LETTER TO THE EDITOR

Quasiparticle pairing at arbitrary densities: a soluble model

R F Bishop† and M R Strayer‡

† Department of Theoretical Physics, University of Manchester, Manchester M13 9PL, UK and Daresbury Laboratory, Science Research Council, Daresbury, Warrington WA4 4AD, UK

‡ Daresbury Laboratory, Science Research Council, Daresbury, Warrington WA4 4AD, UK

Received 5 November 1976

Abstract. Quasiparticle pairing in a many-body medium, with pairwise boundary-condition model interactions, is discussed within a formalism which treats two-particle and two-hole interactions simultaneously. We find the possibility of two sorts of bound pairs, one predominantly particle-like, which corresponds at zero density to a free bound state, and the other predominantly hole-like, which appears only above a certain critical density, and show that they form a continuous spectrum.

In several previous publications (Bishop *et al* 1974, 1975, 1976, Ghassib *et al* 1976) we have considered the Galitskii–Feynman (GF) *T*-matrix approach to dense Fermi systems, both at zero and at finite temperatures. In particular we have stressed the possibility, above some critical density, of the formation of bound pairs of quasiparticles which are themselves linear combinations of bare particle and hole states. In contrast to the more usual Brueckner–Bethe–Goldstone (BG) approach, which considers only two-particle interactions in the many-body medium, the GF formalism explicitly treats particle–particle and hole–hole interactions symmetrically. The role of the hole states in binding quasiparticle pairs in the GF formalism is illustrated by the fact that such binding can occur even when the bare potential supports no free (i.e. *in vacuo*) two-body bound states, or also when there is no possibility of BG bound pairs. In particular we demonstrate here that the inclusion of hole–hole interactions can dramatically alter the properties of a many-body system *even at arbitrarily low densities*, in contradiction to widely-held views to the contrary.

In this Letter we wish to apply previously used techniques (Bishop *et al* 1974, 1975, Ghassib *et al* 1976) to some particularly simple model interactions, namely the pure hard core (HC) and its generalization, the boundary-condition model (BCM) (Breit and Bouricius 1949, Feshbach and Lomon 1956). Such models are of interest both in their own right as simple examples of singular interactions, and also practically, since many excellent models of the internucleon force, for example, have incorporated either the HC or the BCM. For our present purposes the attraction of the BCM is its simplicity. We expect that in the GF formalism, the BCM will display all of the general pairing phenomena discussed above, but will prove tractable to essentially exact solution, just as in the free-interaction (Bishop 1973) and BG cases (Bishop 1976). Furthermore, by adjusting the free parameters of the BCM, we can vary the energy of any free bound states, or exclude them entirely. Thus in the context of nuclear theory, we can, for

example, model the deuteron and attempt to shed some light on the old question as to what happens to a deuteron when it is imbedded in a many-nucleon system whose density may be varied.

The GF T matrix for two bodies interacting via a potential V is the solution to an integral equation which is the analogue of the Bethe–Salpeter equation (Salpeter and Bethe 1951) in the ladder approximation. In the relative momentum representation the fully off-shell GF T matrix has the form $T(\mathbf{p}, \mathbf{p}'; z, \mathbf{P}) \equiv \langle \mathbf{p} | T(z, \mathbf{P}) | \mathbf{p}' \rangle$, where $\mathbf{p}$ and $\mathbf{p}'$ are the relative incoming and outgoing momenta of the pair, $\mathbf{P}$ is their (conserved) average momentum and $\hbar^2 z / M$ is their relative energy in the centre-of-mass frame considered as a complex variable. M is the mass of each (identical) fermion. In operator notation, the GF T -matrix equation is

$$T(z, \mathbf{P}) = u - u[Q(\mathbf{P})g_0(z) - \bar{Q}(\mathbf{P})g_0^\dagger(z)]T(z, \mathbf{P}) \quad (1)$$

