Thermal coherent states in the Bargmann representation

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The thermal coherent states considered previously by the present authors represent an alternative mixed-state generalization of the usual pure-state coherent states. They describe displaced harmonic oscillators in thermodynamic equilibrium with a heat bath at nonzero temperature. We show how they provide a “random” (or “thermal” or “noisy”) basis on a quantum-mechanical Hilbert space $\mathcal{H}$. Their usefulness rests on the fact that the corresponding statistical density operator provides a probability operator measure on $\mathcal{H}$. We thereby show how the thermal coherent states permit a generalization to nonzero temperatures of the well-known P and Q representations of operators in $\mathcal{H}$. Particular emphasis here is placed on imbedding the formulation in the Bargmann or holomorphic representation of $\mathcal{H}$. We examine the corresponding Bargmann representations of both state vectors and operators, and show how the former relate to the usual position and momentum representations and the latter to the usual P , Q , and W (or Weyl) representations. A particularly important and unexpected result is that the present temperature-dependent generalized P and Q representations are the analytic continuations to negative temperatures of each other. The usual Q and P representations thus represent the limits as the temperature approaches zero along the positive and negative real axes, respectively, of the enlarged generalized Q representation, suitably analytically continued to negative temperatures. We discuss the possible physical applications of the present thermal coherent states to both quantum optics situations involving coherent signals in the presence of thermal noise and to signal and image processing.

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I. INTRODUCTION

Coherent states have played an important theoretical role in both the conceptual and practical development of many areas of physics underpinned by quantum mechanics. Since their explicit introduction in quantum optics by Glauber [1], their range of applications has grown enormously [2]. At the mathematical level they provide a nonorthonormal and overcomplete basis in the Hilbert space $\mathcal{H}$ of the associated linear harmonic oscillator. Unlike their orthonormal basis counterparts, which are the eigenvectors of a self-adjoint operator, the coherent states form a continuous manifold over $\mathcal{H}$. Like their orthonormal counterparts, however, the coherent states also permit a resolution of the identity operator in $\mathcal{H}$, and it is this property that makes them of practical use, since arbitrary states in $\mathcal{H}$ can thereby be expanded in terms of them.

In a previous publication [3] we have considered a generalization of the ordinary Glauber coherent states into the so-called *coherent mixed states* or *thermal coherent states*. They describe displaced harmonic oscillators in thermodynamic equilibrium with a heat bath at some finite nonzero temperature T . An equivalent alternative description is in terms of a superposition of coherent states and thermal noise. This latter description is very closely related, for example, to the Glauber-Lachs for-

malism [4–6] in quantum optics for admixtures of incoherent (or, loosely, thermal) and coherent radiation.

In contrast to the various types of coherent states generally studied in the literature, which are pure states, our coherent mixed states are, as the name implies, mixed states in the usual quantum-mechanical sense. They are described by a thermal density operator, and they reduce to pure states only in the limiting case of zero temperature, $T=0$. They can be considered as a “noisy” or “random” basis in the Hilbert space $\mathcal{H}$. In this context we prove that there exists a resolution of the identity for these mixed states. In turn, this makes possible an expansion of an arbitrary state in $\mathcal{H}$ in the coherent mixed-state thermal basis. Furthermore, we show how the usual P and Q representations of operators in $\mathcal{H}$ in terms of pure coherent states may be extended to nonzero temperatures with the corresponding mixed thermal coherent states. In the above context it is interesting to note that other authors have also studied mixed states as overcomplete bases in a Hilbert space, although usually from a rather different vantage point to our own. For example, several authors [7–9] have given detailed analyses of their group-theoretical structure, which nicely complement our own work.

One of the main aims of the present paper is to expand our earlier work, particularly by expressing the analysis within the Bargmann Hilbert space representation [10].

This representation makes possible the exploitation of many of the powerful results of the theory of analytic functions in the complex plane, within the quantum-mechanical context. The coherent mixed states are thus defined in the Bargmann representation in Sec. II, where we also derive certain useful results for this representation. For example, we derive various transformations which connect the Bargmann representation of an arbitrary state in $\mathcal{H}$ with its more usual Schrödinger wave function (or x representation) and its Fourier transform, the momentum space wave function (or p representation). We also explain in Sec. II both how an arbitrary operator in $\mathcal{H}$ can be expressed in the Bargmann representation either in a differential form or in an integral kernel form, and how the present mixed (thermal) coherent states can be considered as a ("noisy" or "random") basis in $\mathcal{H}$. The various relations between the Bargmann integral representation of an operator in $\mathcal{H}$ and its better known P , Q , and W (or Weyl) representations are first discussed in Sec. III before we introduce in Sec. IV the generalizations of the P and Q representations to nonzero temperatures that stem from the extension of the pure coherent states to the thermal mixed coherent states. We examine various relations between these generalized (nonzero temperature) P and Q representations, some of which generalize known results [11–13] for the usual (zero-temperature) representations. We conclude in Sec. V with a discussion of our results, where we also indicate possible applications of them.

II. DISPLACED OSCILLATOR AT NONZERO TEMPERATURE IN THE BARGMANN REPRESENTATION

In terms of the creation and destruction operators $a^\dagger$ and a of the linear harmonic oscillator, which satisfy the basic canonical commutation relation, $[a, a^\dagger] = I$, we first introduce the usual Glauber coherent states $|z\rangle$ in terms of the displacement operator $D(z)$ as follows:

$$|z\rangle \equiv D(z)|0\rangle, \quad D(z) \equiv \exp(za^\dagger - z^*a), \quad (2.1)$$

where $|0\rangle$ is the vacuum state, $a|0\rangle = 0$. In terms of the normalized eigenstates $|n\rangle$ of the number operator $a^\dagger a$,

$$a^\dagger a|n\rangle = n|n\rangle, \quad |n\rangle = (n!)^{-1/2}(a^\dagger)^n|0\rangle, \quad (2.2)$$

the coherent states are easily seen to be given as

$$|z\rangle = \exp(-\frac{1}{2}|z|^2) \sum_{n=0}^{\infty} (n!)^{-1/2} z^n |n\rangle, \quad (2.3)$$

and to satisfy the fundamental relations,

$$a|z\rangle = z|z\rangle, \quad \langle z|a^\dagger = z^*\langle z|, \quad (2.4)$$

where z^* is the complex conjugate of the complex parameter z . From Eq. (2.3) it is trivial to derive the overlap between any two coherent states as

$$\langle z|z'\rangle = \exp(-\frac{1}{2}|z|^2 - \frac{1}{2}|z'|^2 + z^*z'). \quad (2.5)$$

