Displaced negative-binomial mixed states: Generalized thermo-field-dynamics

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Displaced negative-binomial mixed states are introduced and their properties are studied. It is shown that they may be used to construct a resolution of the identity operator and, therefore, that they can be used as a “basis” in the Hilbert space. Generalized P and Q representations with respect to these states are introduced and their properties are discussed. The pure $SU(1,1)$ coherent states discussed by Perelomov are considered in the two-mode harmonic oscillator Hilbert space and it is shown that the partial trace with respect to one of the modes leads to negative-binomial mixed states. Using this observation, the formalism of thermo-field-dynamics is generalized in a corresponding negative-binomial field dynamics.

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I. INTRODUCTION

Coherent states have played an important role in various branches of physics. Although they form a nonorthonormal overcomplete basis in the Hilbert space, the fact that a resolution of the identity can be constructed from them makes them practically useful. The most crucial property that makes an overcomplete set of states a useful basis is the ability to construct a resolution of the identity operator in terms of them. Whenever this property is available, a practical formalism can be built by using this basis. Several such sets of coherent states have been studied in the literature in connection with various groups, but all of them have been pure states.

In previous works by the present authors [1] we have considered mixtures of coherent states with thermal noise (or “thermal coherent states”). Such states have been studied previously in the literature [2] in different contexts. We have investigated some of their formal mathematical properties including the important fact that they form a resolution of the identity operator in the corresponding Hilbert space. Using this we have shown how these states can be used as a “random” or “noisy” basis in the Hilbert space.

Generalized P and Q representations have been defined in terms of these bases and their properties have been studied. In this paper, we show that a similar formalism can be developed in terms of the more general displaced negative-binomial mixed states. The importance of the negative-binomial distribution in quantum statistics has been discussed by earlier authors [3]. Motivated by this, several authors have studied negative-binomial mixed states in a quantum optics context [4]. Here we consider the displaced version of these states and prove that they can be used to form a resolution of the identity. From

this observation we then define generalized P and Q representations based on these states and we study in detail their relationship to the ordinary P and Q representations, to the Wigner function, and to related formal approaches. A special case of our displaced negative-binomial mixed states, when the parameter k characterizing the negative-binomial distribution takes the special value $\frac{1}{2}$, is provided by the thermal coherent states studied in our previous work [1]. In this sense, the formalism presented there is now generalized.

The negative-binomial distribution is intimately related to the discrete series of $SU(1,1)$ representations [5]. Indeed the Hilbert space of a two-mode harmonic oscillator can be expressed as a direct sum of an infinite number of subspaces each of which is related to a particular representation in the discrete series of $SU(1,1)$ representations [6]. The trace of certain two-mode $SU(1,1)$ Perelomov coherent states defined within each of these subspaces, gives just the negative-binomial mixed states [7]. We explain that thermo-field-dynamics [8] is related to the special subspace corresponding to the value $k = \frac{1}{2}$, and show that by going to another subspace we can introduce a similar formalism involving the negative-binomial mixed states, that we call negative-binomial field dynamics.

In Sec. II we introduce the displaced negative-binomial mixed states and prove the corresponding resolution of the identity operator. In Sec. III we calculate the P , Q , and Wigner representations of these states. In Sec. IV we introduce the generalized P and Q representations and study their relations with both the ordinary P and Q representations and with the Wigner function. In Sec. V we generalize the well-known thermo-field-dynamics into the negative-binomial field dynamics. We conclude in Sec. VI with a discussion of our results.

II. DISPLACED NEGATIVE-BINOMIAL MIXED STATES

We consider the mixed states described by the following density operator,

$$\rho(k, r) = (1-r)^{2k} \sum_{N=0}^{\infty} p(k, N) r^N |N\rangle \langle N|, \quad (2.1)$$

$$p(k, N) = \frac{\Gamma(N+2k)}{\Gamma(N+1)\Gamma(2k)}, \quad k = \frac{1}{2}, 1, \frac{3}{2}, \dots, \quad 0 \leq r < 1.$$

We refer to them as negative-binomial mixed states because the distribution defined by the diagonal matrix elements in the number operator basis $\{|N\rangle, N=0, 1, \dots\}$,

$$\langle N | \rho(k, r) | N \rangle = (1-r)^{2k} r^N p(k, N), \quad (2.2)$$

is simply the well-known negative-binomial distribution. The word "mixed" in the terminology is used in order to distinguish them from the pure SU(1,1) Perelomov coherent states [9],

$$|z; k\rangle = (1-|z|^2)^k \sum_{N=0}^{\infty} [p(k, N)]^{1/2} z^N |N\rangle, \quad |z| < 1, \quad (2.3)$$

which are pure states with the negative-binomial distribution. We note also that $\rho(k, r)$ may be expressed in the form

$$\rho(k, r) = \int_0^{2\pi} \frac{d\theta}{2\pi} |z; k\rangle \langle z; k|, \quad z = r^{1/2} e^{i\theta}. \quad (2.4)$$

