Stochastic finite-time consensualization for Markov jump networks with disturbance

Xiaoli Luan, Yang Min, Zhengtao Ding and Fei Liu

Abstract

This paper is devoted to the finite-time consensus control for directed networks with stochastic Markov jump topologies and external disturbances. The purpose of this study is to design a control protocol to ensure that the disagreement dynamics of interconnected networks stays in a given bound over a finite time interval rather than asymptotically converge to zero in infinite settling time. Through utilization of certain features of Laplacian matrix in real Jordan form, sufficient conditions for the existence of finite-time consensus protocol is derived by allowing Lyapunov function to increase in a fixed time interval. Finite-time convergence result for stochastic consensus problem is validated via a simulation study.

Index Terms

Markov jump topology, Consensus control, finite-time convergence, Lyapunov function

I. INTRODUCTION

In recent years, consensus problem for multiple interconnected dynamical systems has attracted a significant research interest due to its broad applications in various areas, such as formation control of satellite flying, cooperative control of unmanned air vehicles, flocking of multiple robotic systems, etc [1], [2], [3], [4]. Roughly speaking, the consensus objective is to reach some kind of agreement between the variables of subsystems that are connected via a network. Since the pioneering works on the consensus problems for distributed decision making and

Xiaoli Luan, Yang Min and Fei Liu are with the Key Laboratory of Advanced Control for Light Industry Processes, Ministry of Education, and the Institute of Automation, Jiangnan University, Wuxi 214122, China. E-mail: xlluan@jiangnan.edu.cn, yangmin1222@yeah.net, fliu@jiangnan.edu.cn

Zhengtao Ding is with the Control Systems Centre, School of Electrical and Electronic Engineering, University of Manchester, Sackville Street Building, Manchester M 13 9PL, UK. E-mail: Zhengtao.Ding@manchester.ac.uk

parallel computing, up to now, considerable amount of results have been reported (see [5], [6] and the references therein).

The consensus problems for linear dynamical systems have been extensively investigated in [7]. However, almost all the physical systems are inherently nonlinear. Recently, much attention has also been paid on consensus problem for systems with nonlinear dynamics [8], [9], [10]. Meanwhile, the inevitable communication delays usually make the consensus analysis and design more challenging. Therefore, the effect of various communication delays and input delays on consensus was considered in [11], [12]. Other works in this area deal with various aspect of consensus control such as variation of the consensus protocol, consensus estimation, and consensus performance [13], [14]. Details refer to the surveys of the latest development [15] and the references therein.

Among the aforementioned works, the communication topology is assumed to be time invariant or fixed. In practical applications, systems are connected via a network that changes with time due to link failures, packet drops, node failure, etc. Such variations in topology can happen randomly, which motivates the investigation of consensus problems under a stochastic framework [16], [17], [18]. For example, by casting the consensus problem into a sample data system, the asymptotic agreement over random undirected graphs was discussed in [16]. This result is further extended to randomly weighted directed graphs to deal with mean square consensus problem [17]. In 2009, the mean square consensus stability was studied in [18] for a network in Markov switching topology.

So far, most of the existing consensus algorithms for subsystems with linear or nonlinear dynamics, or networks over determined or stochastic topologies, or protocols with synchronous or asynchronous information, are restricted to asymptotical consensus. In other words, the states of multiple subsystems asymptotically reach consensus over infinite settling time. But in practical systems, such as communication network systems, biochemistry reaction systems, and robot control systems, where operating times are often relatively short, it is important that the systems achieve some desired performance in a specified finite time. Consider for example the problem of driving a car across a tunnel for which the distance between the boundaries is a known quantity 2β , knowing that the mission lifetime is N , we can reformulate the problem in term of finite time stability since we have specific constraints on state bounds and time. Therefore, finite-time consensus characterization is an important issue for network-connected dynamic systems. Recently, consensus over a finite time interval has been investigated in [19], [20], [21], where

convergence time is theoretically finite but indeterminate. Furthermore, the distinction between finite and infinite time interval is essentially mathematical.

Different from the finite-time consensus algorithms proposed in [19], [20], [21], finite-time consensus problem to be addressed in this paper refers to guaranteeing the state disagreement stays in a given bound during finite horizon of time. To this end, the concept of finite-time stability put forward in 1961 by Dorato [22] is adopted, which has been developed well in the past few years [23], [24], [25], [26]. In the presence of stochastic Markov jump communication topology and external disturbance, a control strategy is designed with the relative state information to ensure finite-time consensus of the network-connected dynamic systems. Compared with other consensus algorithms, the proposed algorithm has three new features: 1) The states disagreement stay within a given bound in a specified time interval rather than asymptotically converges to zero over infinite settling time. 2) Less conservative conditions are derived by allowing Lyapunov function to possibly increase in a fixed time interval. 3) Certain features of Laplacian matrix is exploited in real Jordan form to make the systems finite-time consentable, which facilitates the protocol design.