Furthermore, by taking matrix elements between arbitrary members $\langle n|$ and $|m\rangle$ of the complete number-state basis, one may readily show that a resolution of the

identity operator I can be given in terms of the (over) complete set of coherent states as

$$I = \frac{1}{\pi} \int d^2z |z\rangle\langle z|, \quad (2.6)$$

$$d^2z \equiv d(\operatorname{Re} z) d(\operatorname{Im} z) = (2i)^{-1} dz dz^*,$$

where the integral runs over the entire complex z plane. In terms of the operators a and $a^\dagger$, the position operator $\hat{x}(\rightarrow x)$ and the canonically conjugate momentum operator $\hat{p}(\rightarrow -id/dx)$ are given by

$$\hat{x} = 2^{-1/2}(a^\dagger + a), \quad \hat{p} = 2^{-1/2}i(a^\dagger - a). \quad (2.7)$$

We also define position and momentum eigenfunctions $|x\rangle$ and $|p\rangle$, respectively, as usual,

$$\hat{x}|x\rangle = x|x\rangle, \quad \hat{p}|p\rangle = p|p\rangle, \quad (2.8)$$

with normalizations as implied by the following resolutions of the identity operator:

$$I = \int_{-\infty}^{\infty} dx |x\rangle\langle x|, \quad I = \int_{-\infty}^{\infty} dp |p\rangle\langle p|. \quad (2.9)$$

We now consider arbitrary normalized ket states $|f\rangle$ and $|g\rangle$ in the Hilbert space $\mathcal{H}$ spanned by the oscillator states $|n\rangle$; $n = 0, 1, 2, \dots$, defined as

$$|f\rangle \equiv \sum_{n=0}^{\infty} f_n |n\rangle, \quad \sum_{n=0}^{\infty} |f_n|^2 = 1, \quad (2.10)$$

$$|g\rangle \equiv \sum_{n=0}^{\infty} g_n |n\rangle, \quad \sum_{n=0}^{\infty} |g_n|^2 = 1.$$

The bra state $\langle f^*|$ is defined analogously as

$$\langle f^*| \equiv (|f\rangle)^\dagger = \sum_{n=0}^{\infty} f_n^* \langle n|. \quad (2.11)$$

In the Bargmann (or holomorphic) representation, the ket state $|g\rangle$ is represented by the analytic function $F_B(|g\rangle; z) \equiv g_B(z)$ defined as

$$|g\rangle \rightarrow g_B(z) \equiv F_B(|g\rangle; z) \equiv \exp(\frac{1}{2}|z|^2) \langle z^*|g\rangle$$

$$= \sum_{n=0}^{\infty} (n!)^{-1/2} g_n z^n, \quad (2.12a)$$

and the bra state $\langle f^*|$ is similarly represented by the analytic function $f_B(z)$,

$$\exp(\frac{1}{2}|z|^2) \langle f^*|z\rangle = \sum_{n=0}^{\infty} (n!)^{-1/2} f_n^* z^n = f_B(z), \quad (2.12b)$$

with $\langle f^*|z\rangle = \langle z|f^*\rangle^* = \langle z^*|f\rangle$. As simple examples we consider the coherent state $|A\rangle$ and the number eigenstate $|n\rangle$, which in the Bargmann representation are represented by the respective analytic functions,

$$|A\rangle \rightarrow F_B(|A\rangle; z) = \exp(-\frac{1}{2}|A|^2 + Az), \quad (2.13a)$$

$$|n\rangle \rightarrow F_B(|n\rangle; z) = (n!)^{-1/2} z^n. \quad (2.13b)$$

The scalar product of two states $\langle f^*|$ and $|g\rangle$ is immediately given as

$$\langle f^* | g \rangle = \sum_{n=0}^{\infty} f_n g_n, \quad (2.14)$$

where the assumption that the states are normalizable assures that the sum is convergent. It is also immediately clear that equivalent ways of writing Eq. (2.14) are the alternative forms,

$$\langle f^* | g \rangle = f(d/dz)g(z)|_{z=0} = g(d/dz)f(z)|_{z=0}. \quad (2.15)$$

Finally, we may also make use either of the completeness relation of Eq. (2.6) or of the following elementary integral over the complex plane,

$$\frac{1}{\pi} \int d^2z z^m z^{*n} \exp(-|z|^2) = n! \delta_{mn}, \quad (2.16)$$

to write the scalar product in the forms

$$\begin{aligned} \langle f^* | g \rangle &= \frac{1}{\pi} \int d^2z e^{-|z|^2} f_B(z) g_B(z^*) \\ &= \frac{1}{\pi} \int d^2z e^{-|z|^2} f_B(z^*) g_B(z). \end{aligned} \quad (2.17)$$

