Stability of nonlinear normal modes of symmetric Hamiltonian systems

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Abstract. A nonlinear normal mode of a Hamiltonian system is a periodic solution near equilibrium with period close to that of a periodic trajectory of the linearised vector field. This paper is a sequel to the preceding paper in this issue, in which we developed methods for proving the existence of nonlinear normal modes of a Hamiltonian system that is invariant under the action of a compact Lie group. Here we show how to compute the spectral stability of these nonlinear normal modes. The methods are a natural extension of those used in the previous paper, namely, Birkhoff normal form and singularity theory. In particular we prove that under certain non-degeneracy conditions, Floquet exponents computed from a truncated Hamiltonian in normal form determine the spectral stability for the corresponding solutions of the full Hamiltonian. Applications include systems with $O(2)$ and D_n symmetry, systems with five degrees of freedom and $O(3)$ symmetry (modelling some vibrational modes of a liquid drop), and resonant time-reversible systems with Z_2 symmetry (such as the planar spring pendulum). We give explicit reductions to Birkhoff normal form for Hamiltonians of the form 'kinetic + potential' where the potential is invariant under $O(2)$ or D_n . The analysis of systems with $O(3)$ symmetry uses the dynamical invariance of fixed-point spaces of subgroups to simplify the computations considerably.

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1. Introduction

A nonlinear normal mode of a Hamiltonian system is a family of periodic solutions near an equilibrium point that approximates a periodic solution of the linearisation of the system. If the Hamiltonian system is invariant under a group of symmetries Γ then the nonlinear normal modes break the symmetry in a manner that, in principle, is predictable. In Montaldi *et al* (1988) we proved an equivariant analogue of the Weinstein–Moser theorem on the existence of certain nonlinear normal modes; and in Montaldi *et al* (1990a) we developed general methods for finding nonlinear normal modes explicitly. Here we show how to compute the spectral stability of the nonlinear normal modes obtained by these methods. The approach is a natural extension of that used in Montaldi *et al* (1990a), namely a combination of Birkhoff normal form and singularity theory.

Section 2 recalls basic results on Hamiltonian systems, symplectic geometry, and stability. In section 3 we discuss the problem of truncation of Birkhoff normal form, proving by a perturbation argument that under certain conditions the Floquet exponents computed from a truncated normal form Hamiltonian give the same stability conditions as those for the full Hamiltonian. In sections 4–6 we apply the methods to various examples: systems with $\mathbf{O}(2)$ and $\mathbf{D}_n$ symmetry, systems with five degrees of freedom and $\mathbf{O}(3)$ symmetry, and resonant time-reversible systems with $\mathbf{Z}_2$ symmetry. Section 4 includes explicit reductions to Birkhoff normal form for Hamiltonians of the form ‘kinetic + potential’. The analysis of $\mathbf{O}(3)$ symmetry in section 5 makes systematic use of the isotropy lattice to provide invariant subspaces for the dynamics, and this greatly simplifies the computations. Section 6 completes the proof of theorem 6.3 of Montaldi *et al* (1990a). The appendix derives a formula for the dimension of the space of $\Gamma \times \mathbf{S}^1$ -invariants that are homogeneous of degree k , in terms of the character of the Γ action. Throughout we assume familiarity with the methods and notation of Montaldi *et al* (1988, 1990a).

2. Systems in Birkhoff normal form

In this section we set up notation and basic conventions, and recall various results that we shall need, mainly about systems that are in exact Birkhoff normal form. In particular, we discuss the existence and spectral stability of nonlinear normal modes. In practice such systems arise by transforming a system into Birkhoff normal form to some order and then truncating. The effects of truncation are studied in section 3. This section is based on Montaldi *et al* (1988, 1990a).

2.1. Notation and conventions

We consider a Hamiltonian function $H: V \rightarrow \mathbb{R}$ on a symplectic manifold (phase space) V , defined in the neighbourhood of an equilibrium point, and invariant under a symplectic action of a compact Lie group Γ which fixes the equilibrium. Since we are concerned only with local behaviour, we may assume that V is a linear space with the equilibrium point at 0, and use coordinates $(p, q) = (p_1, \dots, p_n; q_1, \dots, q_n)$ to identify V with $\mathbb{R}^{2n}$. We can make the identification in such a manner that the symplectic form is $\omega = dp \wedge dq$ and Γ acts linearly and symplectically on V .

Hamilton’s equations can be written in the form

$$\frac{dv}{dt} = JDH(v) \equiv f(v) \quad (2.1)$$

where J is the symplectic involution

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

on V and $v = (p, q) \in V$.

For any map $g: \mathbb{R}^m \rightarrow \mathbb{R}^r$ we let $j^k g$ denote the k -jet of g at 0, i.e. the Taylor expansion around zero to order k . We assume throughout that $j^1 H = 0$, i.e. $H(0) = 0$

and $DH(0) = 0$. Then the quadratic part of H is

$$H_2 = j^2 H.$$

Let L be the linearised vector field

$$L = JD^2H(0)$$

and consider the linearised flow

$$\frac{dv}{dt} = Lv. \quad (2.2)$$

We also assume that the equilibrium at 0 is *linearly stable*, i.e. that

$$L \text{ is semisimple and has purely imaginary eigenvalues.} \quad (2.3)$$

If H_2 is definite then (2.3) is automatic.

As in Montaldi *et al* (1990a) section 2, the linearised flow (2.2) defines a symplectic action of a group $\mathbf{S}$ on V , and the Birkhoff normal form of H is invariant under $\mathbf{S}$. Because of (2.3) $\mathbf{S}$ is a torus $\mathbf{T}^k$. Let

$$V = \bigoplus_{j=1}^r V_{\alpha_j}$$

be the decomposition of V into resonance subspaces, so that:

(i) each V_{α_j} is an L -invariant subspace of V , and the eigenvalues of L on V_{α_j} are non-zero integer multiples of $i\alpha_j$;

(ii) the α_j are pairwise linearly independent over the field of rational numbers. When H is in Birkhoff normal form, the resonance subspaces are invariant under the Hamiltonian flow (2.1) and for most purposes we may restrict H to a single V_{α_j} . In other words, we may assume that H is fully resonant, i.e. $V = V_{\alpha}$ for a real number $\alpha > 0$. We make this assumption for the remainder of section 2. Its applicability when H is not in Birkhoff normal form is justified in section 3. If V is an irreducible representation of Γ then H is necessarily fully resonant.

2.2. Existence of nonlinear normal modes

Assume that H is fully resonant and in exact Birkhoff normal form. If $\{f, g\}$ denotes the Poisson bracket of functions f and g on V , then being in Birkhoff normal form is equivalent to $\{H, H_2\} = 0$. All trajectories of (2.2) are periodic with period $2\pi/\alpha$, so there is an action of the circle group $\mathbf{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ on V defined by

$$\theta v = e^{\theta\alpha^{-1}L}v \quad \theta \in \mathbf{S}^1. \quad (2.4)$$

This is what remains of the torus action described above when we assume H to be fully resonant. In Birkhoff normal form, H is invariant under this $\mathbf{S}^1$ -action as well as under Γ .

We wish to find the periodic trajectories of (2.1) with period close to $2\pi/\alpha$. To do this, define

$$H_{\tau} = 2\pi[(1 + \tau)H_2 - H] \quad (2.5)$$

for a new period-scaling parameter τ near 0. The factor 2π is introduced for later convenience. Then we have the following well known result.

Theorem 2.1. If H is in Birkhoff normal form then $\mathbf{S}^1$ -orbits of critical points of H_τ near 0 are periodic trajectories of (2.1) of period $2\pi/\alpha(1+\tau)$.

Proof. Let z be a critical point. Then $DH_\tau(z) = 0$, so

$$JDH(z) = (1+\tau)JDH_2(z). \quad (2.6)$$

Since $\{H, H_2\} = 0$, H is invariant under the $\mathbf{S}^1$ -flow (2.3), so (2.6) holds at each point of the $\mathbf{S}^1$ -orbit through z . Hence the $\mathbf{S}^1$ -orbit of z is a trajectory of H . $\square$

Remarks 2.2. (i) A critical point of H_τ (or the corresponding $\mathbf{S}^1$ -orbit) is called a relative equilibrium for the $\mathbf{S}^1$ -action. Relative equilibria occur in families known as nonlinear normal modes.

(ii) Theorem 2.1 generalises to systems not in Birkhoff normal form, see subsection 2.4.

2.3. Stability

We are especially interested in the spectral stability of these periodic trajectories. To define this we recall some ideas from Floquet theory. Let $u = u(t)$ be a periodic trajectory through $u(0) \in V$ with period T . Let σ_t denote the Hamiltonian flow of H for time t . The linearisation

$$M_u = D(\sigma_T)_{u(0)}: T_{u(0)}P \rightarrow T_{u(0)}P$$

is the Floquet operator of u . We say that u is spectrally stable if all eigenvalues of M_u lie on the unit circle. If, moreover, the image of the nilpotent part of M_u is tangent to u then u is said to be linearly stable. We wish to relate the Floquet operator M_u to the flow of H_τ . We begin with the following result.

Lemma 2.3. Let $T = 2\pi/\alpha(1+\tau)$. Then the time- T flows of H and H_τ coincide.

Proof. More is true: if σ_t is the flow of H and ρ_t the flow of H_τ then

$$\rho_t = \exp[-(1+\tau)tL]\sigma_t \quad (2.7)$$

for all t . To verify this, differentiate with respect to t and use the fact that the vector fields $Lv = JDH_2(v)$ and $JDH(v)$ commute. Putting $t = T$ in (2.7), we obtain the assertion of the lemma. $\square$

Corollary 2.4. The Floquet operator M_u is given by

$$M_u = \exp(TJD^2H_\tau(u(0))).$$

In particular, the periodic trajectory u is spectrally stable if and only if the linear part of the H_τ flow at $u(0)$, $v \mapsto JD^2H_\tau(v)$, has purely imaginary eigenvalues.

Proof. $M_u = D(\sigma^T)u(0) = D(\rho^T)u(0) = \exp(TJD^2H_\tau(u(0)))$. $\square$

2.4. Reduction via the splitting lemma

Birkhoff normal forms provide a powerful tool for calculations, but because they involve Taylor series truncation they do not directly prove the existence of periodic

trajectories in general systems. This deficiency can be remedied using the splitting lemma, see Montaldi *et al* (1990a), section 3. We seek periodic trajectories of the Hamiltonian differential equation

$$\frac{dv}{dt} = f(v) \quad (2.8)$$

near those of its linearisation

$$\frac{dv}{dt} = Lv. \quad (2.9)$$

The idea is to introduce the loop space $X = \mathcal{C}^1(\mathbb{S}^1, V)$ of C^1 maps $\mathbb{S}^1 \rightarrow V$ with C^1 norm. Let $S: X \times \mathbb{R} \rightarrow \mathbb{R}$ be the action functional (see Montaldi *et al* (1990a) section 3) and let $S_\alpha(u) = S(u, \alpha)$. It is well known (see proposition 3.1 of Montaldi *et al* (1990a)) that $u \in X$ is a critical point of S_α if and only if $v(t) = u(\alpha t)$ is a periodic solution, of period $2\pi/\alpha$, of (2.8). The Splitting Lemma allows us to reduce this problem to finite dimensions whilst preserving symmetries.

As well as the original symmetry group Γ , we take into account the $\mathbb{S}^1$ symmetry of Birkhoff normal forms. Further, the Hamiltonian may have a time-reversal symmetry ρ , see Montaldi *et al* (1990a) sections 4.2, 5. These symmetries generate a group $\Gamma \rtimes \hat{\mathbb{S}}^1$ in the notation of Montaldi *et al* (1990a), and S_α is $\Gamma \rtimes \hat{\mathbb{S}}^1$ invariant. Let $Z_\alpha = \{z \in X : \int \omega(z, v) = 0 \forall v \in V_\alpha\}$.

The following result is theorem 3.3 of Montaldi *et al* (1990a).

Theorem 2.5. For fixed α and sufficiently small τ there exists a smooth family of $\Gamma \rtimes \hat{\mathbb{S}}^1$ -equivariant diffeomorphisms $\xi_\tau: V_\alpha \oplus Z_\alpha \rightarrow X$ defined on a neighbourhood of $(0, 0)$ and with image a neighbourhood U of 0 in X , and a smooth family of $\Gamma \rtimes \hat{\mathbb{S}}^1$ -invariant functions $F_\tau: V_\alpha \rightarrow \mathbb{R}$, such that the critical points of $S_{\alpha(1+\tau)}$ are precisely the points $\xi_\tau(v, 0)$ where v is a critical point of F_τ . $\square$

Thus the periodic trajectories of period $2\pi/\alpha(1+\tau)$ near $2\pi/\alpha$ correspond bijectively to the critical points of F_τ . Moreover, this correspondence preserves symmetry groups.

If H is in Birkhoff normal form to sufficiently high order then there is an explicit formula for F_τ to that order.

Proposition 2.6. (a) If H is in Birkhoff normal form to degree k then

$$\frac{1}{2\pi} j^k F_\tau(v) = (1 + \tau) H_2(v) - j^k H(v). \quad (2.9)$$

(b) If H is in exact Birkhoff normal form then

$$F_\tau = H_\tau \quad (2.10)$$

with H_τ as in (2.5).

Proof. See Montaldi *et al* (1990a) theorem 3.4. $\square$

2.5. Constraints on Floquet operators

In this subsection we recall from Montaldi *et al* (1988) some constraints imposed by symmetry on linear operators that commute with a symplectic action of a Lie group,

and in particular on the Floquet operator of a nonlinear normal mode. As before, we assume that the Hamiltonian H is fully resonant and in Birkhoff normal form.

Let $u = u(t)$ be a nonlinear normal mode for (2.1) with period $T = 2\pi/\alpha(1 + \tau)$; let $\Sigma_u \subseteq \Gamma \times \hat{\mathbf{S}}^1$ be its isotropy subgroup and let M_u be its Floquet operator. Because H is fully resonant, all eigenvalues of M_u are near 1 in the sense that they tend to 1 as $\|u\| \rightarrow 0$. Hence for sufficiently small $\|u\|$ we may define the infinitesimal Floquet operator

$$B_u = \frac{1}{T} \log(M_u - I) = \frac{1}{T} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} (M_u - I)^j. \quad (2.11)$$

Then

$$M_u = \exp(TB_u). \quad (2.12)$$

The eigenvalues of M_u are called Floquet multipliers and those of B_u are Floquet exponents. The condition for spectral stability is equivalent to all eigenvalues of B_u lying on the imaginary axis.

We find it convenient to work with B_u because, when H is in Birkhoff normal form, corollary 2.4 yields the explicit formula

$$B_u = JD^2H_\tau(u(0)). \quad (2.13)$$

Suppose now that G is a compact Lie group acting linearly and symplectically on a symplectic space V . Define the Lie algebra $\mathfrak{sp}_G(V)$ of infinitesimally symplectic linear operators on V to consist of all G -equivariant linear operators β on V satisfying

$$\omega(\beta x, y) + \omega(x, \beta y) = 0$$

for all $x, y \in V$, where ω is the symplectic form. We then have the following result.

Theorem 2.7. If H is in Birkhoff normal form and is fully resonant then B_u belongs to $\mathfrak{sp}_\Sigma(V)$, where $V = T_{u(0)}P$ and $\Sigma = \Sigma_u \subseteq \Gamma \times \hat{\mathbf{S}}^1$.

Proof. The flow of H commutes with $\Gamma \times \hat{\mathbf{S}}^1$ and Σ fixes $u(t)$, hence Σ fixes $u(0)$.

To exploit this result (and its more subtle extension to the case when H is not in Birkhoff normal form, see Montaldi *et al* (1988), theorem 6.4) we must analyse the structure of $\mathfrak{sp}_G(V)$. We can read this off from properties of the associated Lie group $\mathbf{Sp}_G(V)$, consisting of all invertible linear transformations β of V for which

$$\omega(\beta x, \beta y) = \omega(x, y).$$

The required properties are proved in Montaldi *et al* (1988), section 2. We first recall some standard ideas from the representation theory of Lie groups, see Adams (1969) or Bröcker and tom Dieck (1985) for details.

A representation of G is irreducible if it has no proper non-zero invariant subspaces. If V is a representation of G we write $\text{Hom}_G(V, V)$ for the set of equivariant linear transformations of V . If V is irreducible then $\text{Hom}_G(V, V)$ is a real division algebra, hence is isomorphic to one of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Correspondingly we say that V is real, complex or quaternionic.