The rest of the paper is organized as follows. In Section 2, the consensus problem is formally stated and some necessary preliminaries are given. Section 3 presents the main results of this paper, that is, sufficient conditions for the existence of the desired controller are given. Numerical example is provided in Section 4 to show the effectiveness of the proposed techniques. Finally, Section 5 gives the conclusions and future directions.

II. PROBLEM DESCRIPTION

The dynamics of N subsystems takes the following form:

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) + B_w w_i(t) \quad (1)$$

for $i = 1, \dots, N$, where $x_i(t) \in R^n$ and $u_i(t) \in R^m$ represent the state and control input of subsystem i , respectively. $w_i(t) \in L_2^p \left[0, +\infty \right)$ is the external disturbance, and A, B, B_w are constant matrices with (A, B) controllable.

The connections between the subsystems are specified by a directed graph G which consists of a set of vertices denoted by Λ and a set of edges denoted by Θ . A vertex represents a subsystem, and each edge represents a connection. Associated with the graph, its adjacency matrix Q with elements q_{ij} denotes the connections such that $q_{ij} = 1$ if there is a connection from subsystem

j to subsystem i , and $q_{ij} = 0$, otherwise. The Laplacian matrix $L = \{l_{ij}\}$ is commonly defined as $l_{ii} = \sum_{j=1, j \neq i}^N q_{ij}$, $l_{ij} = -q_{ij}$ for $i \neq j$. For a directed graph, the Laplacian matrix L has the following properties.

Lemma 2.1: ([10]) Zero is an eigenvalue of L with $\mathbf{1} = \begin{bmatrix} 1, & \dots, & 1 \end{bmatrix}^T \in R^N$ as the corresponding right eigenvector and all the non-zero eigenvalues have positive real parts. Furthermore, zero is a single eigenvalue of L if and only if the graph contains a directed spanning tree.

In this work, we address the consensus problem of system (1) over stochastic switching topologies $G(r_t)$. The switching process $\{r_t, t \geq 0\}$ is governed by a homogeneous Markov chain. Given a probability space (Ω, F, P) where Ω is the sample space, F is the algebra of events and P is the probability measure defined on F . Let the random form process $\{r_t, t \geq 0\}$ be the Markov stochastic process taking values in a finite set $M = \{1, 2, \dots, S\}$ with transition rate matrix $\Pi = \{\pi_{rl}\}$, $r, l \in M$ and define the following transition probability from mode r at time t to mode l at time $t + \Delta t$ as

$$P_{rl} = P_r \{r_{t+\Delta t} = l | r_t = r\} = \begin{cases} \pi_{rl}\Delta t + o(\Delta t), & r \neq l \\ 1 + \pi_{rr}\Delta t + o(\Delta t), & r = l \end{cases}$$

with transition probability rates $\pi_{rl} \geq 0$ for $r, l \in M$, $r \neq l$ and $\sum_{l=1, l \neq r}^S \pi_{rl} = -\pi_{rr}$ where $\Delta t > 0$ and $\lim_{\Delta t \rightarrow 0} o(\Delta t)/\Delta t \rightarrow 0$.

To formulate the problem to be addressed in this paper, let's make the following assumptions:

Assumption 2.1: For any given positive number d , the external disturbance $w_i(t)$ for each subsystem is time varying and satisfies

$$\int_0^t w_i^T(t)w_i(t)dt \leq d, d \geq 0 \quad (2)$$

Assumption 2.2: For the convenience of presentation of the proposed design, the eigenvalues of the Laplacian matrix are distinct.

The objective of this paper is to present conditions that ensure the state disagreement between neighbourhood subsystems to stay in a given bound during finite horizon of time. To this end, let us define $\kappa(r_t) \in R^N$ as the left eigenvector of jump Laplacian $L(r_t)$ associated with the eigenvalue 0 and satisfy $\kappa(r_t)^T \mathbf{1} = 1$.

Based on the vector $\kappa(r_t)$, we introduce the state disagreement variable

$$\varsigma_i = x_i - \sum_{j=1}^N \kappa(r_t)_j x_j \quad (3)$$

for $i = 1, \dots, N$. It is desirable that the state disagreement among subsystems (3) is bounded during a finite time. This motivates the subsequent finite-time consensus concept:

Definition 2.1: For a given time constant $T > 0$, a network-connected dynamic system (1) (setting $u_i(t) = 0, w_i(t) = 0$) is said to be finite-time consentable with respect to $\begin{pmatrix} c_1 & c_2 & T & R & d \end{pmatrix}$, where $c_1 < c_2$ and $R > 0$, if

$$E \{ \varsigma_i^T(0) R \varsigma_i(0) \} \leq c_1 \Rightarrow E \{ \varsigma_i^T(t) R \varsigma_i(t) \} < c_2, \quad \forall t \in [0 \quad T] \quad (4)$$

Remark 2.1: It should be noted that the definition of a finite-time consentable system presented here has the similar form as the finite-time stability definition presented in [27], [28], where the state trajectories of system are required to be bounded in a specified time interval.