The normalizability of the states $\langle f^* |$ and $|g\rangle$ implies, via Eq. (2.17), that both (holomorphic) functions $f_B(z)$ and $g_B(z)$ must be entire functions of order $\rho \leq 2$ (and type $\tau \leq \frac{1}{2}$ if $\rho=2$). In this case, all three of the above expressions given by Eqs. (2.14), (2.15), and (2.17) yield convergent and identical results. The space of all such functions is denoted as the Bargmann Hilbert space [10], and the corresponding representation of the wave functions as the holomorphic or Bargmann representation. In the Bargmann space the algebra of the operators $a^\dagger$ and a is clearly represented directly by the algebra of the complex variable z and its derivative,

$$a^\dagger \rightarrow z, \quad a \rightarrow \frac{d}{dz}. \quad (2.18)$$

By inserting a complete set of either position eigenstates or momentum eigenstates from Eq. (2.9) into the overlap $\langle z^* | f \rangle$, it is not difficult to derive the Fourier-like mappings,

$$f_B(z) = \pi^{-1/4} \exp(-\frac{1}{2}z^2) \int_{-\infty}^{\infty} dx \exp(\sqrt{2}zx - \frac{1}{2}x^2) \times f_x(x), \quad (2.19a)$$

$$f_B(z) = \pi^{-1/4} \exp(\frac{1}{2}z^2) \int_{-\infty}^{\infty} dp \exp(\sqrt{2}izp - \frac{1}{2}p^2) \times f_p(p), \quad (2.19b)$$

between the holomorphic representation $f_B(z)$ of an arbitrary state $|f\rangle$ in $\mathcal{H}$ and either its Schrödinger (or coordinate-space) representation $f_x(x) \equiv \langle x | f \rangle$ or its Fourier-inverse momentum-space representation, $f_p(p) \equiv \langle p | f \rangle$. Similar inverse relations can easily be found by inserting a complete set of coherent states as in Eq. (2.6) into the overlaps $\langle x | f \rangle$ or $\langle p | f \rangle$ respectively,

$$\begin{aligned} f_x(x) &= \pi^{-5/4} \exp(-\frac{1}{2}x^2) \int d^2z \exp(-|z|^2 - \frac{1}{2}z^{*2} \\ &\quad + \sqrt{2}xz^*) f_B(z), \end{aligned} \quad (2.20a)$$

$$\begin{aligned} f_p(p) &= \pi^{-5/4} \exp(-\frac{1}{2}p^2) \int d^2z \exp(-|z|^2 + \frac{1}{2}z^{*2} \\ &\quad - \sqrt{2}ipz^*) f_B(z). \end{aligned} \quad (2.20b)$$

Alternative new inverses can also be obtained rather easily from Eqs. (2.19a and b) directly, in the rather revealing forms

$$f_x(z_R) = \pi^{-3/4} \exp(-\frac{1}{2}z_R^2) \int_{-\infty}^{\infty} dz_I \exp(-z_I^2) f_B(\sqrt{2}z), \quad (2.21a)$$

$$f_p(z_I) = \pi^{-3/4} \exp(-\frac{1}{2}z_I^2) \int_{-\infty}^{\infty} dz_R \exp(-z_R^2) f_B(\sqrt{2}z^*), \quad (2.21b)$$

where $z_R \equiv \text{Re}z$ and $z_I \equiv \text{Im}z$.

We turn now to the representation of operators in the Bargmann Hilbert space. An arbitrary operator Θ ,

$$\Theta = \sum_{m,n=0}^{\infty} \Theta_{mn} |m\rangle \langle n|, \quad (2.22)$$

can be represented as an analytic function of two variables, defined as follows:

$$\begin{aligned} \Theta \rightarrow \Theta_B(z_1, z_2^*) &\equiv B(\Theta; z_1, z_2^*) \\ &\equiv \exp(\frac{1}{2}|z_1|^2 + \frac{1}{2}|z_2|^2) \langle z_1^* | \Theta | z_2^* \rangle \\ &= \sum_{m,n=0}^{\infty} (m!n!)^{-1/2} \Theta_{mn} z_1^m z_2^{*n}. \end{aligned} \quad (2.23)$$

We refer to this as the Bargmann or B representation of Θ . We note that this representation of an arbitrary bounded operator was first introduced by Glauber [4] into quantum optics, where it is known in the special case of the density operator as the Glauber R function. More generally, the B representation provides an integral representation of an operator in the Bargmann formulation as an alternative to the more usual differential representation,

$$\Theta(a^\dagger, a) |f\rangle \rightarrow \Theta(z, d/dz) f_B(z), \quad (2.24)$$

where $|f\rangle$ is an arbitrary state in $\mathcal{H}$ with Bargmann representation $f_B(z)$. Thus the operator Θ is now represented as a kernel of an integral so that, by insertion of a complete set of coherent states $\{|z'^*\rangle\}$ as in Eq. (2.6), the action of Θ on the arbitrary state $|f\rangle$ of Eq. (2.10) can, by use of Eq. (2.12a), be written in the integral form,

$$\Theta |f\rangle \rightarrow \frac{1}{\pi} \int d^2z' e^{-|z'|^2} B(\Theta; z, z'^*) f_B(z'). \quad (2.25)$$

Furthermore, the B representation of the product of two operators is easily seen to be given by

$$\begin{aligned} B(\Theta_1 \Theta_2; z_1, z_2^*) &= \frac{1}{\pi} \int d^2z_3 e^{-|z_3|^2} B(\Theta_1; z_1, z_3^*) \\ &\quad \times B(\Theta_2; z_3, z_2^*). \end{aligned} \quad (2.26)$$

We note that for a certain class of operators their corresponding Bargmann representations $\Theta_B(z_1, z_2^*)$ in Eq.

(2.23) define entire functions of two complex variables z_1 and z_2^* . It is clear that such a function is uniquely determined by its diagonal component $\Theta_B(z_1, z_1^*)$ alone, since each monomial $z_1^m z_1^{*n}$ in the sum may be uniquely extended to $z_1^m z_2^{*n}$ by writing $z_1 = re^{i\phi}$ in the usual polar form and by isolating the separate dependence on r and on ϕ . This ability to employ analytic continuation techniques lies at the heart of the power of the Bargmann representation. Klauder [14] has given a rather detailed discussion concerning the universality of the class of the operators Θ which are uniquely specified by their diagonal B representations $\Theta_B(z, z^*)$.