The average number of photons in the mixed state described by Eq. (2.1) is given by

$$\text{Tr}[\rho(k, r) a^\dagger a] = \frac{2kr}{1-r}, \quad (2.5)$$

where a and $a^\dagger$ are, respectively, the usual bosonic destruction and creation operators in terms of which the states $|N\rangle$ are the eigenstates of the number operator $N = a^\dagger a$. In the case $k = \frac{1}{2}$ the density operator of Eq. (2.1) reduces to the thermal density operator

$$\begin{aligned} \rho_{\text{th}}(r) &= \rho(k = \frac{1}{2}, r) = (1-r) \sum_{N=0}^{\infty} r^N |N\rangle \langle N| \\ &= (1-e^{-\beta}) e^{-\beta a^\dagger a}, \\ \beta &\equiv -\ln r. \end{aligned} \quad (2.6)$$

We note that $\rho(k, r)$ may also be written in terms of $\rho_{\text{th}}(r)$ in the form

$$\rho(k, r) = [\Gamma(2k)]^{-1} \left[\frac{1}{r} - 1 \right]^{2k-1} a^{2k-1} \rho_{\text{th}}(r) (a^\dagger)^{2k-1}. \quad (2.7)$$

The proof of Eq. (2.7) follows straightforwardly by simply acting with the operators a^{2k-1} and $(a^\dagger)^{2k-1}$ on the left and right, respectively, of both sides of Eq. (2.6). In the special case $r=0$ the density operator of Eq. (2.1) reduces to the simple form of the projector onto the vacuum,

$$\rho(k, r=0) = |0\rangle \langle 0|. \quad (2.8)$$

We next introduce the "displaced negative-binomial

mixed states" as

$$\rho(k, r; A) \equiv D(A) \rho(k, r) D^\dagger(A), \quad (2.9)$$

$$D(A) \equiv \exp[AA^\dagger - A^*A]. \quad (2.10)$$

In terms of the displaced number eigenstates $\rho(k, r; A)$ can be expressed as

$$\begin{aligned} \rho(k, r; A) &= (1-r)^{2k} \sum_{N=0}^{\infty} p(k, N) |N; A\rangle \langle N; A|, \\ |N; A\rangle &\equiv D(A) |N\rangle. \end{aligned} \quad (2.11)$$

The average number of photons in the mixed state $\rho(k, r; A)$ is given by

$$\text{Tr}[a^\dagger a \rho(k, r; A)] = |A|^2 + \frac{2kr}{1-r}. \quad (2.12)$$

In the special case $r=0$, Eq. (2.9) reduces to

$$\rho(k, r=0; A) = |A\rangle \langle A|. \quad (2.13)$$

In the special case $k = \frac{1}{2}$, Eq. (2.9) describes coherent states with thermal noise (or thermal coherent states)

$$\begin{aligned} \rho(k = \frac{1}{2}, r; A) &= (1-e^{-\beta}) \exp[-\beta(a^\dagger - A^*)(a - A)], \\ \beta &= -\ln r. \end{aligned} \quad (2.14)$$

These density operators have been studied in the literature [1,2] and our intention here is to study the more general density operators of Eq. (2.9). From Eqs. (2.7) and (2.9) we easily prove the relation

$$\begin{aligned} \rho(k, r; A) &= [\Gamma(2k)]^{-1} \left[\frac{1}{r} - 1 \right]^{2k-1} (a - A)^{2k-1} \\ &\quad \times \rho(k = \frac{1}{2}, r; A) (a^\dagger - A^*)^{2k-1}. \end{aligned} \quad (2.15)$$

We next consider the set of mixed states $\rho(k, r; A)$ with fixed k, r and variable A ($A \in \mathbb{C}$) and prove the important resolution of the identity

$$\int \frac{d^2 A}{\pi} \rho(k, r; A) = I. \quad (2.16)$$

The proof of Eq. (2.16) reduces to the proof of the relation,

$$\begin{aligned} (1-r)^{2k} \sum_{N=0}^{\infty} \rho(k, N) r^N \int \frac{d^2 A}{\pi} \langle M | D(A) | N \rangle \\ \times \langle N | D^\dagger(A) | \Lambda \rangle = \delta_{\Lambda M}. \end{aligned} \quad (2.17)$$

It is known [10] that

$$\begin{aligned} \langle M | D(A) | N \rangle &= \left[\frac{N!}{M!} \right]^{1/2} A^{M-N} \\ &\quad \times \exp(-\frac{1}{2}|A|^2) L_N^{M-N}(|A|^2), \end{aligned} \quad (2.18)$$

where the functions L_N^{M-N} are Laguerre polynomials. Substitution of Eq. (2.18) into Eq. (2.17) and straightforward integration proves the result. Equation (2.16) has already been proved previously [1] for the special case $k = \frac{1}{2}$. In the special case $r=0$, Eq. (2.13) shows that Eq.

(2.16) is simply the well-known resolution of the identity operator in terms of the Glauber coherent states.