Lemma 2.2: ([29]) For a Laplacian matrix that satisfies Lemma 2.1, there exist a similarity transformation F with $F(r_t) = \begin{bmatrix} \mathbf{1} & Y_1(r_t) \end{bmatrix}$ and $F(r_t)^{-1} = \begin{bmatrix} \kappa(r_t) & Y_2(r_t) \end{bmatrix}^T$ such that

$$F(r_t)^{-1} L(r_t) F(r_t) = J(r_t) \quad (5)$$

where $J(r_t)$ is a block-diagonal matrix in the real Jordan form

$$J(r_t) = \begin{bmatrix} 0 & & & & & \\ & \lambda_1(r_t) & & & & \\ & & \ddots & & & \\ & & & \lambda_{n_\lambda}(r_t) & & \\ & & & & \mu_1(r_t) & \\ & & & & & \ddots \\ & & & & & & \mu_{n_\mu}(r_t) \end{bmatrix}$$

with $\lambda_i(r_t) \in R$ for $i = 1, \dots, n_\lambda$ and $\mu_i(r_t) = \begin{bmatrix} \alpha_i(r_t) & \beta_i(r_t) \\ -\beta_i(r_t) & \alpha_i(r_t) \end{bmatrix} \in R^{2 \times 2}$ for $i = 1, \dots, n_\mu$.

In the above expressions, $\lambda_i(r_t)$, $\alpha_i(r_t)$ and $\beta_i(r_t)$ are positive real numbers with $\lambda_i(r_t)$ denoting real eigenvalues of $L(r_t)$ and $\alpha_i(r_t) \pm j\beta_i(r_t)$ denoting complex conjugate eigenvalues of $L(r_t)$ respectively and clearly we have $1 + n_\lambda + 2n_\mu = N$.

Remark 2.2: It is important to underline that there is a great difference between asymptotic consensus and finite-time consensus. In fact, system (1) may be finite-time consentable, i.e. the state disagreement between neighbourhood subsystems starting within a specified bound c_1 does not exceed a specified bound c_2 in a specified time interval $[0 \quad T]$, but may not reach consensus after the specified interval of time. On the other hand, the disagreement between

neighbourhood subsystems might exceed the given bound over a certain time interval, but asymptotically goes to zero.

III. MAIN RESULTS

In this section, sufficient conditions are identified for the existence of finite-time consensus controller. To achieve finite-time consensus, the following protocol is adopted:

$$u_i = K(r_t) \sum_{j=1}^N q_{ij}(r_t) (x_j - x_i) \quad (6)$$

where $q_{ij}(r_t)$ are adjacency elements of the stochastic jump graph $G(r_t)$, and $K(r_t) \in R^{m \times n}$ is the mode-dependent control gain to be designed. According to the relationship between the adjacency matrix and Laplacian matrix, it is easy to get

$$u_i = -K(r_t) \sum_{j=1}^N l_{ij}(r_t) x_j \quad (7)$$

Substituting protocol (7) into the system (1), and denoting $x = \begin{bmatrix} x_1^T & x_2^T & \cdots & x_N^T \end{bmatrix}^T$, $w = \begin{bmatrix} w_1^T & w_2^T & \cdots & w_N^T \end{bmatrix}^T$, the closed-loop system is expressed as follows:

$$\dot{x} = (I_N \otimes A - L(r_t) \otimes BK(r_t)) x + (I_N \otimes B_w) w \quad (8)$$

where $L(r_t)$ is the Laplacian matrix associated with jump graph $G(r_t)$, and $\otimes$ denotes the Kronicker product of matrices.

Based on the state transformation (3), one has

$$\begin{aligned} \varsigma &= x - \left(\left(\mathbf{1} \kappa(r_t)^T \right) \otimes I_n \right) x \\ &= M(r_t) \otimes I_n x \end{aligned} \quad (9)$$

where $M(r_t) = I_N - \mathbf{1} \kappa(r_t)^T$.

Denoting $\varsigma_i = \begin{bmatrix} \varsigma_1^T & \varsigma_2^T & \cdots & \varsigma_N^T \end{bmatrix}^T$, and employing the properties $L(r_t) \mathbf{1} = 0$ and $\kappa(r_t)^T L(r_t) = 0$, the network dynamics can be transformed to the following system

$$\dot{\varsigma} = (I_N \otimes A - L(r_t) \otimes BK(r_t)) \varsigma + (M(r_t) \otimes B_w) w \quad (10)$$

To exploit the structure of $L(r_t)$, let $\xi = T(r_t)^{-1} \otimes I_n \varsigma$, then system (10) can be restated in terms of ξ as follows:

$$\dot{\xi} = (I_N \otimes A - J(r_t) \otimes BK(r_t)) \xi + (T(r_t)^{-1} M(r_t) \otimes B_w) w \quad (11)$$

For the notational convenience, let us partition the state variable ξ into ξ_i for $i = 1, \dots, N$.