An arbitrary representation V can be decomposed as a direct sum of irreducible representations. By collecting together all summands of a given isomorphism type, we

obtain a unique isotypic decomposition

$$V = V_1 \oplus \dots \oplus V_l \quad (2.14)$$

where each V_j is a direct sum of subspaces, each isomorphic to an irreducible W_j , such that if $i \neq j$ then $W_i \not\cong W_j$. We say that V_j is a real, complex, or quaternionic isotypic block according to the type of W_j . If $\beta \in \text{Hom}_G(V, V)$ then β maps each V_j to itself, hence $\beta = \bigoplus \beta_j$ where $\beta_j = \beta|_{V_j}$. If $\mathbf{k}_j$ is the corresponding division algebra $\text{Hom}_G(W_j, W_j)$, then

$$V_j \cong \mathbf{k}_j^{n_j} \otimes_{\mathbf{k}_j} W_j. \quad (2.15)$$

Now we introduce the symplectic structure. If V is a symplectic representation of G then by ignoring the symplectic structure we obtain an ordinary representation, the underlying representation of V . An irreducible symplectic representation is one that has no proper non-zero invariant symplectic subspaces. The underlying representation of an irreducible symplectic representation may or may not be irreducible (i.e. it may have non-symplectic invariant subspaces). The situation is described by the following result.

Theorem 2.8. (Classification of symplectic irreducibles). (a) The underlying representation of an irreducible symplectic representation is either; (i) a complex or quaternionic irreducible representation, or (ii) isomorphic to $W \oplus W$ where W is a real irreducible representation.

(b) In the real and quaternionic cases the isomorphism type of the irreducible symplectic representation is uniquely determined by that of its underlying representation.

(c) In the complex case there are precisely two isomorphism types of irreducible symplectic representation for a given complex irreducible underlying representation.

Proof. This is theorem 2.1 of Montaldi *et al* (1988). $\square$

In case (c) the two types differ by complex conjugation in $\mathbf{k}_j \cong \mathbb{C}$, and we say they are dual to each other.

The symplectic analogue of isotypic decomposition is as below.

Theorem 2.9. (Symplectic isotypic decomposition). Every symplectic representation V has a unique decomposition

$$V = V_1 \oplus \dots \oplus V_l$$

where each V_j is a symplectic subspace of V , and either: (a) V_j is isomorphic to a direct sum of a number of copies of a single real or quaternionic irreducible W_j (the number of copies being even if W_j is real), or (b) V_j is isomorphic to

$$(\oplus^s X) \oplus (\oplus^{s'} Y)$$

where X and Y are dual complex irreducible symplectic representations whose underlying representations W_j are isomorphic. If $i \neq j$ then $W_i \not\cong W_j$.

Proof. See theorem 2.2 of Montaldi *et al* (1988). $\square$

Next we turn to the structure of $\mathfrak{sp}_G(V)$. Let $\mathbf{k} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and equip $\mathbf{k}^n$ with one of the standard antihermitian forms defined by:

$$\begin{aligned}\mathbf{k} = \mathbb{R}: \{x, y\} &= \sum_{j=1}^{n/2} (x_j y_{n/2+j} - x_{n/2+j} y_j) & n \text{ even} \\ \mathbf{k} = \mathbb{C}: \{x, y\} &= \sum_{j=1}^p x_j \overline{iy_j} - \sum_{j=p+1}^n x_j \overline{iy_j} & 0 \leq p \leq n \\ \mathbf{k} = \mathbb{H}: \{x, y\} &= \sum_{j=1}^n x_j \overline{iy_j}.\end{aligned}$$

Here the bars denote complex or quaternionic conjugation. The groups of $\mathbf{k}$ -linear transformations that preserve these forms are denoted by

$$\begin{aligned}\mathbf{Sp}(n; \mathbb{R}) & \quad \text{if } \mathbf{k} = \mathbb{R} \\ \mathbf{U}(p, q; \mathbb{C}) & \quad \text{if } \mathbf{k} = \mathbb{C}, p + q = n \\ \alpha\mathbf{U}(n; \mathbb{H}) & \quad \text{if } \mathbf{k} = \mathbb{H}.\end{aligned}$$

The corresponding real Lie algebras are denoted

$$\mathfrak{sp}(n, \mathbb{R}) \quad \mathfrak{u}(p, q; \mathbb{C}) \quad \alpha\mathfrak{u}(n; \mathbb{H}).$$

Theorem 2.10 (Structure of infinitesimally symplectic equivariants).

Let $V = V_1 \oplus \dots \oplus V_l$ be the isotypic decomposition of V , and let $V_j \simeq \mathbf{k}_j^{n_j} \otimes_{\mathbf{k}_j} W_j$ as in (2.15). Then

$$\mathfrak{sp}_G(V) = s_1 \oplus \dots \oplus s_l$$

where $s_j = \mathfrak{sp}_G(V_j)$ and: (a) if $\mathbf{k}_j = \mathbb{R}$ then $s_j \simeq \mathfrak{sp}(n_j; \mathbb{R})$; (b) if $\mathbf{k}_j = \mathbb{C}$ then $s_j \simeq \mathfrak{u}(p_j, q_j; \mathbb{C})$; (c) if $\mathbf{k}_j = \mathbb{H}$ then $s_j \simeq \alpha\mathfrak{u}(n_j; \mathbb{H})$.

In case (b) $p_j + q_j = n_j$ and p_j is determined by the antiHermitian form defined by V_j .

Proof. This is the Lie algebra version of theorem 2.5 of Montaldi *et al* (1988). $\square$

Of these Lie algebras, $\mathfrak{u}(n, 0; \mathbb{C}) \equiv \mathfrak{u}(n)$ and $\alpha\mathfrak{u}(1; \mathbb{H})$, and only these, are of compact type—i.e. the corresponding Lie groups $\mathbf{U}(n, 0; \mathbb{C})$ and $\alpha\mathbf{U}(1; \mathbb{H})$ are compact. This happens because the antihermitian form is definite. The representation V_j in these cases is said to be cyclospectral: every $\beta \in \mathfrak{sp}_G(V)$ has a purely imaginary spectrum.

For nonlinear normal modes u we will apply theorem 2.10 with $G = \Sigma_u$, in which case $B_u \in \mathfrak{sp}_G(V)$ by theorem 2.6(b). Then theorem 2.10 yields a block decomposition of B_u into blocks of somewhat restricted types. In particular the, eigenvalues of B_u on a cyclospectral isotypic block must be purely imaginary. Even when the block is not cyclospectral, theorem 2.10 simplifies the calculation of the spectrum of B_u .

2.6. Constraints on Floquet operators for continuous groups

Further restrictions on the Floquet operator arise when the Hamiltonian is invariant under an action of a continuous group, which are encapsulated in the requirement

that the momentum map associated with this action be conserved. In this subsection we briefly review these constraints. More details are given in Montaldi *et al* (1988).

Let G be a Lie group acting symplectically on a symplectic space V , with associated momentum mapping $\Phi: V \rightarrow \mathfrak{g}^*$ (where $\mathfrak{g}$ is the Lie algebra of G), and consider a periodic orbit u of a G -invariant Hamiltonian system on V . At any point z of u we decompose $T_z V$ as follows:

$$\begin{aligned} W &= \ker d\Phi_z \cap \mathfrak{g} \cdot z & X &= \mathfrak{g} \cdot z / W \\ Y &= \ker d\Phi_z / W & Z &= T_z V / \ker d\Phi_z + \mathfrak{g} \cdot z \end{aligned}$$

where $\mathfrak{g} \cdot z = T_z(G \cdot z)$, the tangent space at z to the group orbit through z . Each of these subspaces and quotients carries a natural representation of G_z , and X and Y have natural symplectic structures, while the symplectic form on V gives a non-degenerate pairing between W and Z . If G is compact, then as a symplectic representation of G_z we have

$$T_z V \cong W \oplus X \oplus Y \oplus Z.$$

Moreover, if the four representations of G_z have no common irreducible component, then $T_z V$ decomposes uniquely in this way. With respect to this decomposition, the Floquet operator for the periodic orbit is of the form

$$M_u = \begin{bmatrix} I_W & 0 & * & * \\ 0 & I_X & 0 & 0 \\ 0 & 0 & N_u & * \\ 0 & 0 & 0 & I_Z \end{bmatrix}$$

by equivariance and conservation of momentum (see theorem 4.5 of Montaldi *et al* (1988)). Here I_W , I_X and I_Z are the identity maps on W , X and Z . The entries $*$ are arbitrary G_z -equivariant linear maps which do not affect the spectrum of M_u . The space Y is the reduced representation of G_z and N_u is a symplectic G_z -equivariant linear map on Y called the reduced Floquet operator for u . Thus it is the eigenvalues of N_u that determine the spectral stability of u . At regular points of the G -action, Y is the tangent space to the reduced phase space $\Phi^{-1}(\mu)/G_\mu$ where $\mu = \Phi(z)$.

2.7. Complex structure

For the examples in sections 4–6 we employ a convenient complex notation. The phase space V is identified with $\mathbb{C}^n$. The symplectic form becomes

$$\omega = i \sum_{j=1}^n dz_j \wedge d\bar{z}_j$$

which agrees with $\sum dp_j \wedge dq_j$ if we set $z_j = (1/\sqrt{2})(p_j + iq_j)$. Hamilton's equations become

$$\frac{dz_j}{dt} = i \frac{\partial H}{\partial \bar{z}_j} \quad (2.16)$$

where

$$\frac{\partial}{\partial \bar{z}_j} = \frac{\partial}{\partial p_j} + i \frac{\partial}{\partial q_j}$$

and the symplectic involution J is multiplication by i . The group $\mathbf{S}^1$ introduced in Birkhoff normal form acts by $z_j \rightarrow \exp(a_j i \theta) z_j$ for appropriate $a_j \in \mathbb{Z}$, and the time-reversal operator ρ is $z \rightarrow -\bar{z}$.

In this notation we easily obtain a useful formula for the eigenvalues of an element of $\mathfrak{sp}(2; \mathbb{R})$, one of the commonest types of block. If the linearised flow is

$$\frac{dw}{dt} = ia\bar{w} + bw \quad (2.17)$$

with $w \in \mathbb{C}$, $a \in \mathbb{R}$, $b \in \mathbb{C}$, then the eigenvalues are

$$\lambda = \pm(|b|^2 - a^2)^{1/2}. \quad (2.18)$$

Thus (2.17) is spectrally stable if and only if $|b| \leq |a|$.

3. Truncating Birkhoff normal form

In this section we give conditions under which the stability (or instability) of a nonlinear normal mode of a Hamiltonian H may be determined from some truncation $j^k H$, where $j^k H$ is assumed to be in Birkhoff normal form. We do *not* assume that H is in Birkhoff normal form, therefore we cannot reduce the analysis to the case where H is fully resonant by restricting to the appropriate resonance space. Nevertheless such a reduction turns out to be possible for more subtle reasons, as we argue below. In order to avoid complicating the argument we concentrate on two particular contexts for this problem. The first context is when $\dim \text{Fix}(\Sigma) = 2$, which is common in applications and relatively straightforward; the second context is when determinacy results from singularity theory can be applied as in Montaldi *et al* (1990a) section 5. The main hypotheses needed in the method are isolated so that it can be adapted to other contexts; see the remarks preceding theorem 3.4 below.

The underlying idea is straightforward. If a nonlinear normal mode u of H is sufficiently closely approximated by a nonlinear normal mode $\bar{u}$ of $j^k H$, then the Floquet operator M_u of u is closely approximated by the Floquet operator $M_{\bar{u}}$ of $\bar{u}$. Recall that a symplectic operator is said to be strongly stable if it has a neighbourhood of spectrally stable operators. If $M_{\bar{u}}$ is strongly stable, then so is M_u , and similarly if $M_{\bar{u}}$ is strongly unstable.

However, several technical difficulties complicate the proof and statement of a suitable theorem. In particular we cannot consider u as a small perturbation of $\bar{u}$; instead we consider them both as small perturbations of the equilibrium state $v = 0$. Furthermore, we cannot expect M_u itself to be strongly stable. Instead we require the reduced Floquet operator N_u to be strongly stable in an equivariant sense. We then estimate the degree to which $N_{\bar{u}}$ approximates N_u . The strategy is to show that as $u \rightarrow 0$ then N_u approximates $N_{\bar{u}}$ more closely than $N_{\bar{u}}$ tends towards the boundary of the region of strong stability, see figure 1. The result then follows.

Let $H: V \rightarrow \mathbb{R}$ be a Γ -invariant Hamiltonian, as usual, such that $L = JD^2 H|_0$ is semisimple with purely imaginary eigenvalues, and let V_α be a resonance subspace. We consider a nonlinear normal mode u of period near $2\pi/\alpha$. As in Montaldi *et al* (1988), we can divide the eigenvalues of M_u into two types; those 'near 1' whose eigenspace is close to V_α in the appropriate Grassmannian manifold, and the remainder. In Montaldi *et al* (1988) section 5 we prove that as $u \rightarrow 0$ the eigenvalues

Figure 1. Schematic illustration of the manner in which $N_{\bar{u}}$ and N_u approximate the identity matrix I as $n \rightarrow 0$, when $N_{\bar{u}}$ is strongly stable. (The boundary is actually singular at I .)

that are not near 1 are stable, i.e. lie on the unit circle, provided that

$$\text{all eigenvalues of } L \text{ are purely imaginary} \quad (3.1a)$$

$$L \text{ has no eigenvalues } k\alpha/2 \text{ where } k \in \mathbb{Z} \text{ is odd} \quad (3.1b)$$

$$\text{if } \mu \text{ is a multiple eigenvalue of } L \text{ then } D^2H|_0 \text{ is definite} \\ \text{on the real } \mu \text{ eigenspace of } L \quad (3.1c)$$

$$\text{there are no resonances of the form } \mu_1 - \mu_2 = n\alpha, \ n \in \mathbb{Z} \setminus \{0\}, \\ \text{for distinct eigenvalues } i\mu_1, i\mu_2 \text{ of } L. \quad (3.1d)$$

Provided (3.1a) holds, which is our standard hypothesis, then (3.1b)–(3.1d) are generically true. In particular (3.1c) is generic because the only multiple eigenvalues that occur generically are those forced by symmetry, see Montaldi *et al* (1988), section 5. If (3.1a) fails then all nonlinear normal modes are unstable.

We may thus concentrate on the eigenvalues of M_u that are ‘near 1’, which in effect allows us to assume that $V = V_\alpha$ by restricting M_u to (a small deformation of) V_α . See Montaldi *et al* (1988), section 6a, especially lemma 6.2. Thus we may without loss of generality assume V to be fully resonant. In this case all eigenvalues of the Floquet operator are near 1, so the reduced representation and Floquet operator are the same as the residual representation and Floquet operator of Montaldi *et al* (1988), section 5. We therefore use the term ‘reduced’ while applying the results of that paper concerning residual representations and operators.

We incorporate phase-shift and (possibly) time-reversal symmetries by introducing the group $\Gamma \rtimes \hat{S}^1$ as in Montaldi *et al* (1990a), which acts orthogonally on V . By transforming to Birkhoff normal form, the Hamiltonian can be made $\Gamma \rtimes \hat{S}^1$ -invariant to any specified order k .

Let $\Sigma \subset \Gamma \rtimes \hat{S}^1$ be an isotropy subgroup and let $\mathbf{Sp}_\Sigma(V)$ denote the Lie group of all symplectic Σ -equivariant maps $V \rightarrow V$. Recall that $M \in \mathbf{Sp}_\Sigma(V)$ is strongly stable (respectively, strongly unstable) if it has a neighbourhood in $\mathbf{Sp}_\Sigma(V)$ whose elements all have spectrum on (respectively, not entirely on) the unit circle. A nonlinear normal mode u is strongly stable (respectively, strongly unstable) if and only if its Floquet operator M_u is strongly stable (respectively, strongly unstable). We denote the open subsets of strongly stable and strongly unstable operators by $\mathbf{Sp}_\Sigma^{\text{ss}}(V)$ and $\mathbf{Sp}_\Sigma^{\text{su}}(V)$, respectively. The bifurcation variety is the common boundary

$$\mathcal{B}_\Sigma(V) = \mathbf{Sp}_\Sigma(V) \setminus \{\mathbf{Sp}_\Sigma^{\text{ss}}(V) \cup \mathbf{Sp}_\Sigma^{\text{su}}(V)\}$$

of these two subsets. Clearly $\mathcal{B}_\Sigma(V)$ is a real semialgebraic subset of $\mathbf{Sp}_\Sigma(V)$. By

subsection 2.6 M_u takes the form

$$M_u = \begin{bmatrix} I_W & 0 & * & * \\ 0 & I_X & 0 & 0 \\ 0 & 0 & N_u & * \\ 0 & 0 & 0 & I_Z \end{bmatrix}$$

where $N_u \in \mathbf{Sp}_\Sigma(Y)$. It follows that u is strongly stable if

$$N_u \subseteq \mathbf{Sp}_\Sigma^{\text{ss}}(Y)$$

and strongly unstable if

$$N_u \subseteq \mathbf{Sp}_\Sigma^{\text{su}}(Y).$$

Note that all eigenvalues of M_u other than those occurring on Y are 1.