Another resolution of the identity that involves the set of mixed states $\rho(k, r; A)$ with fixed k, A and variable r ($0 \leq r < 1$) is

$$(2k-1) \int_0^1 \rho(k, r; A) (1-r)^{-2} dr = I. \quad (2.19)$$

In the special case $k = \frac{1}{2}$ the above equation should be expressed as a limit. The derivation of Eq. (2.19) is straightforward. We simply substitute Eq. (2.11) into Eq. (2.19) and perform the integration. It should be pointed out that Eq. (2.19) is closely related to the following resolution of the identity for the $SU(1,1)$ Perelomov coherent states of Eq. (2.3),

$$\frac{2k-1}{\pi} \int_D |z; k\rangle \langle z; k| (1-|z|^2)^{-2} d^2z = I, \quad (2.20)$$

where D is the unit disc ($|z| < 1$). Equation (2.20) is known in the literature [9,11]. By expressing the integral

in polar coordinates and by integrating over the angular variable θ we obtain Eq. (2.19) by making use of Eq. (2.4).

III. P, Q , AND WIGNER REPRESENTATIONS OF THE DISPLACED NEGATIVE-BINOMIAL MIXED STATES

In this section, we derive the P, Q [12,13], and Wigner representations [14] of the density operator $\rho(k, r; A)$. These are interesting in their own right, but they will also be useful for later purposes. We first give a useful relation [Eq. (8.977.2) of Ref. [15]],

$$\sum_{N=0}^{\infty} p(k, N) \frac{x^N}{N!} = e^x L_{2k-1}(-x), \quad (3.1)$$

and use it first to derive the Q representation of the operator $\rho(k, r; A)$. This is defined in the usual way as its diagonal matrix element in the usual Glauber coherent state basis $\{|z\rangle\}$,

$$\begin{aligned} Q(\rho(k, r; A); z) &\equiv \langle z | \rho(k, r; A) | z \rangle = \langle z | D(A) \rho(k, r) D^\dagger(A) | z \rangle = \langle z - A | \rho(k, r) | z - A \rangle \\ &= (1-r)^{2k} L_{2k-1}(-r|z-A|^2) \exp[-(1-r)|z-A|^2]. \end{aligned} \quad (3.2)$$

Substitution of Eq. (2.1) into the above defining equation, together with use of Eq. (3.1) gives the above result.

The P representation can also easily be found from Eq. (2.15) and the P representation for the thermal coherent states [1],

$$\begin{aligned} \rho(k = \tfrac{1}{2}, r; A) &= \int \left[\frac{1}{r} - 1 \right] \\ &\times \exp \left[- \left[\frac{1}{r} - 1 \right] \exp |z - A|^2 \right] \\ &\times |z\rangle \langle z| \frac{d^2z}{\pi}. \end{aligned} \quad (3.3)$$

From Eqs. (2.15) and (3.3) we get

$$\begin{aligned} \rho(k, r; A) &= [\Gamma(2k)]^{-1} \left[\frac{1}{r} - 1 \right]^{2k} \\ &\times \int \exp \left[- \left[\frac{1}{r} - 1 \right] |z - A|^2 \right] \\ &\times |z - A|^{2(2k-1)} |z\rangle \langle z| \frac{d^2z}{\pi}, \end{aligned} \quad (3.4)$$

and, therefore,

$$\begin{aligned} P(\rho(k, r; A); z) &= [\Gamma(2k)]^{-1} \left[\frac{1}{r} - 1 \right]^{2k} \\ &\times |z - A|^{2(2k-1)} \\ &\times \exp \left[- \left[\frac{1}{r} - 1 \right] |z - A|^2 \right]. \end{aligned} \quad (3.5)$$

An alternative derivation of Eq. (3.5) can be based on Mehta's formula [16],

$$\begin{aligned} P(\Theta; A) &= \exp(|A|^2) \int \frac{d^2B}{\pi} \langle -B | \Theta | B \rangle \exp(|B|^2) \\ &\times \exp[AB^* - A^*B]. \end{aligned} \quad (3.6)$$

Equations (3.6) and (2.1) also lead to the result of Eq. (3.5). This proof is longer, but its merit is that by repeating it the reader becomes familiar with the analysis that is used throughout the rest of the paper, in which the Laguerre polynomials play a central role.

The Wigner function of an operator is related to its P representation through the relation

$$W(\Theta; A) = 2 \int \frac{d^2B}{\pi} P(\Theta; B) \exp[-2|A-B|^2]. \quad (3.7)$$

Combining Eqs. (3.5) and (3.7), and performing the integral, we get

$$\begin{aligned} W(\rho(k, r; A); B) &= 2 \left[\frac{1-r}{1+r} \right]^{2k} \\ &\times L_{2k-1} \left[- \frac{4r}{(1+r)} |A-B|^2 \right] \\ &\times \exp \left[- \frac{2(1-r)|A-B|^2}{(1+r)} \right]. \end{aligned} \quad (3.8)$$

For later purposes we also calculate the trace of the product of two such density operators,

$$g(k; r_1, r_2; A-B) \equiv \text{Tr}[\rho(k, r_1; A) \rho(k, r_2; B)], \quad (3.9)$$

and its Fourier transform with respect to $A - B$, defined in general as follows for an arbitrary function of the complex parameter z ,

$$\tilde{f}(w) = \int d^2z f(z) e^{iz \cdot w} \leftrightarrow f(z) = \int \frac{d^2w}{(2\pi)^2} \tilde{f}(w) e^{-iz \cdot w}, \quad (3.10)$$

$$z \cdot w \equiv \frac{1}{2}(zw^* + z^*w) = z_R w_R + z_I w_I,$$

where the subscripts R, I indicate the real and imaginary parts, respectively. A useful formula for the calculations that follow is that when $f(z)$ is a function of $|z|$ only, i.e., $f(z) = f(|z|)$, then

$$\tilde{f}(w) \equiv \tilde{f}(|w|) = \int_0^\infty f(|z|) 2\pi J_0(|z||w|) |z| d|z|. \quad (3.11)$$