Then from equations (3) and (9), and for $i = 1$, we have

$$\begin{aligned}\xi_1 &= \kappa(r_t)^T \otimes I_n \varsigma \\ &= \left(\kappa(r_t)^T M(r_t) \right) \otimes I_n x \\ &\equiv 0\end{aligned}\tag{12}$$

and for $i = 2, \dots, n_\lambda$

$$\dot{\xi}_i = (A - \lambda_i(r_t) BK(r_t)) \xi_i + (T(r_t)^{-1} M(r_t))_i \otimes B_w w\tag{13}$$

and for $i = n_\lambda + 1, \dots, N$

$$\dot{\xi}_{i_1} = (A - \alpha_i(r_t) BK(r_t)) \xi_{i_1} - \beta_i(r_t) BK(r_t) \xi_{i_2} + (T(r_t)^{-1} M(r_t))_{i_1} \otimes B_w w\tag{14}$$

$$\dot{\xi}_{i_2} = (A - \alpha_i(r_t) BK(r_t)) \xi_{i_2} + \beta_i(r_t) BK(r_t) \xi_{i_1} + (T(r_t)^{-1} M(r_t))_{i_2} \otimes B_w w\tag{15}$$

where $i_1 = 1 + n_\lambda + 2k - 1$, $i_2 = 1 + n_\lambda + 2k$, $k \in 1, \dots, n_\mu$, and $(T(r_t)^{-1} M(r_t))_i$ denotes the i th row of matrix $T(r_t)^{-1} M(r_t)$. For notational simplicity, when $r_t = r$, $r \in M$, $K(r_t)$, $\lambda_i(r_t)$, $\alpha_i(r_t)$, $\beta_i(r_t)$ and $T(r_t)^{-1} M(r_t)$ are respectively denoted as K_r , $\lambda_{i,r}$, $\alpha_{i,r}$, $\beta_{i,r}$ and $T_r^{-1} M_r$.

Remark 3.1: With the new model transformation method, the network dynamics is transformed into system (11) in the form of Laplacian matrix in real Jordan form and similarity transformation matrix T_r^{-1} . Then for the notational convenience, system (11) is portioned into subsystems (13)-(15), which is concerned with not only the jump modes $M = \{1, 2, \dots, S\}$, but also the number of subsystems $i = \{1, \dots, N\}$. What is more, because of the Kronecker product between $(T_r^{-1} M_r)_i$ and B_w , it is difficult to partition the disturbance $w(t)$ into $w_i(t)$, which makes the controller design problem complicated.

We shall design the control gain matrix K_r to ensure that the states of system (1) can reach finite-time consensus. The following theorem presents a solution to the problem of finite-time consentabilizing controller design.

Theorem 3.1: For a network-connected dynamic system (1) and a given time-constant $T > 0$, the finite-time consensus problem can be solved under mode-dependent protocol (6) with respect to $\begin{pmatrix} c_1 & c_2 & T & R & d \end{pmatrix}$, if there exist positive constant $\eta > 0$ and $\gamma^2 > 0$, mode-

dependent symmetric positive definite matrix X_r , mode-dependent matrix Y_r and $\lambda > 0$ satisfied the following LMIs for each $r \in M$:

$$\begin{bmatrix} X_r A^T + A X_r - \sigma B Y_r - \sigma Y_r^T B^T + \pi_{rr} X_r - \eta X_r & \Psi_{1r} & \Psi_{2r} \\ & \Psi_{1r}^T & -\Psi_{3r} & 0 \\ & \Psi_{2r}^T & 0 & -\Psi_{4r} \end{bmatrix} < 0 \quad (16)$$

$$\lambda R^{-\frac{1}{2}} T_r^{-1} T_r^{-T} R^{-\frac{1}{2}} < X_r < R^{-\frac{1}{2}} T_r^{-1} T_r^{-T} R^{-\frac{1}{2}} \quad (17)$$

$$\begin{bmatrix} -e^{-\eta T} \eta c_2 + (N-1)^2 \gamma^2 d (1 - e^{-\eta T}) & \sqrt{\eta c_1} \\ \sqrt{\eta c_1} & -\lambda \end{bmatrix} < 0 \quad (18)$$

where

$$\Psi_{1r} = \begin{bmatrix} (T_r^{-1} M_r)_{i1} B_w & (T_r^{-1} M_r)_{i2} B_w & \cdots & (T_r^{-1} M_r)_{iN} B_w \end{bmatrix}$$

$$\Psi_{2r} = [\sqrt{\pi_{r1}} X_r \quad \cdots \quad \sqrt{\pi_{r(r-1)}} X_r \quad \sqrt{\pi_{r(r+1)}} X_r \quad \cdots \quad \sqrt{\pi_{rN}} X_r]$$

$$\Psi_{3r} = \text{diag} \left\{ \gamma^2 I, \gamma^2 I, \dots, \gamma^2 I \right\}$$

$$\Psi_{4r} = \text{diag} \{ X_1, \dots, X_{r-1}, X_{r+1}, \dots, X_N \}$$

$$\sigma = \min \{ \lambda_{1,r}, \dots, \lambda_{n_\lambda,r}, \alpha_{1,r}, \dots, \alpha_{n_\mu,r} \}$$

If the above four linear matrix inequalities (LMIs) are feasible, then the feedback matrix of the consensus protocol is given by $K_r = Y_r X_r^{-1}$.