We may now state the main result of this section in its simplest version.

Theorem 3.1. With the usual notation suppose that L satisfies (3.1a)–(3.1d). Assume that u is a nonlinear normal mode of H with isotropy subgroup Σ , and that $\dim \text{Fix}(\Sigma) = 2$. Suppose that u corresponds to a nonlinear normal mode $\tilde{u}$ of a truncated Birkhoff normal form $j^r \tilde{H}$, and that $N_{\tilde{u}}$ is either strongly stable or strongly unstable. If $N_{\tilde{u}}$ is strongly stable then u is strongly stable, and if $N_{\tilde{u}}$ is strongly unstable then u is strongly unstable.

We shall prove a related but more technical result (theorem 3.4 below) which applies to a particular singularity-theoretic context, and indicate how a similar but simpler argument proves theorem 3.1.

For the more technical result it is convenient to refine our notion of ‘nonlinear normal mode’. To this end we first describe the local structure of nonlinear normal modes. In Montaldi *et al* (1990a), section 5 the existence of nonlinear normal modes is proved as follows, see subsection 2.4. The splitting lemma is used to determine a function

$$F: V \times \mathbb{R} \rightarrow \mathbb{R}$$

written $F(v, \tau) = F_\tau(v)$, such that critical points of F_τ correspond via an equivariant diffeomorphism to nonlinear normal modes of H , having period $[2\pi/(1 + \tau)\alpha]$. Thus we seek zeros of the reduced mapping

$$\varphi = D_v F.$$

It turns out that in order to apply singularity theory it is convenient to replace φ by an auxiliary mapping

$$\psi: V \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

such that if H is in Birkhoff normal form to degree k then

$$j^k \psi(v, \tau, \lambda) = j^k \varphi(v, \tau) + \lambda J D_v(H_2). \quad (3.2)$$

See proposition 5.2 of Montaldi *et al* (1990a), to which we also refer for all singularity-theoretic terminology.

Remarks 3.2. (i) The map called ψ in Montaldi *et al* (1990a) is actually $J^{-1}\psi$ in the above notation. However, ψ and $J^{-1}\psi$ are $\mathcal{K} - \Gamma \rtimes \hat{S}^1$ -equivalent.

(ii) This definition of ψ is appropriate when Γ is a *finite* group, as assumed in Montaldi *et al* (1990a), section 5. For continuous groups such as $\mathbf{O}(2)$ and $\mathbf{O}(3)$, encountered later, we use theorem 3.1. The problem of formulating a more general singularity-theoretic approach, suitable for continuous groups Γ , remains open.

Suppose that H is in Birkhoff normal form to order k , i.e. $j^k H$ is $\Gamma \times \hat{\mathbf{S}}^1$ -invariant. Then Montaldi *et al* (1990a), Proposition 5.2 implies that there is an equivariant C^k diffeomorphism

$$\delta: \psi^{-1}(0) \rightarrow \varphi^{-1}(0).$$

By equation (5.4) of that paper we also have

$$j^k \delta = I.$$

In order to apply singularity theory we introduce the *non-degeneracy condition* (NDC) ψ is finitely $\mathcal{H} - \Gamma \times \hat{\mathbf{S}}^1$ -determined.

If l is the appropriate degree of determinacy then nonlinear normal modes of H correspond locally near 0 to the zeros of the polynomial mapping $j^l \psi$, via an equivariant diffeomorphism.

We now define a branch of nonlinear normal modes, or more briefly branch, of ψ with isotropy Σ : it is the germ at 0 of the closure $\mathcal{U}$ of a subset $\mathcal{U}_0$ of $\psi^{-1}(0) \setminus \{0\}$, maximal with respect to: (a) $\mathcal{U}_0$ is connected; (b) if $u \in \mathcal{U}_0$ then $\Sigma_u = \Sigma$.

Note that $\mathcal{U}_0 = \mathcal{U} \setminus \{0\}$. We apply the same term to other maps related to ψ , for example $j^l \psi$, and to the corresponding set of periodic solutions of the Hamiltonian system defined by H . In fact when Γ is finite we expect such a subset $\mathcal{U}$ to be (generically) a surface through 0 which is smooth away from 0, see Duistermaat (1984). The common Σ will be denoted by $\Sigma_{\mathcal{U}}$ and referred to as the isotropy subgroup of the branch $\mathcal{U}$.

A parametrisation of a branch $\mathcal{U}$ is the germ of a homeomorphism $\mathbb{C} \rightarrow \mathcal{U}$, which is a diffeomorphism for $z \neq 0$, mapping circles $|z| = \text{constant}$ onto periodic orbits.

If $\mathcal{U}$ is a branch of H with isotropy Σ , and $u \in \mathcal{U}$, then by Montaldi *et al* (1988), section 6 we can, in a canonical fashion, consider the Floquet operator M_u to be an element of $\mathbf{Sp}_{\Sigma}(V)$, in the above block form. We say that $\mathcal{U}$ is strongly stable (respectively, strongly unstable) if all non-zero $u \in \mathcal{U}$ sufficiently close to 0 are strongly stable (respectively, strongly unstable).

We require one further concept before stating and proving the main theorem of this section. When (NDC) holds we know that ψ is l -determined for $\mathcal{H} - \Gamma \times \hat{\mathbf{S}}^1$ -equivalence. However, such a value l may not be known explicitly or may be too large for practical computation. We therefore make the following definition, which comes in two parts and is motivated by the way we perform calculations in later examples.

Definition 3.3. Let H be a Γ -invariant Hamiltonian and let r be a positive integer such that $H_r = j^r H$ is $\Gamma \times \hat{\mathbf{S}}^1$ -invariant. Let $\mathcal{U}$ be a branch of H with isotropy subgroup $\Sigma \subset \Gamma \times \hat{\mathbf{S}}^1$. Let ξ be an equivariant symplectic diffeomorphism with $j^1 \xi = I$ such that $H'(v) = H(\xi(v))$ is in Birkhoff normal form to degree $s \geq r$ and $j^r H' = j^r H$. We say that H is r -adequate for $\mathcal{U}$ if the following conditions hold.

(a) There exists a branch, $\mathcal{U}_s$ for $j^s H'$ with isotropy subgroup equal to Σ , and a

bijection

$$\chi_s: \mathcal{U}_s \rightarrow \mathcal{U} \quad (3.3)$$

such that

$$\|\chi_s(u) - u\| = o(\|u\|) \quad (3.4)$$

as $u \rightarrow 0$, for $u \in \mathcal{U}_s$.

(b) Let $u_s \in \mathcal{U}_s$. Suppose that z parametrises $\mathcal{U}$ and hence also parametrises $\mathcal{U}_s$ by pullback through χ_s . Then the eigenvalues of N_{u_s} take the form

$$\mu_i = A_i f_i(z) + O(\|z\|^{k_i+1}) \quad (3.5)$$

where f_i is a polynomial of degree k_i and A_i depends only on the coefficients of $j^r H$.

Remarks (i) Conditions (3.3), (3.4) follow from the analogous conditions with $\mathcal{U}$ replaced by $\mathcal{U}' = \xi(\mathcal{U})$, since $j^1 \xi = I$.

(ii) Suppose that in addition to (b) above we know that $\mathcal{U}_r$ is either strongly stable or strongly unstable. Let $k = \max k_i$ and let $\mathcal{B}_\Sigma = \mathcal{B}_\Sigma(Y)$. Then for all $s \geq r$ formula (3.5), together with the criteria (3.1) for strong stability and similar criteria for strong instability, imply an estimate of the form

$$\text{dist}(N_{u_s}, \mathcal{B}_\Sigma) > C \|u_s\|^k \quad (3.6)$$

for u_s near 0, where C and k depend only on r , not on s . This follows from standard properties of semialgebraic sets but is tedious to describe in detail.

(iii) In practice we check part (b) by a computation of the eigenvalues, writing $j^s H$ as $j^r H + O(r+1)$. When the existence of $\mathcal{U}_r$ is proved using the auxiliary mapping ψ we can deduce (a) from a similar assertion about the corresponding zeros of the auxiliary mappings ψ of H and ψ^r of $j^r H$.

We can now state and prove the main theorem of this section.

Theorem 3.4. With the usual notation let H be a Γ -invariant Hamiltonian on V with an equilibrium at 0 such that the linearisation L satisfies (3.1a)–(3.1d). Let $\mathcal{U}$ be a branch of H . Let r be a positive integer; let H be in Birkhoff normal form to degree r , and assume that: (a) H is r -adequate for $\mathcal{U}$; (b) the branch $\mathcal{U}_r$ for $j^r H$ that corresponds to $\mathcal{U}$ is either strongly stable or strongly unstable; (c) the auxiliary mapping ψ for H is finitely $\mathcal{K} - \Gamma \rtimes \hat{\mathbf{S}}^1$ -determined. Then $\mathcal{U}$ is strongly stable (respectively, strongly unstable) if $\mathcal{U}_r$ is.

Proof. We begin with a notational convention. The function F whose critical points determine the nonlinear normal modes of H depends on H . So do the reduced mapping $\varphi = D_v F$ and the auxiliary mapping ψ . To make this dependence clear we write these as F^H , φ^H and ψ^H , respectively.

Let $u_r \in \mathcal{U}_r$ and let $\mathcal{B}_\Sigma = \mathcal{B}_\Sigma(Y)$. By hypothesis (b) of the theorem and (3.6) with $s = r$ there exist a constant C and a positive integer k such that

$$\text{dist}(N_{u_r}, \mathcal{B}_\Sigma) > C \|u_r\|^k$$

as $u_r \rightarrow 0$.

Let l be any positive integer. Denote by $\mathcal{K}$ the group of $\Gamma \rtimes \hat{\mathbf{S}}^1$ - $\mathcal{K}$ -equivalences, and define the subgroups

$$\mathcal{K}_l = \{\eta \in \mathcal{K} : j^l \eta = I\}.$$

Then

$$\mathcal{H} \supset \mathcal{H}_1 \supset \mathcal{H}_2 \supset \dots$$

and each $\mathcal{H}_i$ is of finite codimension in $\mathcal{H}$. By Damon (1984), theorem 10.2, ψ^H is finitely determined for $\mathcal{H}_i$ equivalence. Let $d(l)$ be the degree of determinacy, and if necessary increase it so that $d(l) \geq \max(l, r)$. Using the Birkhoff normal form procedure, choose an equivariant C^∞ symplectic diffeomorphism ξ with $j^1\xi = I$ so that $H'(v) = H(\xi(v))$ is in Birkhoff normal form to degree $d(k)$ and $j^rH' = j^rH$. Then ψ^H , and hence also $\psi^{H'}$, is $d(k)$ -determined for $\mathcal{H}_k$.

Since $j^{d(k)}H'$ is in Birkhoff normal form there exists an equivariant $C^{d(k)}$ diffeomorphism

$$\delta' : (\varphi^{H'})^{-1}(0) \rightarrow (\psi^{H'})^{-1}(0)$$

with $j^{d(k)}\delta' = I$. Let $H'' = j^{d(k)}H'$. By proposition 5.2 of Montaldi *et al* (1990a)

$$(\varphi^{H''})^{-1}(0) \times \{0\} = (\psi^{H''})^{-1}(0) = (j^{d(k)}\psi^{H'})^{-1}(0).$$

Since $\psi^{H'}$ is $d(k)$ -determined for $\mathcal{H}_k$, there exists a diffeomorphism

$$\eta : (\psi^{H'})^{-1}(0) \rightarrow (\varphi^{H''})^{-1}(0)$$

such that $j^k\eta = I$.

Let $u \in \mathcal{U}$, and let $u' = \xi(u)$ be the corresponding nonlinear normal mode of H' . Then u and u' have the same stability and the same isotropy subgroup. Let $u'' = \eta(\delta'(u'))$, the corresponding nonlinear normal mode of H'' . Since H' is r -adequate there exist a branch $\mathcal{U}_r$ of j^rH and maps χ' , χ'' mapping $\mathcal{U}_r$ bijectively to the branches of H' and H'' that contain u' and u'' . We can choose χ' and χ'' so that the following diagram commutes:

$$\begin{array}{ccc} u \in (\varphi^H)^{-1}(0) & & \\ \downarrow \xi & & \\ u' \in (\varphi^{H'})^{-1}(0) & & \\ \nearrow \chi' \quad \searrow \delta' & & \\ u_r \in \mathcal{U}_r & & (\psi^{H'})^{-1}(0) \\ \searrow \chi'' \quad \nearrow \eta & & \\ u'' \in (\varphi^{H''})^{-1}(0) & & \end{array}$$

Here $j^{d(k)}\delta' = I$ and $j^k\eta = I$, so

$$\|u' - u\| = \mathbf{O}(\|u_k\|^k). \quad (3.7)$$

By (3.6), with $s = d(k)$, we have

$$\text{dist}(N_{u''}, \mathcal{B}_\Sigma) > C \|u''\|^k. \quad (3.8)$$

All of u , u_r , u' and u'' are of the same order of magnitude. By Montaldi *et al* (1988), proposition 5.3 the Floquet operator M_u , and hence the reduced operator N_u , depends in C^1 fashion on u . Thus (3.7), (3.8) imply that

$$\text{dist}(N_u, \mathcal{B}_\Sigma) > C \|u\|'^k.$$

Therefore u' , and hence u , is strongly stable (respectively, strongly unstable) if u_r is. $\square$

Proof of theorem 3.1

This is similar to that of theorem 3.4, except that the estimates established using r -adequacy, and the correspondences between nonlinear normal modes u , u_r , u' , u'' , established above using finite determinacy, follow from the implicit function theorem. Alternatively, the arguments used for the corresponding result in Hopf bifurcation in Stewart (1988) may be modified to take the symplectic structure into account.

Remarks 3.5. (i) In practice we apply theorem 3.4 as follows. Choose a plausible value of r . Put H into Birkhoff normal form to order r and compute the leading part of the eigenvalues of the Floquet operators for nonlinear normal modes of $j^r H$. Bearing in mind the criteria of Montaldi *et al* (1988), check whether these are strongly stable or strongly unstable, and show that H is r -adequate. If so, theorem 3.4 implies that the results obtained are the dominant terms in the eigenvalues of the corresponding nonlinear normal modes for H . If not, increase r and repeat.

(ii) When the Hamiltonian is taken to be the most general $\Gamma \times \hat{S}^1$ -invariant function of degree r , it will have arbitrary coefficients $\alpha_1, \dots, \alpha_p$, say. The conditions for strong stability or strong instability are then that certain polynomials $A_i(\alpha_1, \dots, \alpha_p)$ do not vanish. If so, we shall say that H is non-degenerate and refer to the equations $A_i(\alpha_1, \dots, \alpha_p) \neq 0$ as non-degeneracy conditions. Theorem 3.4 then states that provided we choose r large enough to make H r -adequate, the stabilities and non-degeneracy conditions computed from a Birkhoff normal form to order r continue to apply to the full Hamiltonian H .

(iii) We apply theorem 3.1 below in sections 4, 5, and theorem 3.4 in section 6.

4. Circular and dihedral symmetry

Our first set of examples comprises Hamiltonian systems with two degrees of freedom that are invariant under actions of $\mathbf{O}(2)$ and $\mathbf{D}_n$ for $n \geq 3$. In Montaldi *et al* (1988) we proved the existence of nonlinear normal modes with certain symmetry types and in Montaldi *et al* (1990a) we showed that for generic $\mathbf{D}_n$ -invariant systems these are the only ones. A similar, though simpler, argument shows that the same is true for the $\mathbf{O}(2)$ -invariant systems. We also showed in Montaldi *et al* (1988) that some of these nonlinear normal modes are always spectrally stable for purely group theoretical reasons. In this paper we compute the Floquet exponents for all the nonlinear normal modes in terms of the coefficients in the Birkhoff normal form of the Hamiltonian (theorem 4.3). We use theorem 3.1 to deduce the stabilities when the Hamiltonian is not in Birkhoff normal form (corollary 4.4). In subsection 4.2 we apply these results to Hamiltonians describing the motion of a particle in a two-dimensional potential well with circular or dihedral symmetry.