where $u = M\hbar^{-2}V = \frac{1}{2}V^\dagger$. For simplicity we consider the system to be at zero temperature, when $Q(\bar{Q})$ is equal to unity if both particles are outside (inside) the Fermi sea characterized by a Fermi momentum k_F , and zero otherwise. In equation (1), the operator $g_0(z) = (\frac{1}{2}H_0 - z)^{-1}$ is the free two-particle propagator, where H_0 is their relative kinetic energy operator. The physical regime is obtained by letting z approach real values, either positive or negative, to describe the scattering and bound-state regimes respectively. Free two-particle interactions are similarly described by the free t -matrix solution to the Lippmann–Schwinger equation

$$t(z) = u - u g_0(z) t(z) \quad (2)$$

obtained from equation (1) by letting $Q \rightarrow 1$, $\bar{Q} \rightarrow 0$. For singular potentials it is particularly advantageous to eliminate the potential u between equations (1) and (2), giving the result

$$T = t + t[(1 - Q)g_0 + \bar{Q}g_0^\dagger]T. \quad (3)$$

In order to keep the further analysis as simple as possible, we shall specialize to the case $\mathbf{P} = 0$, which is probably the physically most interesting situation. This results in the simplification

$$Q^2 = Q = 1 - \bar{Q} \quad \langle \mathbf{p} | Q | \mathbf{p}' \rangle = (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}') \theta(p - k_F) \quad (4)$$

but results in no loss of generality for the phenomena described below. Equation (3) is then readily decomposed into partial waves as

$$T_l(p, p'; z) = t_l(p, p'; z) + (2\pi^2)^{-1} \int_0^{k_F} q^2 dq t_l(p, q; z) \left(\frac{1}{q^2 - z} + \frac{1}{q^2 - z^*} \right) T_l(q, p'; z). \quad (5)$$

We now turn to the solution of equation (5) for the pure BCM, which comprises for two free particles a core interaction inside which their wavefunction vanishes, and a specified energy-independent logarithmic derivative of the wavefunction just outside the core radius a :

$$\lim_{\delta \rightarrow 0} (\psi'_{k,l}(a + \delta) / \psi_{k,l}(a + \delta)) = \lambda_l. \quad (6)$$

Beyond the core the interaction vanishes. The pure HC interaction is retrieved from the BCM as the limiting case, $\lambda_l \rightarrow \infty$. The fully off-shell BCM t matrix can be expressed

† Henceforth we employ units $2M = \hbar = 1$.

in terms of the half-shell quantity $\tilde{f}_l(p, z^{1/2}) \equiv t_l(p, z^{1/2}; z)$, where $z^{1/2}$ is defined to have positive imaginary component, as (Bishop 1973†)

$$\begin{aligned} t_l(p, p'; z) &= (p'^2 - z)\theta_l(p, p') + b_l(p', z^{1/2})\tilde{f}_l(p, z^{1/2}) \\ \tilde{f}_l(p, p') &= \bar{d}_l(p)\bar{m}_l(p') \\ \theta_l(p, p') &= (d_l(p')c_l(p) - d_l(p)c_l(p'))/(p'^2 - p^2) \\ b_l(p, p')m_l(p') &= c_l(p) + d_l(p)(m_l(p') - c_l(p'))/d_l(p') \\ \bar{d}_l(p) &= d_l(p) - \mu_l c_l(p) \\ \bar{m}_l^{-1}(p) &= m_l^{-1}(p) - \mu_l(c_l(p)m_l^{-1}(p) - 1)/d_l(p) \end{aligned} \quad (7)$$

in terms of the convenient notation

$$\begin{aligned} d_l(p) &= 4\pi a^2 j_l(pa) & c_l(p) &= p j_l'(pa) \\ m_l(p) &= (ipa^2 h_l^+(pa))^{-1} & \mu_l &= 4\pi a^2/\lambda_l \end{aligned} \quad (8)$$

where the functions $j_l(x)$ and $h_l^+(x)$ are the usual spherical Bessel and Hankel functions of the first kind. Any bound states in the channel l show themselves as usual as simple poles, in the variable z , in the matrix t_l . From equation (7) one can show that the BCM supports free bound states of energy $\epsilon_B = -\hbar^2 k_B^2/M$ for solutions $k_B > 0$ to the equation $\bar{m}_l^{-1}(ik_B) = 0$. Thus, for S waves, the BCM has one free bound state with $k_B a = -(1 + \lambda_0 a)$ for $\lambda_0 a < -1$ and none otherwise.