The B representations of the fundamental creation and destruction operators themselves are easily seen to be given by

$$\begin{aligned} B(a^\dagger; z_1, z_2^*) &= z_1 \exp(z_1 z_2^*), \\ B(a; z_1, z_2^*) &= z_2^* \exp(z_1 z_2^*). \end{aligned} \quad (2.27)$$

Furthermore, these representations are readily seen from Eq. (2.25) to be consistent with those of Eq. (2.18) by making use of the relations

$$\begin{aligned} \frac{1}{\pi} \int d^2 z' e^{-|z'|^2} z e^{zz'^*} f_B(z') &= z f_B(z), \\ \frac{1}{\pi} \int d^2 z' e^{-|z'|^2} z'^* e^{zz'^*} f_B(z') &= \frac{d}{dz} f_B(z), \end{aligned} \quad (2.28)$$

which are valid for an arbitrary entire function $f_B(z)$. Equations (2.28) may be proved either directly, by using the fact that $f_B(z)$ is an entire function so that the Cauchy theorem yields

$$\int \frac{dz'}{2\pi i} \frac{f_B(z')}{z' - z} = f_B(z), \quad (2.29)$$

where the integral is taken around some suitable contour enclosing the point z in an anticlockwise direction, or by expanding in powers of z' or z'^* , respectively, both $f_B(z')$ as in Eq. (2.12a) and $\exp(zz'^*)$, and by making use of the fundamental integral of Eq. (2.16) and the fact that the order of the integration and the summation can be interchanged provided $f_B(z)$ is a holomorphic function.

We note from Eqs. (2.16) and (2.23) that the trace of the operator Θ can be expressed in terms of its diagonal B representation as

$$\text{Tr} \Theta = \frac{1}{\pi} \int d^2 z e^{-|z|^2} B(\Theta; z, z^*), \quad (2.30)$$

and that the trace of the product of two operators can be similarly expressed as

$$\begin{aligned} \text{Tr}(\Theta_1 \Theta_2) &= \frac{1}{\pi^2} \int d^2 z \int d^2 z' e^{-|z|^2 - |z'|^2} B(\Theta_1; z, z'^*) \\ &\quad \times B(\Theta_2; z', z^*). \end{aligned} \quad (2.31)$$

For later use we also note that the unit operator I and the displacement operator $D(\xi)$ from Eq. (2.1) have B representations given respectively by

$$B(I; z_1, z_2^*) = \exp(z_1 z_2^*), \quad (2.32a)$$

$$B(D(\xi); z_1, z_2^*) = \exp(-\frac{1}{2}|\xi|^2 + \xi z_1 - \xi^* z_2^* + z_1 z_2^*). \quad (2.32b)$$

Let us now consider an arbitrary density operator ρ with (real) eigenvalues p_n , eigenstates $|e^{(n)}\rangle$, and matrix elements $\rho_{mn} \equiv \langle m | \rho | n \rangle$ with respect to the eigenstates $|n\rangle$ of the number operator $a^\dagger a$,

$$\begin{aligned} \rho &= \sum_{n=0}^{\infty} p_n |e^{(n)}\rangle \langle e^{(n)}| = \sum_{m,n=0}^{\infty} \rho_{mn} |m\rangle \langle n|, \\ 0 \leq p_n &\leq 1, \quad \forall n, \quad \sum_{n=0}^{\infty} p_n = 1. \end{aligned} \quad (2.33)$$

Its B representation is given by

$$\begin{aligned} B(\rho; z_1, z_2^*) &\equiv \rho_B(z_1, z_2^*) = \exp(\frac{1}{2}|z_1|^2 + \frac{1}{2}|z_2|^2) \langle z_1^* | \rho | z_2^* \rangle \\ &= \sum_{n=0}^{\infty} p_n e_B^{(n)}(z_1) [e_B^{(n)}(z_2)]^* \\ &= \sum_{m,n=0}^{\infty} (m!n!)^{-1/2} \rho_{mn} z_1^m z_2^{*n}. \end{aligned} \quad (2.34)$$

For example, the thermodynamic density operator for a canonical ensemble of systems with Hamiltonian H in thermal equilibrium at temperature $T \equiv (k_B \beta)^{-1}$, with k_B the Boltzmann constant, is

$$\rho = (\text{Tr} e^{-\beta H})^{-1} e^{-\beta H}. \quad (2.35)$$

The system of particular importance for our purposes is, as we shall see in Sec. III, the displaced linear harmonic oscillator with Hamiltonian,

$$H_\xi \equiv D(\xi) a^\dagger a D^\dagger(\xi) = (a^\dagger - \xi^* I)(a - \xi I), \quad (2.36)$$

in units where here, and henceforth, $\hbar = c = k_B = 1$ and $\omega = 1$, where ω is the fundamental angular frequency. The thermodynamic density operator for this system is given by

$$\rho(\xi; \beta) = (1 - e^{-\beta}) \exp(-\beta H_\xi), \quad (2.37)$$

which we may readily prove reduces in the zero-temperature limit to the projector onto the coherent state $|\xi\rangle$,

$$\lim_{\beta \rightarrow \infty} \rho(\xi; \beta) = |\xi\rangle \langle \xi|. \quad (2.38)$$

The B representation of this density operator can rather readily be shown [3] to be given by

$$\rho_B(\xi; \beta; z_1, z_2^*) = (1 - e^{-\beta}) \exp[(1 - e^{-\beta})(\xi z_1 + \xi^* z_2^* - |\xi|^2) + e^{-\beta} z_1 z_2^*]. \quad (2.39)$$

The proposal that we introduced in our earlier work [3] is to extend the above ideas by using a set of mixed states as a basis in the Hilbert space. A mixed state described by a density matrix ρ with eigenstates $|e^{(n)}\rangle$ and eigenvalues p_n as in Eq. (2.33) represents a set of pure states $\{|e^{(n)}\rangle\}$ with a probability distribution $\{p_n\}$. Thus the idea of using mixed states as a basis replaces the usual "fixed" vectors in a pure-state basis such as those discussed above with "noisy" or "random" vectors distributed according to a given probability distribution.