In order to calculate the trace in Eq. (3.9) we use the well-known relation for the trace of the product of two arbitrary operators Θ_1 and Θ_2 ,

$$\text{Tr}[\Theta_1 \Theta_2] = \int P(\Theta_1; z) Q(\Theta_2; z) \frac{d^2z}{\pi}. \quad (3.12)$$

Combining Eqs. (3.2), (3.5), and (3.11) we may express this trace as a convolution of two functions. Hence it is easier to start the calculation from the Fourier transform of this trace,

$$\begin{aligned} \bar{g}(k; r_1, r_2; w) &= \frac{1}{\pi} \bar{g}_P(k, r_1; w) \bar{g}_Q(k, r_2; w) \\ &= \frac{1}{\pi} \bar{g}_P(k, r_2; w) \bar{g}_Q(k, r_1; w), \end{aligned} \quad (3.13)$$

where g_P and g_Q are given in terms of the P and Q functions of Eqs. (3.5) and (3.2) as follows:

$$g_P(k, r; z) \equiv P(\rho(k, r; A=0); z), \quad (3.14)$$

$$g_Q(k, r; z) \equiv Q(\rho(k, r; A=0); z). \quad (3.15)$$

It is clear that $\bar{g}(k; r_1, r_2; w)$ is a symmetric function of r_1 and r_2 . Using Eq. (3.11) we may perform the Fourier transformation on these functions to obtain

$$\bar{g}_P(k, r; w) = \pi \exp \left[-\frac{|w|^2 r}{4(1-r)} \right] L_{2k-1} \left[\frac{|w|^2 r}{4(1-r)} \right], \quad (3.16)$$

$$\bar{g}_Q(k, r; w) = \pi \exp \left[-\frac{|w|^2}{4(1-r)} \right] L_{2k-1} \left[\frac{|w|^2 r}{4(1-r)} \right], \quad (3.17)$$

from which we have the simple relation

$$\bar{g}_P(k, r; w) = \bar{g}_Q(k, r; w) \exp(\frac{1}{4}|w|^2). \quad (3.18)$$

In the special case $r=0$ we have

$$\bar{g}_P(k, 0; w) = \pi, \quad (3.19)$$

$$\bar{g}_Q(k, 0; w) = \pi \exp(-\frac{1}{4}|w|^2). \quad (3.20)$$

Hence, we have the important relation

$$\begin{aligned} \bar{g}(k; r_1, r_2; w) &= \pi \exp \left[-\frac{(1-r_1 r_2)}{(1-r_1)(1-r_2)} \frac{|w|^2}{4} \right] \\ &\times L_{2k-1} \left[\frac{|w|^2 r_1}{4(1-r_1)} \right] \\ &\times L_{2k-1} \left[\frac{|w|^2 r_2}{4(1-r_2)} \right], \end{aligned} \quad (3.21)$$

in which form the symmetry under interchange of r_1 and r_2 is manifest. In the special case that r_1 or r_2 is equal to zero we get

$$\bar{g}(k; r, 0; w) = \bar{g}_Q(k, r; w) = \bar{g}_P(k, r; w) \exp(-\frac{1}{4}|w|^2). \quad (3.22)$$

Substitution of $|w|^2$ by $-\Delta_z$, where Δ_z is the usual Laplacian operator, $\Delta_z \equiv \partial^2/\partial z_R^2 + \partial^2/\partial z_I^2$, gives the following useful operators:

$$\begin{aligned} \bar{g}_P(k, r; -\Delta_z) &\equiv G_P(k, r; \Delta_z) \\ &= \pi \exp \left[\frac{r}{4(1-r)} \Delta_z \right] \\ &\times L_{2k-1} \left[-\frac{r}{4(1-r)} \Delta_z \right], \end{aligned} \quad (3.23)$$

$$\begin{aligned} \bar{g}_Q(k, r; -\Delta_z) &\equiv G_Q(k, r; \Delta_z) \\ &= \pi \exp \left[\frac{1}{4(1-r)} \Delta_z \right] \\ &\times L_{2k-1} \left[-\frac{r}{4(1-r)} \Delta_z \right], \end{aligned} \quad (3.24)$$

$$\begin{aligned} \bar{g}(k; r_1, r_2; -\Delta_z) &\equiv G(k; r_1, r_2; \Delta_z) \\ &= \pi \exp \left[\frac{(1-r_1 r_2)}{4(1-r_1)(1-r_2)} \Delta_z \right] \\ &\times L_{2k-1} \left[-\frac{r_1}{4(1-r_1)} \Delta_z \right] \\ &\times L_{2k-1} \left[-\frac{r_2}{4(1-r_2)} \Delta_z \right]. \end{aligned} \quad (3.25)$$

These functions will be useful later, because the Fourier transform of the product of $\bar{g}(k; r_1, r_2; w)$ with an arbitrary function $f(w, w^*)$ is equal to the operator $G(k; r_1, r_2; \Delta_z)$ acting on the Fourier transform of $f(w, w^*)$. A result analogous to Eq. (3.22) is

$$G(k; r, 0; \Delta_z) = G_Q(k, r; \Delta_z) = G_P(k, r; \Delta_z) \exp(\frac{1}{4}\Delta_z), \quad (3.26)$$

which will be used extensively later.