Proof: Consider a stochastic Lyapunov function candidate as

$$V(\xi_i(t), r_t = r) = V(\xi_i, r) = \xi_i^T P_r \xi_i \quad (19)$$

where P_r is mode-dependent positive definite symmetric matrix for each r . For $i = 2, \dots, n_\lambda$, along the trajectories of system (13), the corresponding time derivative of $V(\xi_i, r)$ is given by

$$\begin{aligned} & \frac{d}{dt} E \{ V(\xi_i, r) \} \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} [E \{ V(\xi_i(t + \Delta t), r_{t+\Delta t}, t + \Delta t) | \xi_i(t) = \xi_i, r_t = r \} - V(\xi_i(t), r, t)] \\ &\leq \xi_i^T \left((A - \lambda_{i,r} B K_r)^T P_r + P_r (A - \lambda_{i,r} B K_r) + \sum_{l=1}^S \pi_{rl} P_l \right) \xi_i \\ &+ 2 \xi_i^T P_r (T_r^{-1} M_r)_i \otimes B_w w \end{aligned} \quad (20)$$

For $i = n_\lambda + 1, \dots, N$, we consider $V(\xi_i, r)$ in pairs for the complex conjugate eigenvalues. It can be obtained in a similar way to the real eigenvalue case that

$$\begin{aligned}
& \frac{d}{dt} E \{V(\xi_{i_1}, r)\} + \frac{d}{dt} E \{V(\xi_{i_2}, r)\} \\
& \leq \xi_{i_1}^T \left((A - \alpha_{i,r} B K_r)^T P_r + P_r (A - \alpha_{i,r} B K_r) + \sum_{l=1}^S \pi_{rl} P_l \right) \xi_{i_1} \\
& + \xi_{i_2}^T \left((A - \alpha_{i,r} B K_r)^T P_r + P_r (A - \alpha_{i,r} B K_r) + \sum_{l=1}^S \pi_{rl} P_l \right) \xi_{i_2} \\
& + 2\xi_{i_1}^T P_r (T_r^{-1} M_r)_{i_1} \otimes B_w w + 2\xi_{i_2}^T P_r (T_r^{-1} M_r)_{i_2} \otimes B_w w
\end{aligned} \tag{21}$$

Letting $V(\xi(t), r) = \sum_{i=2}^N V(\xi_i, r)$, if the following condition holds

$$\begin{bmatrix} (A - \sigma B K_r)^T P_r + P_r (A - \sigma B K_r) + \sum_{l=1}^S \pi_{rl} P_l - \eta P_r & P_r (T_r^{-1} M_r)_i \otimes B_w \\ * & -\gamma^2 I \end{bmatrix} < 0 \tag{22}$$

Then combining conditions (22) and (20)-(21), we have

$$\frac{d}{dt} E \{V(\xi(t), r)\} < \eta E \{V(\xi(t), r)\} + (N-1) \gamma^2 w(t)^T w(t) \tag{23}$$

Multiplying both sides of (23) by $e^{-\eta t}$, yields

$$\frac{d}{dt} E \{e^{-\eta t} V(\xi(t), r)\} < (N-1) \gamma^2 e^{-\eta t} w(t)^T w(t) \tag{24}$$

Integrating equation (24) from 0 to t , it follows that

$$\begin{aligned}
E \{V(\xi(t), r)\} & < e^{\eta t} \left(V(\xi(0), r_0) + (N-1) \gamma^2 \int_0^t e^{-\eta \tau} w(\tau)^T w(\tau) d\tau \right) \\
& < e^{\eta t} \left(V(\xi(0), r_0) + (N-1)^2 \gamma^2 d \frac{(1 - e^{-\eta t})}{\eta} \right)
\end{aligned} \tag{25}$$

Defining $\tilde{P}_r = R^{-1/2} T_r^{-T} P_r T_r^{-1} R^{-1/2}$, and according to condition (25) and $\xi = T_r^{-1} \otimes I_n \varsigma$, it follows that

$$E \left\{ \varsigma_i^T R^{1/2} \tilde{P}_r R^{1/2} \varsigma_i \right\} < e^{\eta t} \left[c_1 \lambda_{\max}(\tilde{P}_r) + (N-1)^2 \gamma^2 d \frac{(1 - e^{-\eta t})}{\eta} \right] \tag{26}$$

On the other hand, the following condition holds

$$E \left\{ \varsigma_i^T R^{1/2} \tilde{P}_r R^{1/2} \varsigma_i \right\} \geq \lambda_{\min}(\tilde{P}_r) E \left\{ \varsigma_i^T R \varsigma_i \right\} \tag{27}$$

Putting together (26) and (27), we have

$$E \left\{ \varsigma_i^T R \varsigma_i \right\} < \frac{e^{\eta t} \left[c_1 \eta \lambda_{\max}(\tilde{P}_r) + (N-1)^2 \gamma^2 d (1 - e^{-\eta t}) \right]}{\lambda_{\min}(\tilde{P}_r) \eta} \tag{28}$$

If the following condition is satisfied

$$\frac{e^{\eta t} \left[c_1 \eta \lambda_{\max}(\tilde{P}_r) + (N-1)^2 \gamma^2 d (1 - e^{-\eta t}) \right]}{\lambda_{\min}(\tilde{P}_r) \eta} < c_2 \quad (29)$$

which implies that for $\forall t \in [0, T]$, $E \{ \zeta_i^T R \zeta_i \} < c_2$. Then the network-connected dynamic system (1) is said to be finite-time consentable with respect to $\begin{pmatrix} c_1 & c_2 & T & R & d \end{pmatrix}$.