The actual mixed basis employed in this paper is the set of all density operators $\{\rho(\xi; \beta)\}$ of Eqs. (2.36) and (2.37) for all complex values ξ and at a fixed (but arbitrary) value of β . Equation (2.38) clearly shows that this mixed-state basis generalizes to nonzero temperatures the coherent-state basis. In this case the probability distribution $\{p_n\}$ becomes the thermal Boltzmann distribution for a canonical ensemble of harmonic oscillators at temperature T . In particular, Eq. (2.33) takes the specific form

$$\rho(\xi; \beta) = \sum_{n=0}^{\infty} p_n(\beta) |n; \xi\rangle \langle n; \xi|, \quad (2.40)$$

$$|n; \xi\rangle \equiv D(\xi) |n\rangle, \quad (2.41)$$

$$p_n(\beta) \equiv (1 - e^{-\beta}) e^{-\beta n}. \quad (2.42)$$

We have proved in our earlier work [3] that the mixed states described by the density operators $\{\rho(\xi; \beta)\}$ may be employed to give the very important and nontrivial resolution of the identity operator,

$$\frac{1}{\pi} \int d^2\xi \rho(\xi; \beta) = I. \quad (2.43)$$

Equation (2.43) is readily expressed in the Bargmann representation, using Eqs. (2.5) and (2.34), as

$$\frac{1}{\pi} \int d^2\xi \rho_B(\xi; \beta; z_1, z_2^*) = \exp(z_1 z_2^*), \quad (2.44)$$

where, as explained in Eq. (2.32a), the right-hand side of Eq. (2.44) is the unit operator in this formalism.

III. THE B REPRESENTATION AND ITS RELATIONSHIPS WITH THE P , Q , AND W REPRESENTATIONS

There has been considerable discussion in the literature [11–13] on the P , Q , and W representations of arbitrary operators in $\mathcal{H}$, and the various relationships between them. In the present section we aim to examine the relationship of the B representation with these others. We shall also present some basic material which will be used in Sec. IV, where these various pure-state or zero-temperature representations are generalized to their thermal mixed (coherent) state or finite-temperature counterparts.

The usual Q and P representations of an arbitrary operator Θ are defined with respect to the coherent states $|z\rangle$ as follows:

$$Q(\Theta; z) \equiv \langle z | \Theta | z \rangle, \quad (3.1)$$

$$\Theta = \int \frac{d^2z}{\pi} P(\Theta; z) |z\rangle \langle z|. \quad (3.2)$$

Equations (3.1) and (3.2), together with the overlap relation of Eq. (2.5), lead immediately to the well-known relation

$$Q(\Theta; z) = \int \frac{d^2z'}{\pi} P(\Theta; z') e^{-|z - z'|^2}. \quad (3.3)$$

This may also be written in the equivalent form

$$Q(\Theta; z) = \exp(\frac{1}{4} \Delta_z) P(\Theta; z), \quad (3.4)$$

in terms of the two-dimensional Laplacian operator,

$$\Delta_z \equiv 4 \frac{\partial^2}{\partial z \partial z^*} = \frac{\partial^2}{\partial z_R^2} + \frac{\partial^2}{\partial z_I^2}, \quad (3.5)$$

where the suffices R and I denote the real and imaginary parts of the corresponding complex number. Equation (3.4) follows from the more general relation,

$$\int \frac{d^2z'}{\pi} f(z', z'^*) k e^{-k|z - z'|^2} = \exp\left[\frac{1}{4k} \Delta_z\right] f(z, z^*), \quad k > 0, \quad (3.6)$$

which is valid for arbitrary functions $f(z, z^*) \equiv f(z)$ of the complex variable z , and which is most readily proven by taking the Fourier transform of both sides of the equation. We next introduce the notation $\tilde{f}(w) \equiv \tilde{f}(w, w^*)$ for the Fourier transform of the general function $f(z)$,

$$\tilde{f}(w) \equiv \int d^2z e^{iz \cdot w} f(z) \leftrightarrow f(z) = \int \frac{d^2w}{(2\pi)^2} e^{-iz \cdot w} \tilde{f}(w), \quad (3.7)$$

in terms of the convenient scalar product notation,

$$z \cdot w \equiv \frac{1}{2}(z w^* + z^* w) = z_R w_R + z_I w_I. \quad (3.8)$$

Equations (3.3) and (3.4) may thus be equivalently written in the form

$$\tilde{Q}(\Theta; w) = \exp(-\frac{1}{4}|w|^2) \tilde{P}(\Theta; w), \quad (3.9)$$

as a relation between the Fourier transforms.

The B representation $B(\Theta; z_1, z_2^*)$ may now be expressed in terms of the respective P and Q representations $P(\Theta; z)$ and $Q(\Theta; z)$. In the first place, the definition of Eq. (2.23), together with Eqs. (3.2) and (2.5), trivially leads to the result

$$B(\Theta; z_1, z_2^*) = \int \frac{d^2z}{\pi} P(\Theta; z) \exp(z z_1 + z^* z_2^* - |z|^2). \quad (3.10)$$

Equation (3.10) may also be equivalently written in the form

$$B(\Theta; z_1, z_2^*) = \int \frac{d^2w}{\pi^2} \tilde{P}(\Theta; 2iw) \exp(w z_1 - w^* z_2^* - |w|^2 + z_1 z_2^*). \quad (3.11)$$

The further use of Eq. (3.9) immediately also yields the relation

$$B(\Theta; z_1, z_2^*) = e^{z_1 z_2^*} \int \frac{d^2 w}{(2\pi)^2} \tilde{Q}(\Theta; w) \exp[-\frac{1}{2}(wz_1 + w^* z_2^*)] . \quad (3.12)$$

It is interesting to note that Eq. (3.12) provides an explicit construction of the off-diagonal elements $\langle z|\Theta|z' \rangle$ of an operator Θ in the coherent-state basis wholly in terms of the diagonal elements $\langle z|\Theta|z \rangle$ alone.