It is possible to solve equations (5) and (7) for arbitrary l and z (Bishop and Strayer 1976), but we consider here only bound states (i.e. z real and negative), and with no loss of generality we restrict ourselves to S waves. Just as in the free-interaction case, it is possible to show that for two-body bound states within the many-body medium, with energies $E_B = -\hbar^2 K_B^2/M$, the GF T matrix behaves as (Bishop and Strayer 1976)

$$\begin{aligned} T_l(p, p'; z) &\xrightarrow{z \rightarrow -K_B^2} N_l^{-1} \bar{D}_l(p) \bar{D}_l(p')/(z + K_B^2) \\ N_l &= (2\pi^2)^{-1} \int_0^\infty p^2 dp q(p) \bar{D}_l^2(p) (p^2 + K_B^2)^{-2} \\ q &\equiv Q - \bar{Q} \quad q(p) = 2\theta(p - k_F) - 1. \end{aligned} \quad (9)$$

The bound-state GF wavefunctions

$$\Psi_B(p) = -(\pm N_l)^{-1/2} q(p) \bar{D}_l(p) (p^2 + K_B^2)^{-1} \quad (10)$$

satisfy a generalized orthonormality condition (Bishop and Strayer 1976)

$$\langle \Psi_{B_i} | q | \Psi_{B_j} \rangle = \pm \delta_{ij} \quad (11)$$

where the choice of sign in equations (10) and (11) is made which makes the root $(\pm N_l)^{1/2}$ real (and positive). Equations (5), (7) and (9) readily yield that the possible GF bound-state wavefunctions and energies are given by the equations

$$\bar{D}_l(p) = \bar{d}_l(p) + 2(2\pi^2)^{-1} \int_0^{k_F} q^2 dq \theta_l(p, q) \bar{D}_l(q) \quad (12)$$

$$1 = 2\bar{m}_l(iK_B) \int_0^{k_F} \frac{q^2 dq}{2\pi^2} \frac{b_l(q, iK_B) \bar{D}_l(q)}{(q^2 + K_B^2)}. \quad (13)$$

† This relation (equation (7)) was proved by Bishop (1973) only in the explicit absence of bound states. It is not difficult to prove that it remains valid even in their presence.

The integral equation (12) is first solved numerically by choosing a suitable quadrature rule to approximate the integral term. The resulting set of linear equations is solved by standard matrix inversion techniques. Equation (13) is approximated by the same quadrature rule, and is solved iteratively to find all the solutions $K_B > 0$ for the particular choice of Fermi momentum k_F and the BCM parameters $\lambda (\equiv \lambda_0)$ and a . (These three parameters actually occur only in the dimensionless combinations λa and $k_F a$.)

Numerical results obtained with 96-point gaussian quadrature are displayed in figure 1 for the range $k_F a < 2.38$ and it should be noticed that the point ($k_F a = 2.38, K_B a \rightarrow \infty$) is a focus for the whole BCM family, or at least for those members which do not sustain a free bound state ($\lambda a > -1$). This 'supersingularity'

Figure 1. The density dependence, $\rho = vk_F^3/(6\pi^2)$, of the energy, $E_B = -\hbar^2 K_B^2/M$, of GF quasiparticle S-wave pair bound states for various boundary-condition models with core radius a and logarithmic derivative parameter λ .

occurs for the following reason. Mathematically, at this value of $k_F a$ the solution to equation (12) (or equivalently equation (5)) becomes completely indeterminate, since the homogeneous equation now sustains a solution. That is, the operator eigenvalue equation, $\bar{D}_l = \lambda_l \theta_l \bar{Q} \bar{D}_l$, has an S-wave eigenvalue $\lambda_0 = 2$ at the value $k_F a = 2.38$. Physically, this resonant behaviour, which is independent of the strength of the attraction in the interaction, is probably peculiar to such singular models as the BCM, as opposed to the other effects described below, which we believe are quite general.