4.1. Stability

We use the same notation as in Montaldi *et al* (1988, 1990a). The action of $\mathbf{O}(2)$ on $V = \mathbb{C}^2$ is given by

$$\begin{aligned}\theta \cdot (z_1, z_2) &= (e^{i\theta} z_1, e^{-i\theta} z_2) & \theta \in \mathbf{SO}(2) &\cong \frac{\mathbb{R}}{2\pi\mathbb{Z}} \\ \kappa \cdot (z_1, z_2) &= (z_2, z_1).\end{aligned}$$

The actions of $\mathbf{D}_n$ are obtained by restricting θ to $2\pi r/n$ for $r = 0, \dots, n-1$. We let Γ denote either $\mathbf{O}(2)$ or $\mathbf{D}_n$. If H is a Γ -invariant Hamiltonian on V such that $iD^2H(0)$ has a conjugate pair of imaginary eigenvalues, which by scaling time we may take to be $\pm i$, then the Birkhoff normal form of H is also invariant under the $\mathbf{S}^1$ -action given by

$$\varphi \cdot (z_1, z_2) = (e^{i\varphi} z_1, e^{i\varphi} z_2) \quad \varphi \in \mathbf{S}^1 \cong \frac{\mathbb{R}}{2\pi\mathbb{Z}}.$$

The proper isotropy subgroups of the combined $\Gamma \rtimes \hat{\mathbf{S}}^1$ -actions are listed in table 1, adapted from Montaldi *et al* (1988, 1990a) and references therein. Recall that the superscripts on $\mathbf{Z}_2$ denote generators and $\kappa' = \kappa \circ (-2\pi/n) \in \mathbf{D}_n$. The subgroup of $\mathbf{O}(2) \times \mathbf{S}^1$ consisting of pairs $(\psi, -\psi)$ is denoted $\widetilde{\mathbf{SO}}(2)$ and its intersection with $\mathbf{D}_n \times \mathbf{S}^1$ by $\tilde{\mathbf{Z}}_n$.

The next result gives the Birkhoff normal form of general Hamiltonians with $\mathbf{O}(2)$ and $\mathbf{D}_n$ symmetry.

Proposition 4.1. (a) Every smooth $\mathbf{O}(2) \times \mathbf{S}^1$ -invariant function $h: V \rightarrow \mathbb{R}$ has the form $h(z_1, z_2) = f(N, P)$ where $N = |z_1|^2 + |z_2|^2$, $P = |z_1|^2 |z_2|^2$ and f is a smooth function of N and P .

(b) Suppose $n \geq 3$ and let

$$m = \begin{cases} n & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

Then every smooth $\mathbf{D}_n \times \mathbf{S}^1$ -invariant function $h: V \rightarrow \mathbb{R}$ has the form $h(z_1, z_2) = f(N, P, S, T)$ where N and P are as above, $S = (z_1 \bar{z}_2)^m + (\bar{z}_1 z_2)^m$, $T = i(|z_1|^2 - |z_2|^2)((z_1 \bar{z}_2)^m - (\bar{z}_1 z_2)^m)$, and f is a smooth function of N, P, S and T .

Proof. For (a) see Golubitsky *et al* (1988), chapter XVII section 1b; for (b) see Golubitsky *et al* (1988) chapter XVIII section 1a. $\square$

Remarks. (i) If n is odd then the Birkhoff normal forms of $\mathbf{D}_n$ - and $\mathbf{D}_{2n}$ -invariant Hamiltonians are identical.

Table 1. Representatives of conjugacy classes of proper isotropy subgroups of the actions of $\mathbf{O}(2) \times \mathbf{S}^1$ and $\mathbf{D}_n \times \mathbf{S}^1$ on $V = \mathbb{C}^2$.

Γ	Isotropy subgroup Σ	$\text{Fix}(\Sigma)$	Dimension of $\text{Fix}(\Sigma)$
$\mathbf{O}(2)$	$\widetilde{\mathbf{SO}}(2)$	$z_1 = z, z_2 = 0$	2 ^a
	$\mathbf{Z}_2^{\kappa} \times \mathbf{Z}_2^{(\pi, \pi)}$	$z_1 = z_2 = z$	2
	$\mathbf{Z}_2^{(\pi, \pi)}$	V	4
$\mathbf{D}_n$ n odd	$\tilde{\mathbf{Z}}_n$	$z_1 = z, z_2 = 0$	2
	$\mathbf{Z}_2^{\kappa}$	$z_1 = z_2 = z$	2
	$\mathbf{Z}_2^{(\pi, \pi)}$	$z_1 = -z_2 = z$	2
	$\uparrow$	V	4
$\mathbf{D}_n$ n even	$\tilde{\mathbf{Z}}_n$	$z_1 = z, z_2 = 0$	2
	$\mathbf{Z}_2^{\kappa} \times \mathbf{Z}_2^{(\pi, \pi)}$	$z_1 = z_2 = z$	2
	$\mathbf{Z}_2^{\kappa'} \times \mathbf{Z}_2^{(\pi, \pi)}$	$z_1 = \exp(2\pi i/n) z_2 = z$	2
	$\mathbf{Z}_2^{(\pi, \pi)}$	V	4

(ii) For any n the Taylor series to degree $2m - 1$ of a $\mathbf{D}_n$ -invariant Hamiltonian in Birkhoff normal form is $\mathbf{O}(2)$ -invariant.

Theorem 4.2. Let H be a Γ -invariant Hamiltonian on V such that $iD^2H(0)$ has eigenvalues $\pm i$.

(a) If $\Gamma = \mathbf{O}(2)$ then there exist nonlinear normal modes with isotropy subgroups $\widetilde{\mathbf{SO}}(2)$ and $\mathbf{Z}_2^\kappa \times \mathbf{Z}_2^{(\pi, \pi)}$. If H is in Birkhoff normal form to degree 4, so $H = \hat{H}(N, P) + O(|z|^5)$, and $\beta = (\partial \hat{H} / \partial P)(0) \neq 0$ then these are the only nonlinear normal modes.

(b) If $\Gamma = \mathbf{D}_n$ ($n \geq 3$) there exist nonlinear normal modes with symmetry types $\mathbf{Z}_n$, $\mathbf{Z}_2^\kappa$ and $\mathbf{Z}_2^{\kappa, \pi}$ if n is odd and symmetry types $\mathbf{Z}_n$, $\mathbf{Z}_2^\kappa \times \mathbf{Z}_2^{(\pi, \pi)}$ and $\mathbf{Z}_2^{\kappa'} \times \mathbf{Z}_2^{(\pi, \pi)}$ if n is even. If H is in Birkhoff normal form to degree $2m$, so $H = \hat{H}(N, P, S, T) + O(|z|^{2m+1})$, and $\beta = (\partial \hat{H} / \partial P)(0)$ and $\gamma = (\partial \hat{H} / \partial S)(0)$ satisfy

$$\gamma \neq 0 \text{ and } \begin{cases} \beta \neq 0 & \text{if } n \neq 4 \\ \beta \pm 2\gamma \neq 0 & \text{if } n = 4 \end{cases}$$

then these are the only nonlinear normal modes.

Proof. For (a) see Montaldi *et al* (1988), section 7; for (b) see Montaldi *et al* (1990a), section 4.

We now come to the main theorem of this section, describing the spectral stability of these nonlinear normal modes in generic systems. Recall that two of the Floquet exponents, corresponding to directions tangent to the periodic orbit and normal to the energy level, are always 0.

Theorem 4.3. Let $\Gamma = \mathbf{O}(2)$ or $\mathbf{D}_n$ and let H be a Γ -invariant Hamiltonian in Birkhoff normal form satisfying the conditions of theorem 4.2. Then the nonlinear normal modes of H have the reduced Floquet exponents given in table 2.

Corollary 4.4. Let $\Gamma = \mathbf{O}(2)$ or $\mathbf{D}_n$ and let H be a Γ -invariant Hamiltonian satisfying the conditions of theorem 4.2. Then we have the following.

(a) The nonlinear normal modes of $\mathbf{O}(2)$ -invariant Hamiltonians are always spectrally stable.

(b) For $n \geq 3$, $n \neq 4$, the $\tilde{\mathbf{Z}}_n$ nonlinear normal modes of $\mathbf{D}_n$ -invariant Hamil-

Table 2. Reduced Floquet exponents for nonlinear normal modes of $\mathbf{O}(2)$ - and $\mathbf{D}_n$ -invariant Hamiltonians in Birkhoff normal form.

Γ	Σ	Floquet exponents
$\mathbf{O}(2)$	$\widetilde{\mathbf{SO}}(2)$ $\mathbf{Z}_2^\kappa \times \mathbf{Z}_2^{(\pi, \pi)}$	$\pm i[\beta z ^2 + \mathbf{O}(z ^4)]$ 0 (repeated)
$\mathbf{D}_n$ $n \geq 3$ $n \neq 4$	$\tilde{\mathbf{Z}}_n$ $\mathbf{Z}_2^\kappa$ or $\mathbf{Z}_2^\kappa \times \mathbf{Z}_2^{(\pi, \pi)}$ $\mathbf{Z}_2^{\kappa, \pi}$ or $\mathbf{Z}_2^{\kappa'} \times \mathbf{Z}_2^{(\pi, \pi)}$	$\pm i[\beta z ^2 + \mathbf{O}(z ^4)]$ $\pm \sqrt{2m}(-\beta\gamma)^{1/2} z ^m + \mathbf{O}(z ^{m+1})$ $\pm \sqrt{2m}(\beta\gamma)^{1/2} z ^m + \mathbf{O}(z ^{m+1})$
$\mathbf{D}_4$	$\tilde{\mathbf{Z}}_4$ $\mathbf{Z}_2^\kappa \times \mathbf{Z}_2^{(\pi, \pi)}$ $\mathbf{Z}_2^{\kappa'} \times \mathbf{Z}_2^{(\pi, \pi)}$	$\pm [(2\gamma - \beta)(2\gamma + \beta)]^{1/2} z ^2 + \mathbf{O}(z ^3)$ $\pm [-8\gamma(2\gamma + \beta)]^{1/2} z ^2 + \mathbf{O}(z ^3)$ $\pm [-8\gamma(2\gamma - \beta)]^{1/2} z ^2 + \mathbf{O}(z ^3)$

tonians are always spectrally stable. The stability of the other nonlinear normal modes depends only on sign $\beta\gamma$, as in figure 2(a).

(c) The stability of the nonlinear normal modes of $\mathbf{D}_4$ -invariant Hamiltonians depends on the signs of γ , $2\gamma + \beta$ and $2\gamma - \beta$, as shown in figure 2(b).

Remarks. (i) The $\widetilde{\mathbf{SO}}(2)$ and $\tilde{\mathbf{Z}}_n$ ($n \neq 4$) nonlinear normal modes are always spectrally stable because their reduced representations are cyclospectral, see Montaldi *et al* (1988), section 7.

(ii) The $\mathbf{Z}_2^K \times \mathbf{Z}_2^{(\pi, \pi)}$ nonlinear normal modes of $\mathbf{O}(2)$ -symmetric Hamiltonians are neutrally stable because these orbits are contained in a continuous family of such orbits, resulting from the $\mathbf{SO}(2)$ -action. This action gives a conserved quantity (the angular momentum $|z_1|^2 - |z_2|^2$) which forces the reduced infinitesimal Floquet operator L to have eigenvalues equal to 0. However L is not zero; its image is along the $\mathbf{SO}(2)$ -orbit and should be interpreted as precession. The presence of precession means that the periodic solution is not stable in the usual sense. It is, however, *orbitally stable* in the sense of Montaldi *et al* (1988), section 0.

(iii) The results of this section are unchanged, except perhaps for extra information on isotropy subgroups, if the system also has time-reversal symmetry.

Proof of theorem 4.3. We consider first the truncated $\mathbf{O}(2)$ -invariant Hamiltonian

$$H = N + \alpha N^2 + \beta P.$$

The nonlinear normal modes are the critical orbits of the function $F_\tau = -\tau N + \alpha N^2 + \beta P$; those with symmetry group $\widetilde{\mathbf{SO}}(2)$ satisfy $z_2 = 0$, $z_1 = z$ (say), and those with symmetry group $\mathbf{Z}_2^K \times \mathbf{Z}_2^{(\pi, \pi)}$ have $z_1 = z_2 = z$. Easy calculations give

$$\tau = \begin{cases} 2\alpha |z|^2 & \text{for the } \widetilde{\mathbf{SO}}(2) \text{ solutions} \\ (4\alpha + \beta) |z|^2 & \text{for the } \mathbf{Z}_2^K \times \mathbf{Z}_2^{(\pi, \pi)} \text{ solutions.} \end{cases}$$

Figure 2. Stable nonlinear normal modes. (a) n odd. The case n even, $n \neq 4$ is similar. (b) $n = 4$.

The reduced Floquet exponents are the eigenvalues of $iD^2F_\tau(z)$ on the complement to $\text{Fix}(\Sigma)$. For $\Sigma = \widehat{\mathbf{SO}}(2)$ these are the eigenvalues of the linear map

$$w \rightarrow \frac{i\partial^2 F_\tau}{\partial \bar{z}_2 \partial z_2} w + \frac{i\partial^2 F_\tau}{\partial \bar{z}_2^2} \bar{w}$$

with $\tau = 2\alpha |z|^2$. Evaluating the partial derivatives gives

$$w \rightarrow i\beta |z|^2 \cdot w$$

which as a real mapping has eigenvalues $\pm i\beta |z|^2$ by subsection 2.7. Because the reduced representation of $\widehat{\mathbf{SO}}(2)$ is cyclospectral we know the corresponding nonlinear normal modes of the full system are spectrally stable. This calculation gives the leading-order term in the Taylor series of the non-trivial Floquet exponents. Similar remarks apply to the $\mathbf{Z}_2^\kappa \times \mathbf{Z}_2^{(\pi, \pi)}$ nonlinear normal modes, except that for these the $\mathbf{SO}(2)$ symmetry forces the reduced Floquet exponents to be identically zero (see section 7 of Montaldi *et al* (1988)).

Turning to $\mathbf{D}_n$ -invariant systems, we consider the truncated Hamiltonian

$$H = N + \beta P + \gamma S + f(N, P)$$

where f is a polynomial of order 2 and degree $\leq m$ in N and P . The critical orbits of F_τ are, up to conjugacy,

$$\left. \begin{array}{lll} \tilde{\mathbf{Z}}_n & z_2 = 0 \quad z_1 = z & \tau = f_1 \\ \mathbf{Z}_2^\kappa & (n \text{ odd}) & \\ \mathbf{Z}_2^\kappa \times \mathbf{Z}_2^{(\pi, \pi)} & (n \text{ even}) & \left. \begin{array}{l} z_1 = z_2 = z \\ \tau = f_1 + (\beta + f_2) |z|^2 + m\gamma |z|^{2(m-1)} \end{array} \right\} \\ \mathbf{Z}_2^{\kappa \circ \pi} & (n \text{ odd}) & \\ \mathbf{Z}_2^{\kappa'} \times \mathbf{Z}_2^{(\pi, \pi)} & (n \text{ even}) & \left. \begin{array}{l} z_1 = z_2 = z \\ z_1 = e^{2\pi i/n} z_2 = n \end{array} \right\} \left. \begin{array}{l} \tau = f_1 + (\beta + f_2) |z|^2 - m\gamma |z|^{2(m-1)} \end{array} \right\} \end{array} \right\}$$

where f_j is the derivative of $f(N, P)$ with respect to its j th argument.

We calculate the reduced Floquet exponents for the $\tilde{\mathbf{Z}}_n$ nonlinear normal modes exactly as for the $\widehat{\mathbf{SO}}(2)$ nonlinear normal modes of $\mathbf{O}(2)$ -invariant Hamiltonians. For $n \neq 4$ the result is the same to leading order. For $n = 4$ we obtain eigenvalues $\pm[(2\gamma - \beta)(2\gamma + \beta)]^{1/2} |z|^2 + O(|z|^3)$.

