391	-0.273	-0.481	-0.258	-0.016	-0.009
45	-0.876	-0.617	-0.652	-0.460	0.001	-0.002
60	-1.117	-0.834	-0.643	-0.529	0.013	0.008
75	-0.789	-0.627	-0.397	-0.366	0.019	-0.004
90	$\approx 0.$	$\approx 0.$	$\approx 0.$	$\approx 0.$	$\approx 0.$	$\approx 0.$

Table 3 Numerical values of $\frac{dE_p}{da_2}$ and $I_{S_{pc}}$ for different half crack central angles (α) and biaxiality ratios (η) (Fig. 4)

Half crack central angle (α)	$\eta=0$		$\eta=0.5$		$\eta=1.0$	
Degrees	$\frac{dE_p}{da_2}$ (mJ/mm ²)	$I_{S_{pc}}$ (mJ/mm ²)	$\frac{dE_p}{da_2}$ (mJ/mm ²)	$I_{S_{pc}}$ (mJ/mm ²)	$\frac{dE_p}{da_2}$ (mJ/mm ²)	$I_{S_{pc}}$ (mJ/mm ²)
10	-0.577	-0.501	-0.436	-0.403	-0.287	-0.284
20	-1.051	-0.880	-0.824	-0.728	-0.567	-0.520
30	-1.221	-1.063	-1.041	-0.896	-0.735	-0.677
40	-1.264	-0.973	-1.134	-0.887	-0.920	-0.697

Figure 2 Central crack in a rectangular plate subject to biaxial and shear stresses

Figure 3 Central slant crack in a rectangular plate subject to biaxial tension

Fig. 4 A circular arc crack with the central angle of 2α and chord length of $2a$ in a thin flat plate subject to biaxial tension

Figure 5 Variation of J_1 , $-J_2$, $-d_{EP}/da_2$ and $-I_{SPC}$ normalized with respect to their corresponding analytical values vs the element bias factor near the crack tip (8 elements on each half crack length)

Fig. 7 Variation of the J_k integrals vs half crack central angle (α) for different biaxiality ratios (Fig. 4)