To obtain the gains of the designed protocol, next we will reduce conditions (22) and (29) to feasibility problems involving LMIs.

Pre- and post-multiplying the inequality (22) by block-diagonal matrix $\text{diag} \left\{ P_r^{-1} \quad I \right\}$, letting $X_r = P_r^{-1}$, $Y_r = K_r X_r$ and applying Schur complement, it leads to inequality (16).

According to $\tilde{P}_r = R^{-1/2} T_r^{-T} P_r T_r^{-1} R^{-1/2}$, one has

$$\tilde{X}_r = \tilde{P}_r^{-1} = R^{1/2} T_r X_r T_r^T R^{1/2}$$

$$\lambda_{\max}(\tilde{X}_r) = \frac{1}{\lambda_{\min}(\tilde{P}_r)}$$

Condition (29) follows that

$$\frac{c_1 \eta}{\lambda_{\min}(\tilde{X}_r)} + (N-1)^2 \gamma^2 d (1 - e^{-\eta t}) < \frac{e^{-\eta t} \eta c_2}{\lambda_{\max}(\tilde{X}_r)} \quad (30)$$

Define

$$\lambda_{\max}(\tilde{X}_r) < 1, \lambda < \lambda_{\min}(\tilde{X}_r) \quad (31)$$

From (31), we can have conditions (17) and (18). Putting (30) and (31) together, the desired condition (19) can be obtained, which completes the proof.

Remark 3.2: It should be noted that inequality (23) with $\eta > 0$ in the proof relaxes the requirement on Lyapunov energy function by allowing it increase, which leads to less conservativeness of the results. If we set $\eta = 0$ and make the assumption that the communication topology is fixed, then the condition derived in (16) will reduce to asymptotical consensus results obtained in many literatures [5], [29]. In this case, the states of network-connected dynamic system (1) can also asymptotically reach to an identical value via the designed control protocol (6).

Remark 3.3: From Theorem 3.1 it can be seen that the finite-time consensus of the system under consideration is influenced by not only the eigenvalues of Laplacian matrix, but also the transition rates of communication topology. What is more, to obtain an optimized finite-time consensus controller, the bound of disagreement between neighbourhood subsystems can be

reduced to the minimum possible value such that LMIs (16)-(18) are satisfied. The optimization problem can be described as follows:

$$\begin{aligned} & \min_{X_r, Y_r, \lambda, c_2} c_2 \\ & s. t. \text{ LMIs (16) - (18)} \end{aligned} \quad (32)$$

IV. NUMERICAL EXAMPLE

In this section, we will provide an example to demonstrate the effectiveness of the proposed methods. The system under consideration is a connection of four subsystems, and each of them is described by a second order state-space model as

$$\dot{x}_i(t) = \begin{bmatrix} -1.48 & 0.96 \\ 1.57 & 1.95 \end{bmatrix} x_i(t) + \begin{bmatrix} 1 \\ 5 \end{bmatrix} u_i(t) + \begin{bmatrix} 0.1 \\ -0.3 \end{bmatrix} w_i(t)$$

The interconnection topology jumps between G_1 and G_2 with adjacency matrix described as:

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, Q_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and the transition rate between G_1 and G_2 is

$$\Pi = \begin{bmatrix} -0.7 & 0.7 \\ 0.4 & -0.4 \end{bmatrix}$$

The external disturbance is supposed to be

$$\begin{aligned} w(t) &= \begin{bmatrix} w_1(t) & w_2(t) & w_3(t) & w_4(t) \end{bmatrix}^T \\ &= \begin{bmatrix} 2\omega(t) & -\omega(t) & 3\omega(t) & -1.5\omega(t) \end{bmatrix}^T \end{aligned}$$

with $\omega(t) = \sin(t)$.

According to the relationship between Q and L , the resultant Laplasian matrices are obtained as

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}, L_2 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

TABLE I: Minimum upper bound of c_2 and maximum lower bound of c_1 for different values of T and d

T	d	c_1	c_2
5	2	0.0056	0.1418
5	3	0.0062	0.1326
5	4	0.0066	0.1260
5	5	0.0069	0.1193
6	5	0.0072	0.1115
7	5	0.0075	0.1048
8	5	0.0077	0.0992
9	5	0.0079	0.0946