The inverse Fourier transform of $\bar{g}(k; r_1, r_2; w)$ cannot be performed easily in the general case, but luckily it is not needed. Only the special case $g(k; r_1, r_2; z=0)$ is needed and we now evaluate this directly, rather than through the inverse Fourier transform of Eq. (3.13),

$$\begin{aligned}
g(k; r_1, r_2; z=0) &= \text{Tr}[\rho(k, r_1; A)\rho(k, r_2; A)] = \text{Tr}[\rho(k, r_1)\rho(k, r_2)] \\
&= (1-r_1)^{2k}(1-r_2)^{2k} \sum_{N=0}^{\infty} [p(k, N)]^2 (r_1 r_2)^N \\
&= \left[\frac{(1-r_1)(1-r_2)}{1-r_1 r_2} \right]^{2k} P_{2k-1} \left[\frac{1+r_1 r_2}{1-r_1 r_2} \right] \equiv h(k, r_1, r_2), \quad (3.27)
\end{aligned}$$

where the functions P_l are Legendre polynomials.

IV. GENERALIZED P AND Q REPRESENTATIONS

The P and Q representations of operators are usually introduced in terms of coherent states, where the key feature in the analysis is the resolution of the identity operator in terms of them. In previous work [1] we used the fact that a resolution of the identity can also be written in terms of the thermal coherent states of Eq. (2.14), in order to define generalized P and Q representations in terms of these states. In this section, this idea is extended further by using the resolution of the identity given by Eq. (2.16), in order to introduce even more general P and Q representations in terms of the mixed states of Eq. (2.9).

The Q and P representation of an arbitrary operator Θ with respect to the mixed states of Eq. (2.9) are introduced as follows:

$$Q(\Theta; k, r; A) \equiv \text{Tr}[\Theta \rho(k, r; A)], \quad (4.1)$$

$$\Theta = \int P(\Theta; k, r; A) \rho(k, r; A) \frac{d^2 A}{\pi}. \quad (4.2)$$

In the special case $r=0$, Eq. (2.13) shows that they are simply the ordinary Q and P representations with respect to the Glauber coherent states, and these latter will be denoted as $Q(\Theta; A), P(\Theta, A)$ correspondingly,

$$Q(\Theta; k, r=0; A) \equiv Q(\Theta; A) = \langle A | \Theta | A \rangle, \quad (4.3)$$

$$P(\Theta; k, r=0; A) \equiv P(\Theta; A), \quad (4.4)$$

$$\Theta = \int P(\Theta; A) |A\rangle \langle A| \frac{d^2 A}{\pi}.$$

From Eqs. (4.1) and (4.2) we get the convolution

$$\begin{aligned}
Q(\Theta; k, r_1; z) &= \int P(\Theta; k; r_2; z') \\
&\quad \times g(k; r_1, r_2; z - z') \frac{d^2 z'}{\pi}. \quad (4.5)
\end{aligned}$$

The corresponding Fourier transform version of this relation is

$$\tilde{Q}(\Theta; k, r_1; w) = \frac{1}{\pi} \tilde{g}(k; r_1, r_2; w) \tilde{P}(\Theta; k, r_2; w). \quad (4.6)$$

Using the operator G of Eq. (3.19) we may rewrite Eq. (4.6) in the form

$$Q(\Theta; k, r_1; z) = \frac{1}{\pi} G(k; r_1, r_2; \Delta_z) P(\Theta; k, r_2; z). \quad (4.7)$$

In the special case $r_2=0$, Eq. (4.6) reduces to the relation

$$\tilde{Q}(\Theta; k, r; w) = \frac{1}{\pi} \tilde{g}_Q(k, r; w) \tilde{P}(\Theta; w). \quad (4.8)$$

Using Eqs. (3.13), (4.6), and (4.8), we can now show

$$\begin{aligned}
\tilde{P}(\Theta; w) &= \frac{1}{\pi} \tilde{g}_P(k, r_2; w) \tilde{P}(\Theta; k, r_2; w) \\
&= \pi [g_Q(k, r_1; w)]^{-1} \tilde{Q}(\Theta; k, r_1; w). \quad (4.9)
\end{aligned}$$

Clearly both sides of this equation must be independent of the parameters r_1 and r_2 . Using Eq. (4.9) and the known relation

$$\tilde{P}(\Theta; w) = \exp\left[\frac{1}{4}|w|^2\right] \tilde{Q}(\Theta; w), \quad (4.10)$$

we obtain the relations

$$\tilde{P}(\Theta; k, r; w) = \pi [\tilde{g}_P(k, r; w)]^{-1} \tilde{P}(\Theta; w), \quad (4.11)$$

$$\tilde{Q}(\Theta; k, r; w) = \frac{1}{\pi} \tilde{g}_Q(k, r; w) \exp\left[\frac{1}{4}|w|^2\right] \tilde{Q}(\Theta; w), \quad (4.12)$$

which express the generalized P and Q representations in terms of their "ordinary" ($r=0$) counterparts.