For the $\mathbf{Z}_2^\kappa$ (n odd) and $\mathbf{Z}_2^\kappa \times \mathbf{Z}_2^{(\pi, \pi)}$ (n even) nonlinear normal modes the reduced space is parametrised by

$$w \rightarrow (z_1, z_2) + \frac{1}{\sqrt{2}} (w, -w).$$

The restriction of the infinitesimal Floquet operator to the reduced space is

$$w \rightarrow i \frac{\partial^2 F_\tau}{\partial \bar{w} \partial w} w + i \frac{\partial^2 F_\tau}{\partial \bar{w}^2} \bar{w}.$$

Using

$$\frac{\partial}{\partial w} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2} \right)$$

etc, we find

$$\begin{aligned} i \frac{\partial^2 F_\tau}{\partial \bar{w} \partial w} &= -i[\beta + f_2 + m(m+1)\gamma |z|^{2m-4}] |z|^2 \\ i \frac{\partial^2 F_\tau}{\partial \bar{w}^2} &= -i[\beta + f_2 - m(m-1)\gamma |z|^{2m-4}] z^2 \end{aligned}$$

and hence eigenvalues:

$$\begin{aligned} \pm 2m(-\beta\gamma)^{1/2} |z|^m + O(|z|^{m+1}) & \quad n \neq 4 \\ \pm [-8\gamma(2\gamma + \beta)]^{1/2} |z|^2 + O(|z|^3) & \quad n = 4. \end{aligned}$$

The calculations for the $\mathbf{Z}_2^{\kappa \circ \pi}$ and $\mathbf{Z}_2^{\kappa'} \times \mathbf{Z}_2^{(\pi, \pi)}$ nonlinear normal modes are similar. $\square$

Proof of corollary 4.4. The case $\mathbf{O}(2)$ is clear since the reduced representations are cyclospectral. For $\mathbf{D}_n$, $n \neq 4$, the reduced representation of $\mathbf{Z}_n$ is cyclospectral, so the corresponding nonlinear normal modes of the full Hamiltonian are spectrally stable. The calculations show that the stabilities for the remaining nonlinear normal modes are of the form (3.5) in the truncated Birkhoff normal forms, and so theorem 3.1 applies. $\square$

4.2. Circular and dihedral potential wells

We now describe the computation of Birkhoff normal forms of Hamiltonians for particles in two-dimensional potential wells that are invariant under actions of $\mathbf{O}(2)$ or $\mathbf{D}_n$ ($n \geq 3$). For $\mathbf{D}_n$ -invariant Hamiltonians the two cases n even and n odd behave quite differently, the essential difference being that $\mathbf{O}(2)$ symmetry-breaking terms occur first in the Birkhoff normal form at degree n when n is even, but $2n$ when n is odd. By theorem 3.4 these are the smallest degrees to which we must calculate the normal forms in order to determine the stability of the nonlinear normal modes for generic Hamiltonians. However, in both cases the terms that break the $\mathbf{O}(2)$ symmetry occur first in the potential at degree n . It follows that for even n only one $\mathbf{O}(2)$ symmetry-breaking term is involved in the calculations, and this occurs in the highest degree that must be considered, namely n . The Birkhoff normal form to degree n can therefore be easily calculated from that for $\mathbf{O}(2)$ symmetric potentials. However, for odd n many more $\mathbf{O}(2)$ symmetry-breaking terms occur, scattered between degree n and $2n$. This leads to considerably more complicated coefficients in the normal forms, as the result for $n = 3$ below shows.

Consider actions of $\mathbf{O}(2)$ and $\mathbf{D}_n$ on $\mathbb{R}^2$ given by

$$\theta \cdot (x, y) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta), \quad \kappa \cdot (x, y) = (x, -y)$$

where θ denotes an arbitrary angle for $\mathbf{O}(2)$ and $2r\pi/n$ ($r = 0, \dots, n-1$) for $\mathbf{D}_n$. Extend these actions to $V = \mathbb{C}^2 = \mathbb{R}^2 \otimes \mathbb{C}$, with coordinates (X, Y) on $\mathbb{C}^2$, in the natural way. We consider $\mathbf{O}(2)$ - and $\mathbf{D}_n$ -invariant Hamiltonians of the form

$$H(X, Y) = |X|^2 + |Y|^2 + V(x, y)$$

where $V(x, y)$ is a smooth invariant function on $\mathbb{R}^2 \subset \mathbb{C}$ with $j^2 V = 0$. The proof of the following result is straightforward.

Lemma 4.5. (i) If V is $\mathbf{O}(2)$ -invariant then it is a smooth function of $q = x^2 + y^2$.

(ii) If V is $\mathbf{D}_n$ -invariant then it is a smooth function of q and

$$r = \sum_{\substack{k=0 \\ k \text{ even}}}^n \binom{n}{k} x^{n-k} y^k.$$

Now make a unitary change of coordinates by defining

$$z_1 = \frac{1}{\sqrt{2}}(X + iY) \quad z_2 = \frac{1}{\sqrt{2}}(X - iY).$$

The actions of $\mathbf{O}(2)$ and $\mathbf{D}_n$ become those in subsection 4.1 and the Hamiltonian becomes

$$H(z_1, z_2) = |z_1|^2 + |z_2|^2 + V(q, r)$$

where

$$q = \frac{1}{4}(|z_1|^2 + |z_2|^2) + \frac{1}{2} \operatorname{Re}(z_1 z_2) \quad r = 2^{-n/2} \operatorname{Re}(z_1 + \bar{z}_2)^n.$$

Proposition 4.6. The Birkhoff normal form to degree 6 of the $\mathbf{O}(2)$ -invariant Hamiltonian

$$H(z_1, z_2) = |z_1|^2 + |z_2|^2 + a_1 q^2 + a_2 q^3 + \dots$$

is

$$\hat{H}_2 + \hat{H}_4 + \hat{H}_6 + \dots$$

where

$$\begin{aligned} \hat{H}_2 &= N & \hat{H}_4 &= \frac{1}{16} a_1 (N^2 + 2P) \\ \hat{H}_6 &= \frac{1}{256} (-a_1^2 + 4a_2) N^3 + \frac{1}{512} (-5a_1^2 + 48a_2) NP. \end{aligned}$$

Proof. We follow the procedure described in section 2 of Montaldi *et al* (1990a). Recall that the Poisson bracket of two functions h, f on V is given by

$$\{h, f\} = \operatorname{Im} \left(\frac{\partial h}{\partial \bar{z}_1} \frac{\partial f}{\partial z_1} + \frac{\partial h}{\partial \bar{z}_2} \frac{\partial f}{\partial z_2} \right)$$

and so

$$\operatorname{ad} H_2(f) = \{N, f\} = \operatorname{Im} \left(z_1 \frac{\partial f}{\partial z_1} + z_2 \frac{\partial f}{\partial z_2} \right).$$

Recall also that the actions of the operators π and η on monomials are given by

$$\begin{aligned} \pi(z_1^{k_1} \bar{z}_1^{l_1} z_2^{k_2} \bar{z}_2^{l_2}) &= \begin{cases} z_1^{k_1} \bar{z}_1^{l_1} z_2^{k_2} \bar{z}_2^{l_2} & \text{if } k_1 + k_2 - l_1 - l_2 = 0 \\ 0 & \text{otherwise} \end{cases} \\ \eta(z_1^{k_1} \bar{z}_1^{l_1} z_2^{k_2} \bar{z}_2^{l_2}) &= \frac{i}{(k_1 + k_2 - l_1 - l_2)} z_1^{k_1} \bar{z}_1^{l_1} z_2^{k_2} \bar{z}_2^{l_2} \quad \text{if } k_1 + k_2 - l_1 - l_2 \neq 0. \end{aligned}$$

Note that η is defined only on $\ker(\pi)$, and that both π and η commute with taking real and imaginary parts.

Write the original Hamiltonian as

$$H = H_2 + H_4 + H_6 + \dots$$

where

$$\begin{aligned} H_2 &= N & H_4 &= a_1 q^2 = \frac{1}{16} a_1 (N + 2 \operatorname{Re}(z_1 z_2))^2 \\ H_6 &= a_2 q^3 = \frac{1}{64} a_2 (N + 2 \operatorname{Re}(z_1 z_2))^3. \end{aligned}$$

Then $H_4 = \hat{H}_4 + \tilde{H}_4$ where

$$\hat{H}_4 = \pi(H_4) = \frac{1}{16}a_1(N^2 + 2P)$$

$$\tilde{H}_4 = (I - \pi)(H_4) = \frac{1}{16}a_1(4N \operatorname{Re}(z_1 z_2) + 2 \operatorname{Re}(z_1^2 z_2^2)).$$

Set

$$L_4 = \eta(\tilde{H}_4) = -\frac{1}{32}a_1[4N \operatorname{Im}(z_1 z_2) + \operatorname{Im}(z_1^2 z_2^2)]$$

and let $H^{(4)} = \exp \operatorname{ad} L_4(H)$, the Hamiltonian normalised to degree 4. Then

$$H^{(4)} = H_2^{(4)} + H_4^{(4)} + H_6^{(4)} + \dots$$

where

$$H_2^{(4)} = H_2 = N$$

$$H_4^{(4)} = H_4 + \{L_4, H_2\} = H_4 - \tilde{H}_4 = \hat{H}_4$$

$$\begin{aligned} H_6^{(4)} &= H_6 + \{L_4, H_4\} + \frac{1}{2}\{L_4, L_4, H_2\} = H_6 + \{L_4, H_4\} - \frac{1}{2}\{L_4, \tilde{H}_4\} \\ &= H_6 + \frac{1}{2}\{L_4, \tilde{H}_4\} \end{aligned}$$

since $\operatorname{ad} L_4 \circ \pi \equiv 0$. A routine calculation gives

$$\begin{aligned} \{L_4, \tilde{H}_4\} &= \frac{1}{512}a_1^2[-4N^3 - 2NP - 6N^2 \operatorname{Re}(z_1 z_2) - 16N(\operatorname{Re}(z_1 z_2))^2 \\ &\quad - 8 \operatorname{Re}(z_1 z_2) \operatorname{Re}(z_1^2 z_2^2)] \end{aligned}$$

and so

$$\hat{H}_6^{(4)} = \pi(H_6) + \frac{1}{2}\{L_4, \tilde{H}_4\} = \frac{1}{256}(-a_1^2 + 4a_2)N^3 + \frac{1}{512}(-5a_1^2 + 48a_2)NP.$$

This completes the proof. $\square$

To compute the stability of nonlinear normal modes of generic $\mathbf{D}_n$ -invariant Hamiltonians we must calculate their Birkhoff normal form to degree n for n even and to degree $2n$ for n odd. For the potential well Hamiltonians under consideration, the Taylor series to degree n can be written

$$H(z_1, z_2) = |z_1|^2 + |z_2|^2 + V_{\mathbf{O}(2)}(q) + br + \mathbf{O}(|z|^{n+1})$$

where $V_{\mathbf{O}(2)}(q)$ is an $\mathbf{O}(2)$ -invariant potential with $j^2 V = 0$. The Birkhoff normal form to degree n of this Hamiltonian is easily deduced from that of $H_{\mathbf{O}(2)} = |z_1|^2 + |z_2|^2 + V_{\mathbf{O}(2)}(q)$.

Proposition 4.7. For $n \geq 3$ let H be a $\mathbf{D}_n$ -invariant Hamiltonian

$$H = H_{\mathbf{O}(2)} + br + \mathbf{O}(|z|^{n+1}).$$

If $\hat{H}_{\mathbf{O}(2)}$ is the Birkhoff normal form to degree n of $H_{\mathbf{O}(2)}$, then the Birkhoff normal form to degree n of H is $\hat{H}_{\mathbf{O}(2)}$ for n odd and $\hat{H}_{\mathbf{O}(2)} + 2^{-(1+n/2)} \binom{n}{n/2} bS$ for n even.

Proof. The process of normalising H in degrees less than n leaves the $\mathbf{O}(2)$ symmetry-breaking term br unchanged. The normal form to degree n is then obtained by averaging the degree- n terms in $H^{(n-1)}$, namely

$$H_n^{(n-1)} = H_{\mathbf{O}(2),n}^{(n-1)} + br$$

to yield

$$H_n^{(n)} = H_{\mathbf{O}(2),n}^{(n)} + b\pi(r).$$

A simple calculation shows

$$\pi(r) = \begin{cases} \frac{1}{2^{n/2+1}} \binom{n}{n/2} S & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

hence the result. $\square$

Corollary 4.8. (a) Suppose n is even, $n \neq 4$, and let H be a $\mathbf{D}_n$ -invariant Hamiltonian with

$$H(z_1, z_2) = |z_1|^2 + |z_2|^2 + a_1 q^2 + a_2 q^3 + \dots + a_{n/2} q^{n/2} + br + \mathbf{O}(|z|^{n+1}).$$

Then the $\mathbf{Z}_2^\kappa \times \mathbf{Z}_2^{(\pi, \pi)}$ nonlinear normal modes of H are spectrally stable and the $\mathbf{Z}_2^{\kappa'} \times \mathbf{Z}_2^{(\pi, \pi)}$ nonlinear normal modes are spectrally unstable if $a_1 b > 0$. The stabilities are interchanged if $a_1 b < 0$.

(b) Let H be a $\mathbf{D}_4$ -invariant Hamiltonian with

$$H(z_1, z_2) = |z_1|^2 + |z_2|^2 + a_1 q^2 + br + \mathbf{O}(|z|^5).$$

Then the $\mathbf{Z}_4$ (respectively $\mathbf{Z}_2^\kappa \times \mathbf{Z}_2^{(\pi, \pi)}$, $\mathbf{Z}_2^{\kappa'} \times \mathbf{Z}_2^{(\pi, \pi)}$) nonlinear normal modes are spectrally stable if $(12b - a_1)(12b + a_1) < 0$ (respectively, $b(12b + a_1) > 0$, $b(12b - a_1) > 0$) and unstable if the inequality is reversed. $\square$

By way of contrast, we now give the Birkhoff normal form to degree 6 of $\mathbf{D}_3$ -invariant potential-well Hamiltonians, and hence the stabilities of nonlinear normal modes in generic systems of this type. The proof is a straightforward, though lengthy, computation.

Proposition 4.9. The Birkhoff normal form to degree 6 of the $\mathbf{D}_3$ -invariant Hamiltonian

$$H = |z_1|^2 + |z_2|^2 + c_1 r + a_1 q^2 + c_2 qr + a_2 \dot{q}^3 + br^2 + \mathbf{O}(|z|^7)$$

is

$$\hat{H} = \hat{H}_2 + \hat{H}_4 + \hat{H}_6 + \mathbf{O}(|z|^7)$$

where

$$\begin{aligned} \hat{H}_2 &= N & \hat{H}_4 &= \frac{1}{32}(2a_1 + 3c_1^2)N^2 + \frac{1}{16}(2a_1 - 21c_1^2)P \\ \hat{H}_6 &= \frac{1}{1024}(-4a_1^2 + 16a_2 + 64b - 33c_1^4 - 20a_1c_1^2 + 64c_1c_2)N^3 \\ &\quad + \frac{1}{2048}(-20a_1^2 + 192a_2 + 768b + 171c_1^4 + 452a_1c_1^2 - 2304c_1c_2)NP \\ &\quad + \frac{1}{128}(160b - 63c_1^4 + 26a_1c_1^2 - 56c_1c_2)S. \end{aligned}$$

The special case $b = c_1 = c_2 = 0$ is the general $\mathbf{O}(2)$ -invariant potential-well Hamiltonian of degree 6, while $c_1 = c_2 = 0$ gives the general $\mathbf{D}_6$ -invariant system of degree 6. The Birkhoff normal form given here agrees with those in propositions 4.6 and 4.7. The Birkhoff normal form for the case $a_1 = a_2 = b = c_2 = 0$, the Hénon–Heiles Hamiltonians, has also been computed in Churchill *et al* (1983).

Corollary 4.10. (a) Let H be the general $\mathbf{D}_3$ -invariant potential-well Hamiltonian as in proposition 4.5. Then the $\mathbf{Z}_2^\kappa$ nonlinear normal modes of H are spectrally stable

Figure 3. Stable nonlinear normal modes (in addition to $\tilde{Z}_3$) for $\mathbf{D}_3$ symmetric potential-well Hamiltonians: (a) $b = c_2 = 0$; (b) $b = 0$, $c_2 > 0$.

and the $\mathbf{Z}_2^{\kappa\pi}$ nonlinear normal modes spectrally unstable if

$$(2a_1 - 21c_1^2)(160b - 63c_1^4 + 26a_1c_1^2 - 56c_1c_2) > 0.$$

The stabilities are interchanged if the inequality is reversed.

(b) For the Hénon–Heiles Hamiltonians the $\mathbf{Z}_2^{\kappa}$ nonlinear normal modes are spectrally stable and the $\mathbf{Z}_2^{\kappa\pi}$ nonlinear normal modes are spectrally unstable.

The spectrally stable nonlinear normal modes for two further special cases are illustrated in figure 3. Note that (a) corresponds to truncating the potential well at fourth order while (b) is the fifth-order truncation. The Hénon–Heiles Hamiltonians sit on the c_1 axis in (a). These results demonstrate that for n odd it is not sufficient to truncate a $\mathbf{D}_n$ -invariant potential at the lowest-order $\mathbf{O}(2)$ symmetry-breaking term.

5. Five degrees of freedom with spherical symmetry

In section 8 of Montaldi *et al* (1988) we prove the existence of five types of nonlinear normal mode for a Hamiltonian that is invariant under the natural symplectically irreducible action of $\mathbf{O}(3)$ on $\mathbb{C}^5$. In this section we apply theorem 3.1 to derive conditions for the spectral stability of these nonlinear normal modes.

It is well known that the irreducible representations of the group $\mathbf{O}(3)$ of orthogonal transformations of $\mathbb{R}^3$ are parametrised by the positive integers l and the signs ± 1 . Namely, the subgroup $\mathbf{SO}(3)$ acts naturally on the $(2l+1)$ -dimensional space U_l of spherical harmonics of degree l , and $-I \in \mathbf{O}(3)$ acts as either $+I$ or $-I$ on U_l . The resulting representations are denoted U_l^+ and U_l^- respectively; they are

all absolutely irreducible, i.e. $\text{Hom}_{\mathbf{O}(3)}(U_l^\pm, U_l^\pm) \cong \mathbb{R}$. The ‘natural’ representations—those arising from the action of $\mathbf{O}(3)$ on the space of functions on the two-dimensional sphere—have sign $(-1)^l$. For details on these representations and their maximal isotropy subgroups see Ihrig and Golubitsky (1984), Michel (1980) and Golubitsky *et al* (1988).