According to the eigenvalues of L_r for $r = 1, 2$, it is easy to get that

$$J_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3/2 & \sqrt{3}/2 \\ 1 & 0 & -\sqrt{3}/2 & 3/2 \end{bmatrix}, J_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

with the transformation matrices

$$T_1 = \begin{bmatrix} 1 & 0 & -1/2 & 0 \\ 1 & 0 & 1/4 & -\sqrt{3}/2 \\ 1 & 1 & -1/2 & 0 \\ 1 & 0 & 1/4 & \sqrt{3}/2 \end{bmatrix}, T_2 = \begin{bmatrix} 1 & -1 & -1/2 & 0 \\ 1 & 0 & 1/2 & 0 \\ 1 & 1/2 & -1/2 & 1/2 \\ 1 & 0 & -1/2 & 0 \end{bmatrix}$$

and

$$T_1^{-1} = \begin{bmatrix} 1/3 & 1/3 & 0 & 1/3 \\ -1 & 0 & 1 & 0 \\ -4/3 & 2/3 & 0 & 2/3 \\ 0 & -2\sqrt{3}/3 & 0 & 2\sqrt{3}/3 \end{bmatrix}, T_2^{-1} = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 2 & -1 \end{bmatrix}$$

Our purpose here is to design a finite-time consensus controller in the form of (6) such that the disagreement among subsystems stays in a given bound in the presence of external disturbance and Markov jump topology. In order to investigate the effect of T and d in the search of maximum lower bound of c_1 and minimum upper bound of c_2 , the result of c_1 and c_2 with the change of T and d are summarized in Table I. As shown in Table I, the increasing of T and d lead to the

Fig. 1: Evolution of topology versus time

bigger bound of c_1 and smaller bound of c_2 , which is consistent with the practical situation.

Introducing the initial value for $c_1 = 0.1$, $c_2 = 0.3$, $T = 5$, $R = I_2$, $d = 2$ and $\eta = 1$, and by solving (16)-(18) in Theorem 3.1, the corresponding mode-dependent controller gains are computed as follows:

$$K_1 = [0.2868 \quad 0.7346]$$

$$K_2 = [0.3332 \quad 0.8326]$$

To demonstrate the effectiveness of the design method, the state disagreement among subsystems under control protocol (6) in Markov jump topologies $\{G_1, G_2\}$ in finite-time interval is shown in Fig. 2 by generating a possible mode evolution of topology shown in Fig. 1. It can be seen that the state disagreement stays within the specified bound $c_2 = 0.3$ over the finite-time horizon $T = 5$ by the designed controller for given initial states $x_{10} = [0.3 \quad 0.5]^T$, $x_{20} = [0.2 \quad 0.4]^T$, $x_{30} = [0.25 \quad 0.45]^T$ and $x_{40} = [0.28 \quad 0.47]^T$.

In order to verify the asymptotical consensus does not imply finite-time consensus, we give another example as follows:

$$\dot{x}_i(t) = \begin{bmatrix} 1.48 & -1.96 \\ 1.57 & 1.95 \end{bmatrix} x_i(t) + \begin{bmatrix} 1 \\ 5 \end{bmatrix} u_i(t) + \begin{bmatrix} 0.1 \\ -0.3 \end{bmatrix} w_i(t)$$

The interconnection topology with adjacency matrix is assumed to be same as the above example. Solving condition (16) in Theorem 1 with $\eta = 0$, we get the following asymptotical consensus controller:

$$K_1 = [-21.5685 \quad 15.5759]$$

$$K_2 = [-2.7860 \quad 2.5697]$$

Fig. 2: Disagreement trajectory of the system in finite-time interval

The disagreement state response is shown in Fig. 3, and the state disagreement trajectory with $c_1 = 0.1, c_2 = 0.8$ and $T = 5$ is shown in Fig. 4. From Fig. 3 and Fig. 4, we can see that the asymptotical consensus does not imply finite-time consensus.

Remark 4.1: It should be pointed out that in the simulation example, as long as the choice of initial condition is satisfied with $E \{ \varsigma_i^T(0) R \varsigma_i(0) \} \leq c_1$, then the network-connected dynamic system (1) is finite-time consentable, i.e. the states disagreement of all subsystems stay within a given bound $E \{ \varsigma_i^T(t) R \varsigma_i(t) \} < c_2$ in finite-time interval.

V. CONCLUSION

This paper considers the problem of finite-time consensus controller design for network-connected dynamic systems. The controller is obtained under the assumption that zero is a single eigenvalue of Laplasian matrix. If this assumption is satisfied, the designed controller guarantees

Fig. 3: Disagreement state response in infinite time interval

that the state disagreement of all subsystems stays within a given bound during finite time under the presence of bounded disturbances and Markov jump network topologies. It is shown that less conservative conditions are derived by allowing Lyapunov function to increase in a certain time interval. Note that this article focuses on the finite-time consensus under bounded external disturbance with the power distributed in full frequency domain. Future direction can investigate the finite frequency controller design for network-connected systems subject to specific frequency ranges of external disturbance.