The Wigner function $\tilde{W}(\Theta; w)$ and its Fourier transform (or Weyl function) $\tilde{\mathcal{W}}(\Theta; w)$ have been studied extensively in a previous work [1], where we have shown

$$\tilde{Q}(\Theta; w) = \exp\left[-\frac{1}{8}|w|^2\right] \tilde{W}(\Theta; w), \quad (4.13)$$

$$\tilde{P}(\Theta; w) = \exp\left[+\frac{1}{8}|w|^2\right] \tilde{W}(\Theta; w). \quad (4.14)$$

Combining Eqs. (4.11) and (4.14) and Eqs. (4.12) and (4.13), we obtain, respectively,

$$\begin{aligned}
\tilde{P}(\Theta; k, r; w) &= \exp\left[\frac{(1+r)}{8(1-r)}|w|^2\right] \\
&\quad \times \left[L_{2k-1}\left[\frac{|w|^2 r}{4(1-r)}\right]\right]^{-1} \tilde{W}(\Theta; w), \quad (4.15)
\end{aligned}$$

$$\begin{aligned}
\tilde{Q}(\Theta; k, r; w) &= \exp\left[-\frac{(1+r)}{8(1-r)}|w|^2\right] \\
&\quad \times L_{2k-1}\left[\frac{|w|^2 r}{4(1-r)}\right] \tilde{W}(\Theta; w). \quad (4.16)
\end{aligned}$$

Equations (4.15) and (4.16) show that the product

$\tilde{P}(\Theta; k, r; w) \tilde{Q}(\Theta; k, r; w)$ is independent of the parameter r . Furthermore they also imply the relationship

$$\begin{aligned} \tilde{Q}(\Theta; k, r_1; w) = & \exp \left[-\frac{|w|^2}{4} \frac{(1-r_1 r_2)}{(1-r_1)(1-r_2)} \right] \\ & \times L_{2k-1} \left[\frac{|w|^2 r_1}{4(1-r_1)} \right] \\ & \times L_{2k-1} \left[\frac{|w|^2 r_2}{4(1-r_2)} \right] \tilde{P}(\Theta; k, r_2; w), \end{aligned} \quad (4.17)$$

which is a generalization of the well-known relation between the usual P and Q representations given in Eq. (4.10).

V. GENERALIZATION OF THERMO-FIELD-DYNAMICS

We consider the two-mode harmonic oscillator Hilbert space $\mathcal{H}_A \times \mathcal{H}_B$ and its subspaces spanned by the following joint number eigenstates:

$$\mathcal{H}_l = \{ |N+l, N\rangle; N = \max(0, -l), \dots, \infty \}, \quad (5.1)$$

where l is an integer, and where $|N+l, N\rangle \equiv |N+l\rangle_A |N\rangle_B$. We easily see that

$$\mathcal{H}_A \otimes \mathcal{H}_B = \sum_{l=-\infty}^{\infty} \mathcal{H}_l. \quad (5.2)$$

We also introduce the discrete series of $SU(1,1)$ representations, whose generators are expressed in terms of the two-mode harmonic oscillator creation and annihilation operators as [6]

$$K_+ \equiv a_1^\dagger a_2^\dagger, \quad (5.3)$$

$$K_- \equiv a_1 a_2, \quad (5.4)$$

$$K_0 \equiv \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1), \quad (5.5)$$

$$K^2 \equiv K_0^2 - \frac{1}{2}(K_+ K_- + K_- K_+) = \frac{1}{4}(a_1^\dagger a_1 - a_2^\dagger a_2)^2 - \frac{1}{4}, \quad (5.6)$$

where the operators $(a_1, a_1^\dagger)$ and $(a_2, a_2^\dagger)$ act respectively in the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. The operators K_+ , K_- , and K_0 are easily seen to obey the $SU(1,1)$ commutation relations

$$[K_0, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_0. \quad (5.7)$$

In the Hilbert space $\mathcal{H}_l$ the eigenstates of the operators K_0, K^2 are the states $|N+l, N\rangle$,

$$K_0 |N+l, N\rangle = [N + \frac{1}{2}(l+1)] |N+l, N\rangle, \quad (5.8)$$

$$K^2 |N+l, N\rangle = \frac{1}{4}(l^2 - 1) |N+l, N\rangle. \quad (5.9)$$

Comparison of these equations with the equations for the discrete series of $SU(1,1)$ representations in a general context, namely,

$$K_0 |k; \mu\rangle = \mu |k; \mu\rangle, \quad (5.10)$$

$$K^2 |k; \mu\rangle = k(k-1) |k; \mu\rangle, \quad k = \frac{1}{2}, 1, \frac{3}{2}, \dots, \quad (5.11)$$

shows that the operators given by Eqs. (5.2)–(5.5), acting on the Hilbert space $\mathcal{H}_l$, form the

$$k = \frac{1+|l|}{2} \quad (5.12)$$

representation of $SU(1,1)$. Hence, the direct sum in Eq. (5.2) contains one $k = \frac{1}{2}$ representation, two $k = 1$ representations, two $k = \frac{3}{2}$ representations, etc.