The above relations are also readily inverted to yield the P and Q representations in terms of the B representation. Clearly, by comparison with Eq. (3.7), Eq. (3.12) in the case $z_1 = z_2 = z^*$, is compatible with the relation

$$Q(\Theta; z) = e^{-|z|^2} B(\Theta; z^*, z) , \quad (3.13)$$

which follows immediately from a comparison of Eqs. (2.23) and (3.1). The further use of Eq. (3.4) immediately yields the formal relation

$$P(\Theta; z) = \exp(-\frac{1}{4}\Delta_z) \exp(-|z|^2) B(\Theta; z^*, z) . \quad (3.14)$$

An alternative relation to Eq. (3.14) is found by putting $z_2 = -z_1 = z^*$ in Eq. (3.10) and by taking the resulting Fourier transform. A simple calculation yields

$$P(\Theta; z) = e^{|z|^2} \int \frac{d^2 z'}{\pi} B(\Theta; -z'^*, z') \exp(zz'^* - z^* z') , \quad (3.15)$$

a relation which was first given by Mehta [15].

We finally introduce the (Fourier transform of the) Wigner function $\Theta_W(z)$ corresponding to an operator Θ , defined in terms of the displacement operator $D(z)$ from Eq. (2.1) as,

$$\Theta_W(z) \equiv \text{Tr}[\Theta D(z)] . \quad (3.16)$$

By using Eq. (2.7) to express $D(z)$ in terms of the operators $\hat{x}$ and $\hat{p}$, and by making use of the simple relationship for position eigenstates $|x \rangle$,

$$e^{-i\alpha\hat{p}}|x \rangle = |x + \alpha \rangle , \quad (3.17)$$

it is not difficult to show that the definition of Eq. (3.16) leads to the expression

$$\begin{aligned} \Theta_W(z) &= \text{Tr}[\Theta D(z)] \\ &= \int_{-\infty}^{\infty} dx \langle x - z_R/\sqrt{2} | \Theta | x + z_R/\sqrt{2} \rangle \\ &\quad \times \exp(i\sqrt{2}x_I z) . \end{aligned} \quad (3.18)$$

It is convenient for later purposes to introduce the trivially related function $\tilde{W}(\Theta; w)$ as follows:

$$\tilde{W}(\Theta; w) \equiv \pi \text{Tr}[\Theta D(\frac{1}{2}iw)] = \pi \Theta_W(\frac{1}{2}iw) , \quad (3.19)$$

with corresponding Fourier transform $W(\Theta; z)$. Equations (3.7) and (3.18) easily lead to the much more familiar expression

$$\begin{aligned} W(\Theta; z) &= 2 \int_{-\infty}^{\infty} dx \langle x + \sqrt{2}z_R | \Theta | -x + \sqrt{2}z_R \rangle \\ &\quad \times \exp(-i2\sqrt{2}z_I x) , \end{aligned} \quad (3.20)$$

which is particularly well known as the Wigner representation in the case that Θ is a density operator. The importance to us of this W representation is that an arbitrary operator Θ can be expressed in terms of it by the following basic relation due to Moyal [11],

$$\begin{aligned} \Theta &= \int \frac{d^2 \xi}{\pi} \text{Tr}[D^\dagger(\xi)\Theta] D(\xi) \\ &= \int \frac{d^2 w}{(2\pi)^2} \tilde{W}(\Theta; w) D(-\frac{1}{2}iw) . \end{aligned} \quad (3.21)$$

The use of Eqs. (2.23) and (2.32b) immediately yields the relationship

$$B(\Theta; z_1, z_2^*) = e^{z_1 z_2^*} \int \frac{d^2 w}{(2\pi)^2} \tilde{W}(\Theta; w) \exp[-\frac{1}{8}|w|^2 - \frac{1}{2}i(wz_1 + w^* z_2^*)] . \quad (3.22)$$

Comparison of Eqs. (3.12) and (3.22), and the use of Eq. (3.9), yields the fundamental relations between the P , Q , and W representations of an arbitrary operator Θ ,

$$\begin{aligned} \tilde{Q}(\Theta; w) &= \exp(-\frac{1}{8}|w|^2) \tilde{W}(\Theta; w) , \\ \tilde{P}(\Theta; w) &= \exp(+\frac{1}{8}|w|^2) \tilde{W}(\Theta; w) . \end{aligned} \quad (3.23)$$

The comparable relations between their Fourier transforms are given by

$$\begin{aligned} Q(\Theta; z) &= \exp(+\frac{1}{8}\Delta_z) W(\Theta; z) , \\ P(\Theta; z) &= \exp(-\frac{1}{8}\Delta_z) W(\Theta; z) . \end{aligned} \quad (3.24)$$

Equation (3.22) may also be formally inverted to give the relation

$$W(\Theta; z) = \exp(-\frac{1}{8}\Delta_z) \exp(-|z|^2) B(\Theta; z^*, z) , \quad (3.25)$$

which also follows from Eqs. (3.14) and (3.24). An alternative inverse follows from the definition of Eq. (3.19) and the

use of Eqs. (2.31) and (2.32b),

$$\tilde{W}(\Theta; w) = \frac{1}{\pi} \int d^2z \int d^2z' B(\Theta; z, z'^*) \exp(-|z|^2 - |z'|^2 + z^* z' - \frac{1}{8}|w|^2 + \frac{1}{2}i w z' + \frac{1}{2}i w^* z^*) . \quad (3.26)$$

Equation (3.26) may also be Fourier transformed to yield the equivalent relation

$$W(\Theta; \xi) = \frac{2}{\pi^2} \int d^2z \int d^2z' B(\Theta; z, z'^*) \exp(-|z|^2 - |z'|^2 - z^* z' - 2|\xi|^2 + 2z^* \xi + 2z'^* \xi^*) . \quad (3.27)$$

IV. GENERALIZED (FINITE-TEMPERATURE) P AND Q REPRESENTATIONS

From the discussion in Sec. III it is clear that since the formalism of P and Q representations is based on coherent states, it would seem that any generalization of the formalism would require the concept of coherent states to be generalized. In this context we recognize that although the coherent states form an overcomplete basis, the key feature that makes them useful in practice is the resolution of the identity operator in terms of them, namely, Eq. (2.6). However we have already seen in Sec. II that the set of density operators $\{\rho(z; \beta); z \in \mathbb{C}\}$, for a fixed (real and positive) value of β , provides a mixed-state generalization to nonzero temperatures of the set of pure-state (zero-temperature) coherent states $\{|z\rangle; z \in \mathbb{C}\}$. Furthermore, these thermal coherent states have also been shown to provide a resolution of the identity, namely, Eq. (2.43).