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Fig. 4: Disagreement trajectory in finite time interval

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REFERENCES

- [1] J. A. Fax and R. M. Murray. Information flow and cooperative control of vehicle formation. *IEEE Transactions on Automatic Control*, 49:1465–1476, 2004.
- [2] J. Huang, S. M. Farritor, A. Qadi, and S. Goddard. Localization and follow-the-leader control of a heterogeneous group of mobile robots. *IEEE/ASME Transactions on Mechatronics*, 11(2):205–215, 2006.
- [3] A. Jadbabaie and A. Morse. Coordination of groups of autonomous mobile agents using nearest rules. *IEEE Transactions on Automatic Control*, 48(6):988–1001, 2003.
- [4] H Du and S Li. Attitude synchronization control for a group of flexible spacecraft. *Automatica*, 50(2):646–651, 2014.
- [5] Z. Li, Z. Duan, G. Chen, and L. Huang. Consensus of multi-agent systems and synchronization of complex networks, a unified viewpoint. *IEEE Transactions on Circuits and Systems I*, 57(1):213–224, 2010.
- [6] Y. Su and J. Huang. Stability of a class of linear switching systems with applications to two consensus problems. *IEEE Transactions on Automatic Control*, 57:1420–1430, 2012.

- [7] L. Scardovi and R. Sepulchre. Synchronization in networks of identical linear systems. *Automatica*, 45(11):2557–2562, 2009.
- [8] Z. Ding. Consensus output regulation of a class of heterogeneous nonlinear systems. *IEEE Transactions on Automatic Control*, 58:2648–2653, 2013.
- [9] H. Du and Y. He and Y. Cheng. Finite-time synchronization of a class of second-order nonlinear multi-agent systems using output feedback control. *IEEE Transactions on Circuits and Systems IRegular Paper*, 61(6):1778–1788, 2014.
- [10] W. W. Yu, G. R. Chen, and M. Gao. Consensus in directed networks of agents with nonlinear dynamics. *IEEE Transactions on Automatic Control*, 56(6):1436–1441, 2011.
- [11] R. O. Saber and R. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9):1520–1533, 2004.
- [12] P. Lin, Y. Jia, and L. Li. Distributed robust H_∞ consensus control in directed networks of agents with time-delay. *System and Control Letters*, 57(8):643–653, 2008.
- [13] Z. Li, Z. Duan, and G. Chen. On H_∞ and H_2 performance regions of multi-agent systems. *Automatica*, 47(4):797–803, 2011.
- [14] V. Ugrinovskii. Distributed robust filtering with H_∞ consensus of estimates. *Automatica*, 47:1–13, 2011.
- [15] Y. C. Cao, W. W. Wu, W. Ren, and G. R. Chen. An overview of recent process in the study of distributed multi-agent coordination. *IEEE Transactions on Industrial Informatics*, 9(1):427–438, 2013.
- [16] Y. Hatano and M. Mesbahi. Agreement over random networks. *IEEE Transactions on Automatic Control*, 50:1867–1872, 2005.
- [17] M. Porfiri and D. J. Stilwell. Consensus seeking over random weighted directed graphs. *IEEE Transactions on Automatic Control*, 52:1767–1773, 2007.
- [18] Y. Zhang and Y. P. Tian. Consentability and protocol design of multi-agent systems with stochastic switching topology. *Automatica*, 45:1195–1201, 2009.
- [19] J. Cortes. Finite-time convergence gradient flows with applications to network consensus. *Automatica*, 42:1993–2000, 2006.
- [20] Q. Hui, W. M. Haddad, and S. P. Bhat. Finite-time semistability and consensus for nonlinear dynamical networks. *IEEE Transactions on Automatic Control*, 53(8):1887–1900, 2008.
- [21] S. H. Li, H. B. Du, and X. Z. Lin. Finite-time consensus algorithm for multi-agent systems with double-integrator dynamics. *Automatica*, 47:1706–1712, 2011.
- [22] P. Dorato. Short time stability in linear time-varying systems. In *Proceedings of the IRE International Convention Record Part 4*, pages 83–87, New York, 1961.
- [23] F. Amato, G. Tommasi, and A. Pironti. Finite-time control of linear systems subject to parameteric uncertainties and disturbances. *Automatica*, 49:2546–2550, 2013.
- [24] Y. Shen. Finite-time control of linear parameter-varying systems with norm-bounded exogenous disturbance. *Journal of Control Theory and Applications*, 6:184–188, 2008.
- [25] G. Chen and Y. Yang. Finite-time stability of switched positive linear systems. *International Journal of Robust and Nonlinear Control*, 24(1):179–190, 2014.
- [26] Y. Wang and X. Shi and G. Wang and Z. Zuo. Finite-time stability for continuous-time switched systems in the presence of impulse effects. *IET Control Theory and applications*, 6(11):1741–1744, 2014.
- [27] F. Amato and M. Ariola. Finite-time control of discrete-time linear systems. *IEEE Transactions on Automatic Control*, 50(5):724–729, 2005.

- [28] X. L. Luan, P. Shi, and F. Liu. Finite-time stabilization for Markov jump systems with Gaussian transition probabilities. *IET Control Theory and applications*, 7(2):298–304, 2013.
- [29] Z. Ding. Consensus control of a class of Lipschitz nonlinear systems. *International Journal of Control*, 87(11):2372–2382, 2014.