We denote by Π_l the projection operators onto the Hilbert subspaces $\mathcal{H}_l$,

$$\Pi_l \equiv \sum_{N=\max(0, -l)}^{\infty} |N+l, N\rangle \langle N+l, N|, \quad (5.13)$$

$$\Pi_l \Pi_m = \Pi_l \delta_{lm}, \quad (5.14)$$

$$\sum_{l=-\infty}^{\infty} \Pi_l = 1. \quad (5.15)$$

They commute with the generators given by Eqs. (5.3)–(5.5) and consequently also with the two-mode squeezing operator [17,18],

$$S(\rho, \theta, \lambda) \equiv \exp \left[-\frac{1}{2} \rho e^{-i\theta} K_+ + \frac{1}{2} \rho e^{i\theta} K_- \right] \exp(i2\lambda K_0), \quad (5.16)$$

$$[S(\rho, \theta, \lambda), \Pi_l] = 0. \quad (5.17)$$

A consequence is that the two-mode squeezing operators leave the Hilbert spaces $\mathcal{H}_l$ invariant, in the sense that when they act on a state that belongs to $\mathcal{H}_l$, they produce another state which belongs in the same space.

We next define $SU(1,1)$ Perelomov [9] coherent states within each of the Hilbert spaces $\mathcal{H}_l$. For positive or zero values of l we get,

$$\begin{aligned} |l0; \rho\theta\lambda\rangle & \equiv S(\rho\theta\lambda) |l, 0\rangle \\ & = \exp \left[i\lambda \left[\frac{1+l}{2} \right] \right] (1-|z|^2)^{1/2(1+l)} \\ & \quad \times \sum_{N=0}^{\infty} \left[\frac{(N+l)!}{N!!} \right]^{1/2} z^N |N+l, N\rangle, \end{aligned} \quad (5.18)$$

$$z = -\tanh(\frac{1}{2}\rho) \exp(-i\theta), \quad l \geq 0.$$

A similar formula can be given for negative values of l . In the rest of this section we present relations which are valid only for positive or zero l . Their extension for negative l is straightforward. We consider the density operator

$$\rho(l0; \rho\theta\lambda) \equiv |l0; \rho\theta\lambda\rangle \langle l0; \rho\theta\lambda|. \quad (5.19)$$

By taking the trace with respect to one of the modes we may readily prove [7],

$$\begin{aligned} \rho_B(l0; \rho\theta\lambda) & \equiv \text{Tr}_A \rho(l0; \rho\theta\lambda) \\ & = \rho \left[k = \frac{1+l}{2}, r = \tanh^2(\frac{1}{2}\rho) \right], \end{aligned} \quad (5.20)$$

$$\begin{aligned}
\rho_A(l0; \rho\theta\lambda) &\equiv \text{Tr}_B \rho(l0; \rho\theta\lambda) \\
&= \left[\text{sech}\left(\frac{1}{2}\rho\right) \right]^{2(l+1)} \\
&\quad \times \sum_{N=0}^{\infty} \frac{(N+l)!}{N!l!} [\tanh(\frac{1}{2}\rho)]^{2N} \\
&\quad \times |N+l\rangle \langle N+l|. \quad (5.21)
\end{aligned}$$

Equation (5.20) is just the negative-binomial mixed state of Sec. II. On the other hand, Eq. (5.21) is a “shifted negative-binomial mixed state” (which is not to be confused with the displaced negative-binomial state of Sec.

II). The term shifted expresses the fact that the series in Eq. (5.21) starts with the term proportional to $|l\rangle\langle l|$ rather than with the term proportional to $|0\rangle\langle 0|$. In the special case $l=0$, corresponding to $k=\frac{1}{2}$, both Eqs. (5.20) and (5.21) give the thermal density operator,

$$\begin{aligned}
\rho_A(0,0; \rho\theta\lambda) &= \rho_B(0,0; \rho\theta\lambda) = \rho(k=\frac{1}{2}, r=\tanh^2(\frac{1}{2}\rho)) \\
&\equiv \rho_{\text{th}}(r=\tanh^2(\frac{1}{2}\rho)). \quad (5.22)
\end{aligned}$$

We now let Θ be an operator acting on the Hilbert space $\mathcal{H}_B$ and I_A be the unit operator acting on the Hilbert space $\mathcal{H}_A$. Then,

$$\begin{aligned}
\langle l0; \rho\theta\lambda | I_A \otimes \Theta | l0; \rho\theta\lambda \rangle &= \text{Tr} \{ [I_A \otimes \Theta] \rho(l0; \rho\theta\lambda) \} = \text{Tr}_B \{ \Theta \rho_B(l0; \rho\theta\lambda) \} \\
&= \text{Tr}_B \left\{ \Theta \rho \left[k = \frac{1+l}{2}, r = \tanh^2(\frac{1}{2}\rho) \right] \right\}. \quad (5.23)
\end{aligned}$$