From this discussion it is therefore natural to attempt generalizations of the P and Q representations of an arbitrary operator Θ , based on $\rho(z; \beta)$. More specifically, we define a generalized Q representation by analogy with Eq. (3.1) as

$$Q(\Theta; z; \beta) \equiv \text{Tr}[\rho(z; \beta)\Theta] , \quad (4.1)$$

and we also hypothesize that Θ has an expansion in terms of the operators $\{\rho(z; \beta); z \in \mathbb{C}\}$ by analogy with Eq. (3.2) as

$$\Theta = \int \frac{d^2z}{\pi} P(\Theta; z; \beta) \rho(z; \beta) . \quad (4.2)$$

If the representation of Eq. (4.2) exists (and is unique), $P(\Theta; z; \beta)$ is the generalized P representation of Θ . It is clear from Eq. (2.38) that in the zero-temperature ($\beta \rightarrow +\infty$) limit the above generalized representations reduce to the ordinary Q and P representations,

$$\begin{aligned} \lim_{\beta \rightarrow \infty} Q(\Theta; z; \beta) &= Q(\Theta; z) , \\ \lim_{\beta \rightarrow \infty} P(\Theta; z; \beta) &= P(\Theta; z) . \end{aligned} \quad (4.3)$$

The existence of the generalized P representation of Eq. (4.2) has been discussed in detail in our earlier work [3]. More specifically, we proved both that an operator Θ is uniquely determined by its generalized Q representation (at a fixed but arbitrary non-negative value of the temperature), and that all operators in the trace class (namely, those whose trace is finite and basis independent) either directly have the generalized P representation of Eq. (4.2)

or may be regarded as the limit of a sequence of such operators. Just as in the zero-temperature case, the generalized P representation exists most generally only as a Schwartz distribution.

Since the above theorems involved only existence proofs we turn now to the specific construction of the generalized P representation, and to a discussion of the relationships between it and the generalized Q representation and between both of them and their zero-temperature analogues. From Eqs. (4.1) and (4.2) we immediately have

$$Q(\Theta; z; \beta) = \int \frac{d^2z'}{\pi} \text{Tr}[\rho(z; \beta)\rho(z'; \beta')] P(\Theta; z'; \beta') . \quad (4.4)$$

The trace in Eq. (4.4) may be calculated straightforwardly from Eqs. (2.31) and (2.39) as,

$$\text{Tr}[\rho(z; \beta)\rho(z'; \beta')] = \frac{2}{g(\beta, \beta')} \exp\left[-\frac{2|z - z'|^2}{g(\beta, \beta')}\right] , \quad (4.5)$$

where

$$g(\beta, \beta') \equiv \frac{\sinh[\frac{1}{2}(\beta + \beta')]}{\sinh(\frac{1}{2}\beta)\sinh(\frac{1}{2}\beta')} . \quad (4.6)$$

Equations (4.4) and (4.5) then immediately yield the fundamental relation,

$$Q(\Theta; z; \beta) = \exp[\frac{1}{8}g(\beta, \beta')\Delta_z] P(\Theta; z; \beta') , \quad (4.7)$$

where we have also made use of Eq. (3.6). The Fourier transform of Eq. (4.7) yields the equivalent relation,

$$\tilde{Q}(\Theta; w; \beta) = \exp[-\frac{1}{8}g(\beta, \beta')|w|^2] \tilde{P}(\Theta; w; \beta') . \quad (4.8)$$

An important special case of Eq. (4.8) is when $\beta' \rightarrow +\infty$,

$$\tilde{Q}(\Theta; w; \beta) = \exp[-\frac{1}{4}(1 - e^{-\beta})^{-1}|w|^2] \tilde{P}(\Theta; w) , \quad (4.9)$$

which is clearly a generalization of our previous Eq. (3.9). The further use of Eq. (3.9) in conjunction with Eqs. (3.13) and (3.14) yields the important relations

$$\begin{aligned} \tilde{Q}(\Theta; w; \beta) &= \exp[-\frac{1}{4}(e^\beta - 1)^{-1}|w|^2] \tilde{Q}(\Theta; w) \\ &= \exp[-\frac{1}{4}(e^\beta - 1)^{-1}|w|^2] \\ &\quad \times \exp(\Delta_w) \tilde{B}(\Theta; w^*, w) , \end{aligned} \quad (4.10)$$

$$\begin{aligned}\bar{P}(\Theta; w; \beta) &= \exp[+ \tfrac{1}{4}(e^\beta - 1)^{-1} |w|^2] \bar{P}(\Theta; w) \\ &= \exp[+ \tfrac{1}{4}(e^\beta - 1)^{-1} |w|^2] \\ &\quad \times \exp(\tfrac{1}{4}|w|^2) \exp(\Delta_w) \bar{B}(\Theta; w^*, w),\end{aligned}\quad (4.11)$$

which express the generalized Q and P representations directly in terms of their respective zero-temperature counterparts as well as in terms of the B representation. In turn these relations also readily yield their more general equivalents,

$$\begin{aligned}\bar{Q}(\Theta; w; \beta) &= \exp[- \tfrac{1}{8}g(\beta, -\beta') |w|^2] \bar{Q}(\Theta; w; \beta'), \\ \bar{P}(\Theta; w; \beta) &= \exp[+ \tfrac{1}{8}g(\beta, -\beta') |w|^2] \bar{P}(\Theta; w; \beta').\end{aligned}\quad (4.12)$$