To study nonlinear normal modes of Hamiltonian systems with $\mathbf{O}(3)$ symmetry we pass to the symplectic representations $V_l^\pm = U_l^\pm \otimes_{\mathbb{R}} \mathbb{C}$. For details see Montaldi *et al* (1988) section 2b. Here we restrict to the simplest interesting case, that of V_2^+ . Since $-I \in \mathbf{O}(3)$ acts trivially in this representation, we refer from now on to the space as $V_2 = U_2 \otimes \mathbb{C}$ and consider it as a representation of $\mathbf{SO}(3)$. Thus we consider a Hamiltonian system with five degrees of freedom and a natural $\mathbf{SO}(3)$ -action.

Recall from Montaldi *et al* (1988), section 8 that there are five maximal isotropy subgroups in this case, corresponding to five different types of nonlinear normal mode. We show that their spectral stability is determined by the Birkhoff normal form of the Hamiltonian truncated at order 6, and compute the Floquet exponents to leading order. By theorem 3.1 the resulting stability conditions apply even if the full Hamiltonian is *not* in Birkhoff normal form.

For a physical model of such a Hamiltonian, consider oscillations near equilibrium of a spherical liquid drop. Of course, this is an infinite-dimensional system having infinitely many oscillatory modes, but individual modes may be expected to correspond to symplectically irreducible representations of $\mathbf{O}(3)$, i.e. to the spaces $V_l^\pm$. We do not carry out such a reduction here, nor do we consider a realistic Hamiltonian for a liquid drop: we merely use the liquid drop picture to make the system more vivid.

5.1. Isotropy subgroups

It is well known that the action of $\mathbf{SO}(3)$ on the space U_2 of harmonic functions of degree 2 is isomorphic to an action on the space of 3×3 real symmetric trace-zero matrices, see Golubitsky and Schaeffer (1982). Thus we may identify V_2 with the space of 3×3 complex symmetric trace-zero matrices. The action of $\mathbf{SO}(3)$ on U_2 and V_2 is by conjugation,

$$\gamma \cdot M = \gamma^{-1} M \gamma$$

and the action of $\mathbf{S}^1$ is by scalar multiplication by unit complex numbers

$$\theta \cdot M = e^{i\theta} M.$$

This is equivalent to viewing the $\mathbf{SO}(3) \times \mathbf{S}^1$ action on V_2 as the tensor product of the actions of $\mathbf{SO}(3)$ on U_2 and $\mathbf{S}^1$ on $\mathbb{C}$. The appropriate symplectic form on V_2 is

$$\omega(M_1, M_2) = \text{Im}(\text{tr } M_1 \bar{M}_2).$$

Recall from Golubitsky *et al* (1988), chapter XVI, section 7, the notation for isotropy subgroups of a $\Gamma \times \mathbf{S}^1$ -action with elements of $\mathbf{S}^1$ fixing only 0. Each proper isotropy subgroup $\Sigma \subset \Gamma \times \mathbf{S}^1$ projects isomorphically onto a subgroup H of Γ . The subgroup Σ is then recaptured as the graph $\{(h, \varphi(h)) : h \in H\}$ of a homomorphism $\varphi : H \rightarrow \mathbf{S}^1$, called the twist homomorphism. The kernel of φ is denoted K and consists of the purely spatial isotropy: $K = \Sigma \cap \Gamma$. In particular cases, Σ is denoted by H if φ is trivial, and by $\tilde{H}$ otherwise, with a superscript added in cases of

ambiguity. Thus $\widetilde{\mathbf{SO}}(2)^k$ means $H = \mathbf{SO}(2)$ and $K = \mathbf{Z}_k \subset \mathbf{SO}(2)$. Note that if Σ is an isotropy subgroup of $\Gamma \times \mathbf{S}^1$ then K is an isotropy subgroup of Γ , though possibly with a larger fixed-point space.

To determine the isotropy subgroups Σ of the $\mathbf{SO}(3) \times \mathbf{S}^1$ -action on V_2 we start by determining the isotropy subgroups K of the $\mathbf{SO}(3)$ -action on V_2 .

Lemma 5.1. Any isotropy subgroup of the $\mathbf{SO}(3)$ -action is conjugate to one of $\mathbf{SO}(3)$, $\mathbf{O}(2)$, $\mathbf{D}_2$, $\mathbf{Z}_2$, $\mathbb{1}$, defined in table 3.

Proof. First consider $\mathbf{SO}(3)$ acting on U_2 . The isotropy subgroup of a given matrix $M \in U_2$ depends on the number of coincident eigenvalues: it is $\mathbf{SO}(3)$ if $M = 0$, $\mathbf{O}(2)$ if two eigenvalues are equal, and $\mathbf{D}_2$ if all three are distinct. Viewing V_2 as $U_2 \oplus U_2$, the isotropy subgroup of (M_1, M_2) is just the intersection of the isotropy subgroups of M_1 and M_2 . The result now follows. $\square$

Theorem 5.2. Any isotropy subgroup Σ of the $\mathbf{SO}(3) \times \mathbf{S}^1$ -action on V_2 is conjugate to one of the subgroups listed in table 4.

The table also lists the fixed-point sets of the isotropy subgroups. Figure 4 shows the partial ordering on the conjugacy classes induced by inclusion of subgroups (or equivalently, by the reversal of inclusion of their fixed-point spaces).

If Σ is an isotropy subgroup then the normaliser $N(\Sigma)$ leaves $\text{Fix}(\Sigma)$ invariant (and is the largest subgroup to do so). Since Σ fixes $\text{Fix}(\Sigma)$ pointwise, there is a natural action of $N(\Sigma)/\Sigma$ on $\text{Fix}(\Sigma)$, giving the effective symmetry of $\text{Fix}(\Sigma)$ as a subspace. We therefore list $N(\Sigma)/\Sigma$ in the table.

Proof. The subgroups of $\mathbf{SO}(3)$ are well known: see Armstrong (1988), chapter 19. They are $\mathbf{SO}(3)$, $\mathbb{1}$, $\mathbb{O}$, $\mathbb{T}$, $\mathbf{O}(2)$, $\mathbf{D}_n$ ($n \geq 2$), $\mathbf{SO}(2)$, $\mathbf{Z}_n$ ($n \geq 2$) and $\mathbb{1}$. Thus for any isotropy subgroup Σ , its projection H in $\mathbf{SO}(3)$ must be one of these. Moreover, the kernel K of the twist homomorphism $\varphi: H \rightarrow \mathbf{S}^1$ must contain H' , the commutator

Table 3. Isotropy subgroups of the $\mathbf{SO}(3)$ -action on U_2 .

K	generators of K	$\text{Fix}(K; V)$
$\mathbf{SO}(3)$		0
$\mathbf{O}(2)$	$\begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\left\{ \begin{pmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & -2z \end{pmatrix} \middle z \in \mathbb{C} \right\}$
$\mathbf{D}_2$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\left\{ \begin{pmatrix} z & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & -(z+w) \end{pmatrix} \middle z, w \in \mathbb{C} \right\}$
$\mathbf{Z}_2$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\left\{ \begin{pmatrix} z & v & 0 \\ v & w & 0 \\ 0 & 0 & -(z+w) \end{pmatrix} \middle z, w, v \in \mathbb{C} \right\}$
$\mathbb{1}$	I	V

Table 4. Isotropy subgroups and their fixed point spaces for the action of $\mathbf{SO}(3) \times \mathbf{S}^1$ on V_2 . Here $\omega = e^{2\pi i/3}$.

Σ	K	Twist image	dim Fix Σ	Fix Σ	$N(\Sigma)/\Sigma$
$\mathbf{SO}(3) \times \mathbf{S}^1$	—	—	0	0	1
$\mathbf{O}(2)$	$\mathbf{O}(2)$	1	2	$\left\{ \begin{pmatrix} z & & \\ & z & \\ & & -2z \end{pmatrix} \middle z \in \mathbb{C} \right\}$	$\mathbf{S}^1$
$\widetilde{\mathbf{SO}}(2)^1$	1	$\mathbf{S}^1$	2	$\left\{ \begin{pmatrix} 0 & 0 & z \\ 0 & 0 & iz \\ z & iz & 0 \end{pmatrix} \middle z \in \mathbb{C} \right\}$	$\mathbf{S}^1$
$\widetilde{\mathbf{SO}}(2)^2$	$\mathbf{Z}_2$	$\mathbf{S}^1$	2	$\left\{ \begin{pmatrix} z & iz & 0 \\ iz & -z & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle z \in \mathbb{C} \right\}$	$\mathbf{S}^1$
$\tilde{\mathbf{T}}$	$\mathbf{D}_2$	$\mathbf{Z}_3$	2	$\left\{ \begin{pmatrix} z & & \\ & \omega z & \\ & & \omega^2 z \end{pmatrix} \middle z \in \mathbb{C} \right\}$	$\mathbf{S}^1$
$\tilde{\mathbf{D}}_4$	$\mathbf{D}_2$	$\mathbf{Z}_2$	2	$\left\{ \begin{pmatrix} z & & \\ & -z & \\ & & 0 \end{pmatrix} \middle z \in \mathbb{C} \right\}$	$\mathbf{S}^1$
$\mathbf{D}_2$	$\mathbf{D}_2$	1	4	$\left\{ \begin{pmatrix} z & & \\ & w & \\ & & -(w+z) \end{pmatrix} \middle w, z \in \mathbb{C} \right\}$	$\mathbf{D}_3 \times \mathbf{S}^1$
$\tilde{\mathbf{Z}}_4$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	4	$\left\{ \begin{pmatrix} z & w & 0 \\ w & -z & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle w, z \in \mathbb{C} \right\}$	$\mathbf{O}(2) \times \mathbf{S}^1$
$\tilde{\mathbf{Z}}_3$	1	$\mathbf{Z}_3$	4	$\left\{ \begin{pmatrix} z & -iz & w \\ -iz & -z & iw \\ w & -w & 0 \end{pmatrix} \middle w, z \in \mathbb{C} \right\}$	$\mathbf{SO}(2) \times \mathbf{S}^1$
$\mathbf{Z}_2$	1	$\mathbf{Z}_2$	4	$\left\{ \begin{pmatrix} 0 & 0 & z \\ 0 & 0 & w \\ z & w & 0 \end{pmatrix} \middle w, z \in \mathbb{C} \right\}$	$\mathbf{O}(2) \times \mathbf{S}^1$
$\mathbf{Z}_2$	$\mathbf{Z}_2$	1	6	$\left\{ \begin{pmatrix} z & v & 0 \\ v & w & 0 \\ 0 & 0 & -(z+w) \end{pmatrix} \middle z, w, v \in \mathbb{C} \right\}$	$\mathbf{O}(2) \times \mathbf{S}^1$
1	1	1	10	V_2	$\mathbf{SO}(3) \times \mathbf{S}^1$

subgroup of H , since $H/K \subset \mathbf{S}^1$ is Abelian. The commutator subgroups are given by:

H	$\mathbf{SO}(3)$	1	$\mathbf{O}$	$\mathbf{T}$	$\mathbf{O}(2)$	$\mathbf{D}_n (n \geq 2)$	$\mathbf{SO}(2)$	$\mathbf{Z}_n (n \geq 2)$	1
H'	$\mathbf{SO}(3)$	1	$\mathbf{T}$	$\mathbf{D}_2$	$\mathbf{SO}(2)$	$\mathbf{Z}_n (n \text{ odd})$ $\mathbf{Z}_{n/2} (n \text{ even})$	1	1	1

Combining the above observation that $H' \subset K \subset H$ with the list of possible groups K given in lemma 5.1, we readily obtain the first three columns of table 3. The remainder of the theorem can be established by straightforward calculations. $\square$

Figure 4. Partial ordering of conjugacy classes of isotropy subgroups of $\mathbf{SO}(3) \times \mathbf{S}^1$ -action on V_2 .

Remarks. (i) The actions of $N(\Sigma)/\Sigma$ on $\text{Fix } \Sigma$ for the interesting cases are as in table 5. Note that $(\gamma, \theta) \in N(\Sigma)$ if and only if $\gamma \in N(H)$ and $\varphi(h) = \varphi(\gamma^{-1}h\gamma)$ for all $h \in H$.

(ii) There are therefore five maximal isotropy subgroups (up to conjugacy), namely $\mathbf{O}(2)$, $\mathbf{D}_4$, $\widetilde{\mathbf{SO}}(2)^1$, $\widetilde{\mathbf{SO}}(2)^2$ and $\mathbb{T}$. The first two correspond to standing waves, the next two to rotating waves, and the last to a discrete rotating wave. Schematic pictures of the corresponding oscillations are given in figure 5.

We believe that these are the only periodic solutions near equilibrium in generic systems with this symmetry. However, there are many other relative periodic orbits, see Montaldi *et al* (1990b).

Table 5. Actions of $N(\Sigma)/\Sigma$ on $\text{Fix } \Sigma$.

Σ	$N(\Sigma)/\Sigma$	Description of action
$\mathbf{D}_2$	$\mathbf{D}_3 \times \mathbf{S}^1$	Usual in dim 4 (as in section 3)
$\tilde{\mathbf{Z}}_4$	$\mathbf{O}(2) \times \mathbf{S}^1$	Usual in dim 4 (as in section 3)
$\tilde{\mathbf{Z}}_2$	$\mathbf{O}(2) \times \mathbf{S}^1$	Usual in dim 4 (as in section 3)
$\tilde{\mathbf{Z}}_3$	$\mathbf{SO}(2) \times \mathbf{S}^1$	$(\theta, \psi) \cdot (z, w) = (\exp[i(2\theta + \psi)]z, \exp[i(\theta + \psi)]w)$
$\mathbf{Z}_2$	$\mathbf{O}(2) \times \mathbf{S}^1$	Trivial $\oplus$ (usual in dim 4)

5.2. Spectral stability

We now study the spectral stability of the five classes of periodic solution corresponding to two-dimensional fixed-point spaces. First we find the general form of $\mathbf{SO}(3) \times \mathbf{S}^1$ -invariant Hamiltonians on V_2 .

Lemma 5.3. Let $M \in V_2$. Up to degree 6, the real-valued invariants for the $\mathbf{SO}(3) \times \mathbf{S}^1$ -action on V_2 are linear combinations of:

- (a) quadratic: $N = \text{tr}(M\bar{M})$
- (b) quartic: $P = \text{tr}(M^2) \cdot \text{tr}(\bar{M}^2)$; $Q = \text{tr}(M^2\bar{M}^2)$; N^2
- (c) sextic: $R = \text{tr}(M^3) \cdot \text{tr}(\bar{M}^3)$; $S = \text{tr}(M^2\bar{M}) \cdot \text{tr}(M\bar{M}^2)$. N^3 , NP , NQ .

Proof. See example 2 in the appendix. □

We write the general (normalised) $\mathbf{SO}(3) \times \mathbf{S}^1$ -invariant Hamiltonian as

$$H = N + aN^2 + bP + cQ + dN^3 + eNP + fNQ + gR + hS + \mathcal{O}(8). \quad (5.1)$$

Figure 5. Schematic representation of oscillations of a sphere corresponding to the five maximal isotropy subgroups of $SO(3) \times S^1$.

By putting a general $\mathbf{SO}(3)$ -invariant Hamiltonian into Birkhoff normal form to order 6 we may write it in the form (5.1) with the $\mathbf{O}(8)$ term $\mathbf{SO}(3)$ -invariant but not necessarily $\mathbf{S}^1$ -invariant.

Let u be a complex parameter for the nonlinear normal mode, with $|u|$ constant along a periodic orbit. We will calculate expressions in the coefficients $a, b, \dots, h$ which in Birkhoff normal form give the Floquet exponents of the five classes of periodic orbits to leading order in $|u|$. In fact, it turns out that these expressions depend only on b, c, g and h . The spectral stabilities of the five classes can then be deduced.

Theorem 5.4. Let H be an $\mathbf{SO}(3)$ -invariant Hamiltonian on V_2 in exact Birkhoff normal form, whose coefficients to order 6 are as in (5.1). Then the Floquet exponents as functions of $|u|$ of the five classes of nonlinear normal modes are given in Table 6, where u is a complex parameter on the nonlinear normal mode as in section 3.

Table 6. Floquet exponents in Birkhoff normal form to leading order in $|u|$ for the five classes.