The family of curves in figure 1 divide into two distinct classes, according as there is no ($\lambda a > -1$) or one ($\lambda a < -1$) free bound state present. In the former case, the general behaviour is exactly as reported elsewhere (Bishop *et al* 1974, 1975, Ghassib *et al* 1976) for ^3He or nuclear matter in the $^1\text{S}_0$ channel. For these cases there exists a critical density below which no binding occurs, and above which one bound pair is present. On the other hand, for $\lambda a < -1$ we have the possibility of two, one or no bound states, depending on the density. In these models, as the density is increased from zero the free bound state gradually becomes less bound. At a certain lower critical density a new bound state appears whose binding energy increases with density. Finally, an upper critical density is reached at which the two bound states merge and annihilate, in the sense that above this density no bound pairs are found (at least until the density appropriate to the first supersingularity is exceeded). A revealing indication of the role played by hole-hole interactions in the formation of these bound pairs is provided by the behaviour of the so-called q -norm, N_0 , of the bound-state wavefunction, given by equations (9) and (10). For the curves with $\lambda a > -1$, the q -norm is everywhere negative, starting at minus infinity at the critical density and monotonically approaching zero as the density is increased. Conversely, for the curves with $\lambda a < -1$, the q -norm starts at a positive value at zero density (where it is equal to the usual norm), and monotonically decreases as the curve is traced along. It passes through zero at the upper critical density, and again approaches minus infinity at the lower critical density. The q -norm thus provides a quantitative estimate of whether the quasiparticles responsible for the pairing are predominantly particle-like ($N_l > 0$) or hole-like ($N_l < 0$).

We should mention that all of the behaviour observed here occurs well within the accessible physical regime for this model, namely below the classical close-packing limit, which corresponds to $k_F a < 3.43$ (for $\nu = 2$) or $k_F a < 2.76$ (for $\nu = 4$) where ν is the internal spin-isospin degeneracy of the fermions. Finally, we stress that the phenomena described above can clearly occur at arbitrarily low densities with an appropriate choice of the BCM parameter λa , which would seem to cast serious doubt on the widely-held view that hole-hole interactions are unimportant except at high densities. On the other hand, the formalism used here is a two-body approximation to the many-body problem, and it remains to be seen whether the higher-order cluster interactions merely quantitatively alter the general behaviour described, or destroy it. At present we believe the former to be true for at least certain ranges of the interaction parameters.

Further details and extensions of the present work will appear elsewhere (Bishop and Strayer 1976), together with a discussion of the physical implications of these results for real systems.

References

- Bishop R F 1973 *Phys. Rev. C* **7** 479
 —1976 *Ann. Phys., NY* **99** 318

Bishop R F, Ghassib H B and Strayer M R 1976 *Phys. Rev. A* **13** 1570

Bishop R F and Strayer M R 1976 to be published

Bishop R F, Strayer M R and Irvine J M 1974 *Phys. Rev. A* **10** 2423

—1975 *J. Low Temp. Phys.* **20** 573

Breit G and Bouricius W G 1949 *Phys. Rev.* **75** 1029

Ghassib H B, Bishop R F and Strayer M R 1976 *J. Low Temp. Phys.* **23** 393

Feshbach H and Lomon E 1956 *Phys. Rev.* **102** 891

Salpeter E E and Bethe H A 1951 *Phys. Rev.* **84** 1232