The special case $k=\frac{1}{2}$ ($l=0$) of this relation is the basic relation of thermo-field-dynamics [8],

$$\langle 00; \rho\theta\lambda | I_A \otimes \Theta | 00; \rho\theta\lambda \rangle = \text{Tr}_B \{ \Theta \rho_{\text{th}}(r = \tanh^2(\frac{1}{2}\rho)) \}. \quad (5.24)$$

We now explain briefly the correspondence between the terminology of thermo-field-dynamics and the terminology of two-mode squeezed states. The mode A is the so-called fictitious mode in thermo-field-dynamics. The $\text{SU}(1,1)$ Perelomov coherent state $|00; \rho\theta\lambda\rangle$ (that belongs to the Hilbert subspace $\mathcal{H}_0$) is the so-called “thermo-field vacuum.” Equation (5.24) states that the expectation value of an arbitrary operator Θ with respect to the (one-mode) thermal density matrix is equal to the expectation value of $I_A \otimes \Theta$ with respect to the (two-mode) pure state $|00; \rho\theta\lambda\rangle$. This result, combined with what is now known as “two-mode squeezing formalism” gives almost all of the results of thermo-field-dynamics. For example, the Bogoliubov transformations that appear frequently in thermo-field-dynamics are simply the Bogoliubov transformations associated with two-mode squeezing. Of course, when the thermo-field-dynamics was developed, the two-mode squeezing formalism was not available, and in this sense several ideas of this formalism were undoubtedly first developed in the context of thermo-field-dynamics.

Equation (5.24) has thus been generalized into Eq. (5.23), where the expectation value of an arbitrary operator Θ with respect to the (one-mode) negative-binomial mixed state density matrix is equal to the expectation value of $I_A \otimes \Theta$ with respect to the (two-mode) pure state $|l0; \rho\theta\lambda\rangle$. We call this result *negative-binomial field dynamics*. Thermo-field-dynamics is, by definition, a formalism within the Hilbert subspace $\mathcal{H}_0$ associated with the $k=\frac{1}{2}$ representation of $\text{SU}(1,1)$. By comparison, negative-binomial field dynamics is a formalism within the more general Hilbert subspace $\mathcal{H}_l$ associated with the k representation of $\text{SU}(1,1)$, with $k=\frac{1}{2}(1+|l|)$.

The significance of the result expressed by Eq. (5.24) and its generalization proposed here in Eq. (5.23), lies in the fact that in many cases the right-hand sides of these equations will be easier to calculate than the corresponding left-hand sides. This is especially true since there is a lot of experience in the analysis of these types of calculations in the “squeezing formalism.” Moreover, in many cases the so-called “fictitious” mode is actually physically very real and has important physical meaning. This is the case, for example, in the “accelerated observer formalism” [19], where the two modes correspond to observers moving in opposite directions, and each of whom has only partial knowledge due to observing only a partial region of the whole of space time. Furthermore, in many problems we only have partial knowledge about the parameters, and the “real” and “fictitious” modes represent the known and unknown parameters, respectively. The trace is then taken with respect to the unknown modes and it is this trace that creates mixed states from the original pure states. This indicates clearly the important practical need for bases of mixed states, since whenever we start from a basis of pure states in the original “big” Hilbert space, we will be required to take the trace with respect to the unknown variables. In this way we always end up with a basis of mixed states in the “small” Hilbert space, to which our observations or knowledge are confined.

VI. DISCUSSION

Several types of coherent states have been considered in the literature as sets of overcomplete and nonorthogonal bases, but all of them are pure states. Our main contribution in both the present and earlier work has been to extend the formalism to deal with mixed states. Thermal coherent states have been considered in our earlier work [1], and that work has been generalized here into the displaced negative-binomial states. We have shown that these latter can be used to form a resolution of the identity

ty operator, and using this fact we have defined generalized P and Q representations. Several relations between these generalized P and Q representations and the ordinary P and Q representations have been studied in detail.

Thermo-field-dynamics has also been generalized into negative-binomial field dynamics by showing that the trace of an operator Θ in $\mathcal{H}_B$ times the (one-mode) thermal density matrix is equal to the expectation value of $I_A \otimes \Theta$ with respect to the two-mode pure state $|00; \rho\theta\lambda\rangle$ in $\mathcal{H}_A \otimes \mathcal{H}_B$. This important relation has been generalized in Eq. (5.23), where it is shown that the trace of an operator Θ times the negative-binomial mixed state is equal to the expectation value of $I_A \otimes \Theta$ with respect to the two-mode pure state $|10; \rho\theta\lambda\rangle$. This places thermo-field-dynamics into a much wider context where mixed states are viewed as resulting from pure states in bigger Hilbert spaces, by taking the partial trace with respect to

certain parameters. Furthermore, this partial trace is often necessitated by a lack of knowledge about these parameters. Such lack of information may either be intrinsically unavoidable or it may arise due to practical limitations, depending on the case at hand.

The formalism presented in this paper and in our previous work [1] exemplifies the practical usefulness of mixed states as bases in Hilbert spaces. We believe that they will find many uses in practice.

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