Σ	Isotypic components of reduced space Y	Corresponding Floquet exponents
$\mathbf{O}(2)$	$\left\{ \begin{pmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle a, b \in \mathbb{C} \right\}$	$\pm 2[-(6b + c)(g + h)]^{1/2} u ^3 + \mathbf{O}(4)$
	$\mathbb{R} \left\{ \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} i & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & i \end{pmatrix} \right\}$	$\pm \frac{1}{3} ic u ^2 + \mathbf{O}(4)$
$\widetilde{\mathbf{SO}}(2)^1$	$\mathbb{C} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & -i & 0 \end{pmatrix}$	$\pm i(4b + c) u ^2 + \mathbf{O}(4)$
	$\mathbb{C} \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\pm \frac{1}{2} ic u ^2 + \mathbf{O}(4)$
	$\mathbb{C} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	$\pm \frac{2}{3} ic u ^2 + \mathbf{O}(4)$
$\widetilde{\mathbf{SO}}(2)^2$	$\mathbb{C} \begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\pm ic u ^2 + \mathbf{O}(4)$
	$\mathbb{C} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & -i & 0 \end{pmatrix}$	$\pm 2i(2b + c) u ^2 + \mathbf{O}(4)$
$\tilde{\mathbf{D}}_4$	$\mathbb{C} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	$\pm 2[(6b + c)(g + h)]^{1/2} u ^3 + \mathbf{O}(4)$
$\tilde{\mathbb{T}}$	$\mathbb{C} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \bar{\omega} & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\pm \frac{2}{3} i(6b + c) u ^2 + \mathbf{O}(4)$

Corollary 5.5. Let H be an $\mathbf{SO}(3)$ -invariant Hamiltonian on V_2 , not necessarily in Birkhoff normal form, whose Birkhoff normal form to order 6 is as in (5.1). Then the spectral stability of the five classes of nonlinear normal modes is determined by table 7.

Table 7. Spectral stability for general $\mathbf{SO}(3)$ -invariant Hamiltonians on $\mathbb{C}^5$.

Σ	Condition for spectral stability
$\mathbf{O}(2)$	$(6b + c)(g + h) < 0$
$\widetilde{\mathbf{SO}}(2)^1$	always
$\widetilde{\mathbf{SO}}(2)^2$	always
$\tilde{\mathbf{D}}_4$	$(6b + c)(g + h) > 0$
$\tilde{\mathbb{T}}$	always

Remarks. (i) The $\widetilde{\mathbf{SO}}(2)^1$, $\widetilde{\mathbf{SO}}(2)^2$ and $\tilde{\mathbb{T}}$ periodic orbits are always spectrally stable because the reduced representations of those isotropy subgroups are cyclospectral, see Montaldi *et al* (1988).

(ii) We have been careful to say that the periodic orbits are spectrally stable rather than linearly stable because the Floquet matrix may (and generically does) have a nilpotent part. However, provided 1 is not an eigenvalue of N_u (the reduced Floquet operator), the image of the nilpotent part is along the group orbit, so these should be considered as being relatively linearly stable, see remark (ii) following corollary 4.4.

In order to prove theorem 5.4 we need an explicit form for the momentum map for $\mathbf{SO}(3) \times \mathbf{S}^1$. The result is well known but we give a proof for completeness.

Lemma 5.6. The momentum map for the action of $\mathbf{SO}(3) \times \mathbf{S}^1$ on V_2 is

$$\Phi(M) = (-\tfrac{1}{2}i[M, \bar{M}], \tfrac{1}{2} \operatorname{tr} M\bar{M}) \in \mathfrak{so}(3)^* \oplus \mathbb{R}.$$

Remark. The commutator $[\cdot, \cdot]$ of two symmetric matrices is antisymmetric, so that $-\tfrac{1}{2}i[M, \bar{M}] \in \mathfrak{so}(3)$; while $\mathfrak{so}(3)$ and $\mathfrak{so}(3)^*$ are identified by the invariant inner product $\langle M, N \rangle = \operatorname{tr} M\bar{N}$.

Proof. We already know the $\mathbb{R}$ component of the map, because the $\mathbf{S}^1$ -action is precisely the action associated with the Hamiltonian $H_2 = N = \operatorname{tr} M\bar{M}$. The $\mathfrak{so}(3)^*$ component requires a simple calculation, using the general fact that if V is a symplectic representation of G then for $\xi \in \mathfrak{g}$, $v \in V$, we have $\Phi_\xi(V) = \tfrac{1}{2}\omega(A_\xi v, v)$, where A_ξ is the image of ξ under the representation of $\mathfrak{g}$. In our case, $A_\xi \cdot M = [\xi, M]$, so

$$\Phi_\xi(M) = \langle \Phi(M), \xi \rangle = \tfrac{1}{2} \operatorname{Im} \operatorname{tr}([\xi, M], \bar{M}).$$

It is easy to see, using the standard properties of the trace, that this is equal to $\operatorname{tr}(-\tfrac{1}{2}i[M, \bar{M}]\xi)$, as required. $\square$

Proof of theorem 5.4. We begin by calculating the reduced representation Y of the five classes of periodic orbit (see subsection 2.6). It turns out that the Floquet exponents for three of these classes, namely $\mathbf{O}(2)$, $\tilde{\mathbf{D}}_4$ and $\tilde{\mathbb{T}}$, can then be

determined by looking at the single subspace $\text{Fix } \mathbf{D}_2$. The exponents for the two rotating waves, however, have to be calculated separately. Since the technique is similar to that of section 4, we do not actually perform the calculation for the two rotating waves.

$\mathbf{O}(2)$ standing wave: Let

$$u = z \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \in V_2 \quad z \neq 0,$$

a point in $\text{Fix } \mathbf{O}(2)$. Let $G = \mathbf{SO}(3) \times \mathbf{S}^1$. The tangent space to the G -orbit through u is

$$\mathbf{g} \cdot u = \mathbb{R} \cdot iu \oplus \mathbb{C} \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}.$$

and $\ker d\Phi_u$ is the symplectic orthogonal to this. Therefore

$$\ker d\Phi_u = \mathbb{R} \cdot iu \oplus \mathbb{C} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

whence, by subsection 2.6,

$$Y = \left\{ \begin{pmatrix} & & 0 \\ \mathbf{A} & & \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

where $\mathbf{A}$ is any complex symmetric traceless 2×2 matrix. Note that Y can be viewed as a subspace of V_2 , rather than just a quotient, since G is compact. The representation of $\mathbf{O}(2)$, on Y is the symplectification of the action on $\mathbb{R}^2$ with kernel $\{\pm I\}$. This symplectic representation is of real type (subsection 2.5), and $\mathfrak{sp}_{\mathbf{O}(2)}(Y) \cong \mathfrak{sp}(2; \mathbb{R})$. Since there are only two distinct Floquet exponents (of multiplicity two) it is sufficient to calculate the exponents along any symplectic subspace of Y which is invariant under the linearised dynamics.

$\tilde{\mathbf{D}}_4$ standing wave: Here

$$u = z \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{g} \cdot u = z \cdot \mathbb{R} \left\{ \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}$$

$$\ker d\Phi_u = \mathbf{g} \cdot u \oplus \mathbb{C} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Thus the reduced space is

$$Y = \mathbb{C} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

on which $\tilde{\mathbf{D}}_4$ acts through $\{\pm I\}$.

$\tilde{\mathbb{T}}$ discrete rotating wave: Here

$$u = z \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \bar{\omega} \end{pmatrix}.$$

$$\begin{aligned} \mathbf{g} \cdot u = z \cdot \mathbb{R} \left\{ \begin{pmatrix} i & 0 & 0 \\ 0 & i\omega & 0 \\ 0 & 0 & i\bar{\omega} \end{pmatrix}, \begin{pmatrix} 0 & 1-\omega & 0 \\ 1-\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 0 & 0 & 1-\bar{\omega} \\ 0 & 0 & 0 \\ 1-\bar{\omega} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega-\bar{\omega} \\ 0 & \omega-\bar{\omega} & 0 \end{pmatrix} \right\} \\ \ker d\Phi_u = \mathbf{g} \cdot u \oplus \mathbb{C} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \bar{\omega} \end{pmatrix} \end{aligned}$$

whence

$$Y = \mathbb{C} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \bar{\omega} \end{pmatrix}$$

on which the action of $\tilde{\mathbb{T}}$ factors through the usual one of $\mathbf{Z}_3$ on $\mathbb{R}^2$.

$\tilde{\mathbf{SO}}(2)^1$ rotating wave: With

$$u = z \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix},$$

we have

$$\mathbf{g} \cdot u = \mathbb{R} \left\{ iu, z \begin{pmatrix} 2 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, z \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2i & 0 \\ 0 & 0 & -2i \end{pmatrix} \right\}.$$

Then $\ker d\Phi_u = \mathbb{R} \cdot iu \oplus Y$, where

$$Y = \mathbb{R} \left\{ \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} i & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & i \end{pmatrix} \right\} \oplus \mathbb{C} \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & -i & 0 \end{pmatrix}, \begin{pmatrix} i & 1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

with character $\chi_1 \oplus \chi_2 \oplus \chi_3$, where χ_a refer to the action of $\mathbf{SO}(2)$ on $\mathbb{R}^2 \simeq \mathbb{C}$ as $\theta \cdot z = e^{ia\theta}z$.

$\widetilde{\text{SO}}(2)^2$ rotating wave: Here

$$u = z \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So

$$g \cdot u = \mathbb{R} \cdot iu \oplus \mathbb{C} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix}.$$

Then $\ker d\Phi_u = \mathbb{R} \cdot iu \oplus Y$, where

$$Y = \mathbb{C} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & -i & 0 \end{pmatrix} \right\}$$

with character $\chi_2 \oplus \chi_3 \oplus \chi_4$.

The space $\text{Fix}(\mathbf{D}_2)$, which is the four-dimensional subspace of V_2 consisting of diagonal matrices, is invariant under the dynamics of any $\text{SO}(3)$ -invariant Hamiltonian. Moreover, for the $\mathbf{O}(2)$, $\tilde{\mathbf{D}}_4$ and $\tilde{\mathbb{T}}$ nonlinear normal modes, the non-zero Floquet exponents can be calculated by looking at the system restricted to $\text{Fix}(\mathbf{D}_2)$. Indeed for $\tilde{\mathbf{D}}_4$ and $\tilde{\mathbb{T}}$ we have $Y \subset \text{Fix}(\mathbf{D}_2)$, whereas for $\mathbf{O}(2)$ both non-zero Floquet exponents occur on $Y \cap \text{Fix}(\mathbf{D}_2)$.

The restricted dynamics on $\text{Fix}(\mathbf{D}_2)$ is invariant under the normaliser of $\mathbf{D}_2$ in $\text{SO}(3) \times \mathbf{S}^1$, which is $\mathbb{T} \times \mathbf{S}^1$. Of course, $\mathbf{D}_2 \subset \mathbb{T}$ acts trivially, so the effective symmetry on $\text{Fix}(\mathbf{D}_2)$ is $\mathbb{T} \times \mathbf{S}^1 / \mathbf{D}_2 \simeq \mathbf{D}_3 \times \mathbf{S}^1$. We can therefore calculate the Floquet exponents for the $\mathbf{O}(2)$, $\tilde{\mathbf{D}}_4$ and $\tilde{\mathbb{T}}$ solutions by applying the results of section 4.

We now proceed with this calculation. We parametrise $\text{Fix}(\mathbf{D}_2)$ as

$$\frac{1}{\sqrt{3}} \begin{pmatrix} s+t & 0 & 0 \\ 0 & s\omega + t\bar{\omega} & 0 \\ 0 & 0 & s\bar{\omega} + t\omega \end{pmatrix}$$

with $s, t \in \mathbb{C}$. In this coordinate system, $\mathbf{D}_3 \times \mathbf{S}^1$ acts on $(s, t) \in \mathbb{C}^2$ as on (z_1, z_2) in section 4, which is easy to see since $\mathbf{D}_3 \simeq N(\mathbf{D}_2) / \mathbf{D}_2$ acts as permutations of the diagonal elements of the matrices in $\text{Fix}(\mathbf{D}_2)$. More abstractly, we may observe that $\mathbf{D}_3$ is the Weyl group of $\text{SO}(3)$, and some of the calculations performed here can be generalised to other Weyl groups.

In these coordinates, the restrictions to $\text{Fix}(\mathbf{D}_2)$ of the $\text{SO}(3) \times \mathbf{S}^1$ invariants in lemma 5.3 are

$$\begin{aligned} N &= |s|^2 + |t|^2 = \hat{N} \\ P &= 4 |s|^2 |t|^2 = 4\hat{P} \\ Q &= \frac{1}{3}(|s|^4 + 4 |s|^2 |t|^2 + |t|^4) = \frac{1}{3}(\hat{N}^2 + 2\hat{Q}) \\ R &= \frac{1}{3} |s^3 + t^3| = \frac{1}{3}(\hat{N}^3 - 3\hat{N}\hat{P} + \hat{S}) \\ S &= \frac{1}{3} |s^2\bar{t} + \bar{s}t^2|^2 = \frac{1}{3}(\hat{N}\hat{P} + \hat{S}). \end{aligned}$$

Here the hat is added to the $\mathbf{D}_3 \times \mathbf{S}^1$ invariants of section 4 to distinguish them from the $\text{SO}(3) \times \mathbf{S}^1$ invariants. We wish to write the restricted Hamiltonian $\hat{H}$ on $\text{Fix}(\mathbf{D}_2)$

as

$$\hat{H} = \hat{N} + \beta \hat{P} + \gamma \hat{S} + \dots$$

as in section 4. A simple substitution gives

$$\beta = 4b + \frac{2}{3}c \quad \gamma = \frac{1}{3}(g + h).$$

The Floquet exponents for the three nonlinear normal modes in V_2 that we are considering can be read off from table 2, once the symmetry of each restricted to $\text{Fix}(\mathbf{D}_2)$ is known. We derive this symmetry as follows. In general, if $\Sigma_1 \subset \Sigma_0$ are two isotropy subgroups of G , so that $\text{Fix}(\Sigma_0) \subset \text{Fix}(\Sigma_1)$, then the subgroup of $N(\Sigma_1)/\Sigma_1$ that leaves $\text{Fix}(\Sigma_0)$ invariant is $(\Sigma_0 \cap N(\Sigma_1))/\Sigma_1$. In this case $\Sigma_1 = \mathbf{D}_2$, and $\Sigma_0 = \mathbf{O}(2)$, $\tilde{\mathbf{D}}_4$ and $\tilde{\mathbb{T}}$, respectively. Thus $\text{Fix}(\mathbf{O}(2))$ has symmetry $\mathbf{Z}_2$, $\text{Fix}(\tilde{\mathbf{D}}_4)$ has symmetry $\tilde{\mathbf{Z}}_2$, and $\text{Fix}(\tilde{\mathbb{T}})$ has symmetry $\tilde{\mathbf{Z}}_3$. $\square$

Proof of corollary 5.5. This follows by theorem 3.1. $\square$

Remark. The trick of restricting H to $\text{Fix}(\Sigma)$ for suitable submaximal isotropy subgroups Σ , employed to advantage in the above proof, can often be used to pick out a number of Floquet exponents, and reduce their calculation to a smaller system invariant under a smaller group. However, ‘hidden symmetries’ in the sense of Golubitsky *et al* (1985), may mean that there are more restrictions on $H|_{\text{Fix}(\Sigma)}$ than just $N(\Sigma)/\Sigma$ equivariance, and this may create unexpected degeneracies. For example, in the above calculation we cannot assume without further argument that $H|_{\text{Fix}(\mathbf{D}_2)}$ is an arbitrary $\mathbf{D}_3$ -invariant Hamiltonian. The details show that this difficulty does not actually arise in this case.

6. The time-reversible 1:1 resonance with $\mathbf{Z}_2$ symmetry

In this section we continue the discussion of the time-reversible 1:1 resonance with $\mathbf{Z}_2$ symmetry begun in section 6 of Montaldi *et al* (1990a). In particular, we use theorem 3.4 to complete the proof of theorem 6.3 of that paper, restated below as theorem 6.1.

As in Montaldi *et al* (1990a), we consider Hamiltonians H on $V = \mathbb{C}^2$ that are invariant under the action

$$\kappa \cdot (z_1, z_2) = (z_2, z_1)$$

of $\mathbf{Z}_2 = \langle \kappa \rangle$, and are *time-reversible*, i.e. invariant under

$$\rho \cdot (z_1, z_2) = (-\bar{z}_1, -\bar{z}_2).$$

We suppose that $L = JD^2H(0)$ has double eigenvalues $\pm i$. Because L commutes with the $\mathbf{Z}_2$ action, it is necessarily semisimple, and so generates the $\mathbf{S}^1$ -action

$$\theta \cdot (z_1, z_2) = (e^{i\theta}z_1, e^{i\theta}z_2) \quad \theta \in \mathbb{R}/2\pi\mathbb{Z}.$$

These three group actions combine to give an action of $\mathbf{Z}_2 \times \mathbf{O}(2)$.

Suppose that H is in Birkhoff normal form to degree 4, so that its Taylor series at 0 is

$$H = H_2 + H_3 + H_4 + \hat{H}$$

where H_j is a $\mathbf{Z}_2 \times \mathbf{O}(2)$ -invariant homogeneous polynomial of degree j ($j = 2, 3, 4$), $\tilde{H}$ is $\mathbf{Z}_2$ -invariant, and $j^4 \tilde{H} = 0$. By Montaldi *et al* (1990a), proposition 6.1, the ring of $\mathbf{Z}_2 \times \mathbf{O}(2)$ -invariant functions is generated by

$$N = |z_1|^2 + |z_2|^2 \quad P = |z_1|^2 |z_2|^2 \quad Q = \operatorname{Re}(z_1 \bar{z}_2).$$

Therefore

$$H_2 = N$$

$$H_3 = 0$$

$$H_4 = \alpha_1 N^2 + \alpha_2 P + \beta_1 Q^2 + \beta_2 NQ$$

where $\alpha_j, \beta_j \in \mathbb{R}$. Note that Q does not appear in H_2 because of the assumption that L has repeated eigenvalues $\pm i$.

By proposition 6.2 of Montaldi *et al* (1990a) the possible symmetry types of nonlinear normal modes are those in figure 6, which also shows fixed-point spaces.

That paper also contains the first half of the proof of the following theorem on existence and stability of nonlinear normal modes with these symmetry types.

Figure 6. Isotropy lattice for the action of $\mathbf{Z}_2 \times \mathbf{O}(2)$.

Theorem 6.1. Let $H = H_2 + H_4 + \tilde{H}$ be a $\mathbf{Z}_2$ -invariant time-reversible Hamiltonian with $H_2 = N$, $H_4 = \alpha_1 N^2 + \alpha_2 P + \beta_1 Q^2 + \beta_2 NQ$ and $j^4 \tilde{H} = 0$. Suppose that

$$\begin{aligned} \alpha_2 \neq 0 \quad \beta_1 \neq 0 \quad \alpha_2 + \beta_1 \neq 0 \\ \beta_1 \pm \beta_2 \neq 0 \quad \text{and} \quad \alpha_2 + \beta_1 \pm \beta_2 \neq 0. \end{aligned}$$

Then the nonlinear normal modes near 0 are as follows.

(a) There is one nonlinear normal mode with symmetry group $\mathbf{Z}_2^k \times \mathbf{Z}_2^\rho$. It is stable if $(\beta_1 + \beta_2)(\alpha_2 + \beta_1 + \beta_2) > 0$ and unstable if the inequality is reversed.

(b) There is one nonlinear normal mode with symmetry group $\mathbf{Z}_2^{k+\pi} \times \mathbf{Z}_2^\rho$. It is stable if $(\beta_1 - \beta_2)(\alpha_2 + \beta_1 - \beta_2) > 0$ and unstable if the inequality is reversed.

(c) There are two nonlinear normal modes with symmetry group $\mathbf{Z}_2^\rho$ if $|\beta_2| < |\alpha_2 + \beta_1|$ and none otherwise. These bifurcate from the $\mathbf{Z}_2^k \times \mathbf{Z}_2^\rho$ mode when $\beta_2 = -(\alpha_2 + \beta_1)$, and from the $\mathbf{Z}_2^{k+\pi} \times \mathbf{Z}_2^\rho$ mode when $\beta_2 = (\alpha_2 + \beta_1)$. They are stable if $\alpha_2(\alpha_2 + \beta_1) > 0$ and unstable if the inequality is reversed.

(d) There are two nonlinear normal modes with symmetry group $\mathbf{Z}_2^{k+\rho}$ if $|\beta_2| < |\beta_1|$ and none otherwise. These bifurcate from the $\mathbf{Z}_2^k \times \mathbf{Z}_2^\rho$ mode when

$\beta_2 = -\beta_1$ and from the $\mathbf{Z}_2^{k\circ\pi} \times \mathbf{Z}_2^p$ mode when $\beta_2 = \beta_1$. They are stable if $\alpha_2\beta_1 > 0$ and unstable if the inequality is reversed.

(e) There are no nonlinear normal modes with trivial symmetry.

Proof. Let $\hat{H} = H_2 + H_4$ be the truncation of H at degree 4. Then the nonlinear normal modes of $\hat{H}$ are the $\mathbf{S}^1$ -orbits of critical points of $\hat{H}_\tau = 2\pi((1 + \tau)H_2 - \hat{H})$ or, equivalently, the $\mathbf{S}^1$ -orbits of solutions of $\hat{\phi}(z, \tau) = 0$ where $\hat{\phi}(z, \tau) = iD_{\bar{z}}\hat{H}_\tau(z): \mathbb{C}^2 \times \mathbb{R} \rightarrow \mathbb{C}^2$. These are listed in table 3 of Montaldi *et al* (1990a). There we proved that these correspond precisely to the nonlinear normal modes of the full Hamiltonians by checking the non-degeneracy condition (NDC2) of section 5 of that paper. Thus we defined $\psi: \mathbb{C}^2 \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}^2$ by

$$\psi(z, \tau, \lambda) = -\frac{1}{2\pi} [\hat{\phi}(z, \tau) - 2\pi\lambda D_{\bar{z}}H_2(z)]$$

and complexified this to obtain $\psi_{\mathbb{C}}: \mathbb{C}^4 \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}^4$. Note that as in section 6 of Montaldi *et al* (1990a) the complex notation is being used in two different ways, but this should cause no confusion. We showed that, with the non-degeneracy conditions of the theorem, on each solution of $\psi_{\mathbb{C}} = 0$ with $\lambda = 0$, $D_z\psi_{\mathbb{C}}$ has two non-zero eigenvalues, the maximum possible. On real solutions, i.e. solutions of $\hat{\phi}(z, \tau) = 0$, $D_z\psi_{\mathbb{C}}$ is the complexification of $iD_{\bar{z}}^2H_\tau$ (see the remarks preceding the proof of proposition 5.2 of Montaldi *et al* (1990a)) and so has the same eigenvalues. By corollary 2.4, the corresponding nonlinear normal mode of $\hat{H}$ is linearly stable if these eigenvalues are imaginary, and unstable if they are real. Hence the calculation of eigenvalues given in Montaldi *et al* (1990a), combined with theorem 3.4, gives a proof of the stability statements in theorem 6.1. $\square$

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Appendix. Invariant theory

We show how to use character theory and Haar integration to compute the dimension of the space of $\Gamma \times \mathbf{S}^1$ invariants that are homogeneous of degree k , giving explicit formulas when $k = 2, 4$ and 6 .

Let Γ act on V and let $\Gamma \times \mathbf{S}^1$ act on $V \otimes_{\mathbb{R}} \mathbb{C}$. Let $\chi(\gamma, \theta)$ be the corresponding character and let $\chi(\gamma)$ be the character of the Γ action. By Golubitsky and Stewart (1985) we have

$$\chi(\gamma, \theta) = 2 \cos \theta \cdot \chi(\gamma). \quad (\text{A.1})$$

The action of a group G on a space W induces an action of G on the k th symmetric tensor power $S^k W$, for each $k \geq 1$. Then the set of G -invariant homogeneous polynomials of degree k is the same as

$$\text{Fix}(G, S^k W).$$

So we must compute the character $\chi_k(g)$ of G acting on $S^k W$.

There is a recursive formula for χ_k :

$$k\chi_k = \sum_{i=0}^{k-1} p_{k-i}\chi_i$$

where $p_i(g) = \chi(g^i)$. See MacDonald (1979), p. 160, where h_i is used in place of our χ_i . By the trace formula, see, for example, Golubitsky and Stewart (1985), we have

$$\dim \text{Fix}(G, S^k W) = \int_G \chi_k(g)$$

where $\int_G$ is the normalised Haar integral. For $k \leq 6$ we have the following explicit results:

$$\chi_2(g) = \frac{1}{2}[\chi(g)^2 + \chi(g^2)]$$

$$\chi_3(g) = \frac{1}{6}[\chi(g)^3 + 3\chi(g)\chi(g^2) + 2\chi(g^3)]$$

$$\chi_4(g) = (1/4!)[\chi(g)^4 + 3\chi(g)^2\chi(g^2) + 6\chi(g)\chi(g^3) + 6\chi(g^4)]$$

$$\chi_5(g) = (1/5!)[\chi(g)^5 + 10\chi(g)^3\chi(g^2) + 20\chi(g)^2\chi(g^3) + 15\chi(g)\chi(g^2)^2 \\ + 30\chi(g)\chi(g^4) + 20\chi(g^2)\chi(g^3) + 24\chi(g^5)]$$

$$\chi_6(g) = (1/6!)[\chi(g)^6 + 15\chi(g)^4\chi(g^2) + 40\chi(g)^3\chi(g^3) + 45\chi(g)^2\chi(g^2)^2 \\ + 90\chi(g)^2\chi(g^4) + 120\chi(g)\chi(g^2)\chi(g^3) + 144\chi(g)\chi(g^5) \\ + 90\chi(g)^2\chi(g^4) + 15\chi(g^2)^3 + 40\chi(g^3)^2 + 120\chi(g^6)].$$

We apply these formulae to $G = \Gamma \times S^1$ and use (A.1). Thus when $k = 2$ we have $g = (\gamma, \theta)$, $\chi(g) = \chi(\gamma, \theta) = 2(\cos \theta)\chi(\gamma)$, and

$$\dim \text{Fix}(\Gamma \times S^1, S^2(V \otimes \mathbb{C})) = \int_G \chi_2(g) = \frac{1}{2} \int_{S^1} 4 \cos^2 \theta \int_{\Gamma} \chi(\gamma)^2 + \frac{1}{2} \int_{S^1} 2 \cos 2\theta \int_{\Gamma} \chi(\gamma^2).$$

The second term vanishes and we are left with $\int_{\Gamma} \chi(\gamma)^2$. Similarly, when $k = 4$ we compute

$$\dim \text{Fix}(\Gamma \times S^1, S^4(V \otimes \mathbb{C})) = \frac{1}{4!} \left(2^4 \int_{S^1} \cos^4 \theta \int_{\Gamma} \chi(\gamma)^4 \right. \\ \left. + 3 \times 2^2 \int_{S^1} \cos^2 2\theta \int_{\Gamma} \chi(\gamma^2)^2 + 6 \times 2^3 \int_{S^1} \cos^2 \theta \cos 2\theta \int_{\Gamma} \chi(\gamma)^2 \chi(\gamma^2) \right. \\ \left. + 8 \times 2^2 \int_{S^1} \cos \theta \cos 3\theta \int_{\Gamma} \chi(\gamma) \chi(\gamma^3) + 6 \times 2 \int_{S^1} \cos 4\theta \int_{\Gamma} \chi(\gamma^4) \right).$$

The last two integrals over S^1 vanish, and we are left with

$$\frac{1}{4!} \left(16 \times \frac{3}{8} \int_{\Gamma} \chi(\gamma)^4 + 12 \times \frac{1}{2} \int_{\Gamma} \chi(\gamma^2)^2 + 48 \times \frac{1}{4} \int_{\Gamma} \chi(\gamma)^2 \chi(\gamma^2) \right) = \frac{1}{4} \int_{\Gamma} [\chi(\gamma)^2 + \chi(\gamma^2)]^2.$$

The case $k = 6$ is similar but more complicated. We tabulate the results in table 8.

Example 1. $D_3 \times S^1$ acting on $\mathbb{C}^2$. The conjugacy classes of D_3 are $\{1\}$, $\{\xi, \xi^2\}$, $\{\kappa, \kappa\xi, \kappa\xi^2\}$ where $\xi = R_{2\pi/3}$ and κ is the flip. We have the values presented in table 9. The number of quadratic invariants is

$$\int_{\Gamma} \chi(\gamma)^2 = \frac{1}{6}(4 + 2.1) = 1$$

Table 8. Dimension of space of invariants of degree k for $\Gamma \times \mathbf{S}^1$ on $V \otimes \mathbb{C}$.

k	$\dim \text{Fix}(\Gamma \times \mathbf{S}^1, S^k(V \otimes \mathbb{C}))$
2	$\int_{\Gamma} \chi(\gamma)^2$
4	$\frac{1}{4} \int_{\Gamma} [\chi(\gamma)^2 + \chi(\gamma^2)]^2$
6	$\frac{1}{36} \int_{\Gamma} [\chi(\gamma)^6 + 6\chi(\gamma)^4\chi(\gamma^2) + 4\chi(\gamma)^3\chi(\gamma^3) + 9\chi(\gamma)^2\chi(\gamma^2)^2 + 12\chi(\gamma)\chi(\gamma^2)\chi(\gamma^3) + 4\chi(\gamma^3)^2]$

Table 9. Conjugacy classes of $\mathbf{D}_3$ where $\xi = R_{2\pi/3}$ and κ is the flip.

γ	$\{1\}$	$\{\xi, \xi^2\}$	$\{\kappa, \kappa\xi, \kappa\xi^2\}$
$\chi(\gamma)$	2	-1	0
$\chi(\gamma^2)$	2	-1	2
$\chi(\gamma^3)$	2	2	0

recalling that $\int_{\Gamma} = (1/|\Gamma|) \sum_{\Gamma}$ when Γ is finite. The number of quartic invariants is $\frac{1}{24}[(4+2)^2 + 3(0+2)^2 + 2(1-1)^2] = 2$.

The number of sextic invariants is

$$\frac{1}{36} \times \frac{1}{6} [(2^6 + 2 \times 1) + 6(2^4 \times 2 + 0 + 2(-1)) + 4(16 + 0 + 2(-2)) + 9(16 + 0 + 2) + 12(8 + 0 + 2 \times 2) + 4(4 + 0 + 2 \times 4)] = 3.$$

We know from section 4 that these numbers are correct; the relevant bases are $\{N\}$, $\{N^2, P\}$ and $\{N^3, NP, S\}$.

Example 2. $\mathbf{SO}(3) \times \mathbf{S}^1$ acting on V_2 . The conjugacy classes of Γ are parametrized by $\varphi \in [0, \pi]$, representing all rotations through φ . If ρ is a class function, constant on conjugacy classes, then the normalized Haar integral of ρ is

$$\frac{1}{\pi} \int_0^{\pi} \rho(\varphi)(1 - \cos \varphi) d\varphi$$

by Wigner (1959), page 156. Characters are class functions; and if χ is the character on U_2^+ then

$$\chi(\varphi) = 1 + 2 \cos \varphi + 2 \cos 2\varphi$$

by the Cartan decomposition, see Golubitsky *et al* (1988), p. 110. We have,

$$\int_0^{\pi} \cos m\varphi \cos n\varphi d\varphi = \begin{cases} 0 & m \neq n \\ \pi & m = n = 0 \\ \frac{\pi}{2} & m = n > 0. \end{cases}$$

This greatly simplifies the computations, which yield the following results:

Quadratics $\frac{1}{\pi} \int_0^{\pi} (1 + 2 \cos \varphi + 2 \cos 2\varphi)^2 (1 - \cos \varphi) d\varphi = 1$

Quartics $\frac{1}{\pi} \int_0^{\pi} [(1 + 2 \cos \varphi + 2 \cos \varphi)^2 + (1 + 2 \cos 2\varphi + 2 \cos 4\varphi)]^2 (1 - \cos \varphi) d\varphi = 3$

Sextics A lengthy computation gives the result 5.

Remark. In practice there is a useful check on the computation: the integrals can be evaluated numerically (say using Simpson's rule). Since we know the answer is an integer, rounding errors can be ignored. On a desktop PC such a calculation takes about a minute for the sextic invariants of $\mathbf{SO}(3) \times \mathbf{S}^1$.

The following functions of 3×3 complex trace-zero symmetric matrices M are clearly $\mathbf{SO}(3) \times \mathbf{S}^1$ -invariant and linearly independent:

Quadratic	$N = \operatorname{tr} M \bar{M}$
Quartic	$P = (\operatorname{tr} M^2)(\operatorname{tr} \bar{M}^2)$
	$Q = \operatorname{tr} M^2 \bar{M}^2$
	N^2
Sextic	$R = (\operatorname{tr} M^3)(\operatorname{tr} \bar{M}^3)$
	$S = (\operatorname{tr} M^2 \bar{M})(\operatorname{tr} M \bar{M}^2)$
	$N^3, NP, NQ.$

Hence we have found all $\mathbf{SO}(3) \times \mathbf{S}^1$ invariants up to degree 6.

Alternatively, the same results can be obtained by applying results of Spencer and Rivlin (1958) on $\mathbf{SO}(3)$ invariants of pairs (A, B) of symmetric real matrices. Take $A = \operatorname{Re} M$, $B = \operatorname{Im} M$, and then symmetrise over the $\mathbf{S}^1$ action. See Haaf *et al* (1990).

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