Finally, we may relate the generalized P and Q representations to the W representation introduced in Sec. III, by making use of Eq. (3.23). In this way we find the especially important relations

$$\begin{aligned}\bar{Q}(\Theta; w; \beta) &= \exp[- \tfrac{1}{8} \coth(\tfrac{1}{2}\beta) |w|^2] \bar{W}(\Theta; w), \\ \bar{P}(\Theta; w; \beta) &= \exp[+ \tfrac{1}{8} \coth(\tfrac{1}{2}\beta) |w|^2] \bar{W}(\Theta; w),\end{aligned}\quad (4.13)$$

or, equivalently, by taking Fourier transforms,

$$\begin{aligned}Q(\Theta; z; \beta) &= \exp[+ \tfrac{1}{8} \coth(\tfrac{1}{2}\beta) \Delta_z] W(\Theta; z), \\ P(\Theta; z; \beta) &= \exp[- \tfrac{1}{8} \coth(\tfrac{1}{2}\beta) \Delta_z] W(\Theta; z).\end{aligned}\quad (4.14)$$

Equations (4.13) and (4.14) clearly generalize their zero-temperature ($\beta \rightarrow +\infty$) counterparts of Eqs. (3.23) and (3.24). Their real importance, however, lies in showing that the generalized P representation may be viewed as the analytic continuation to negative temperatures of the generalized Q representation, and vice versa. In particular, the usual (zero-temperature) Q and P representations $Q(\Theta; z)$ and $P(\Theta; z)$ may be obtained from $Q(\Theta; z; \beta)$ in the respective limits $\beta \rightarrow +\infty$ (i.e., $T \rightarrow 0^+$) and $\beta \rightarrow -\infty$ (i.e., $T \rightarrow 0^-$), in the latter case after the analytic continuation has been performed.

V. CONCLUDING REMARKS

Most previous generalizations of the original Glauber coherent states have been based on replacing the Weyl displacement operators of Eq. (2.1), which provide a unitary representation of the Heisenberg-Weyl group associated with the operators $\{a, a^\dagger, I\}$, by an alternative unitary operator connected with another group. Well-known examples include, for example, the groups $SU(1,1)$, $SU(2)$, and $Sp(4, \mathbb{R})$, all of which are associated with various aspects of two-photon or squeezed coherent states of either a single-mode or two-mode field. Indeed, the latter case of $Sp(4, \mathbb{R})$ incorporates the most general squeezed coherent states of a two-mode field [16], where the corresponding algebra is realized by the complete set of operators bilinear in the creation and destruction operators of the two independent modes.

All of the above generalized coherent states are pure states. Our main contribution in both the present and earlier work [3] has been to provide an entirely different generalization, namely, to the mixed states, which have

been discussed in different contexts by other authors [7–9]. We have demonstrated how the coherent mixed states can be regarded as providing a “random” (or “noisy” or “thermal”) basis in the quantum-mechanical Hilbert space. Their practical usefulness arises basically from the fact that we were able to prove that there exists a resolution of the identity operator in terms of them. An alternative but wholly equivalent description is that we have introduced the present *probability operator measure* (POM) in terms of the density operator $\rho(z; \beta)$, as an alternative to the more usual POM proportional to $|z\rangle\langle z|$ for the Glauber coherent states. We have focussed attention here on coherent mixed states associated with the displaced oscillator in thermal equilibrium at nonzero temperature. However, as we showed previously [3], the resulting thermal coherent states may also be based on the squeezed and displaced harmonic oscillator, thereby providing a mixed-state generalization of the two-photon or squeezed coherent states for a single-mode field. The generalization of *any* of the alternative pure-state coherent states to their thermal mixed-state counterparts is, in principle, possible by techniques quite analogous to those used here.

One of the key features of the present work has been the generalization of the usual (zero-temperature) P and Q representations of operators to their nonzero-temperature counterparts, and the exploration of the various relationships both between these representations and with the W representation. We have also shown how our whole formalism can be imbedded into the Bargmann Hilbert representation; and we have specifically examined the Bargmann representation of both state vectors and operators, and shown how these relate to the P , Q , and W representations. A particularly important and unexpected result was the discovery that the temperature-dependent P and Q representations are the analytic continuations to negative temperatures of each other. We strongly suspect that further examination of the analyticity properties of the generalized P and Q representations in the entire complex-valued temperature plane will shed further light on the intricate relationships between the various representations. Such studies may also be of practical importance in extending the usefulness of the P representation, for example, to operators whose usual P representation is only given by a Schwartz distribution or generalized function rather than by an ordinary regular function. We intend to examine such possibilities in future work.

From our previous discussions one may physically intuit that our thermal coherent states are intimately related to the common practical situation where we have a field which is an admixture of a coherent field described by a pure coherent state and a random incoherent field described by a Poisson distribution of photons characterized only by an average number. (Although the present formalism is restricted to the case of a single-mode or single-frequency field, this restriction is not vital to the formalism.) We showed in detail in our earlier work [3] how our thermal coherent states can be used in the so-called Glauber-Lachs [4,5] formalism in quantum optics for a coherent field with an admixture of noise described

by a thermal blackbody radiation field at a given temperature. From the practical viewpoint we therefore expect that our coherent mixed states, and the resulting analysis based on them given in this paper, should be especially relevant to the description of coherent signals in the presence of thermal noise. This is a field of great practical importance and considerable recent activity [17–25]. We believe that the work presented here should provide valuable theoretical support to such studies.

We note finally that although we are concerned in this work primarily with quantum-mechanical applications, the same ideas are potentially useful in the areas of signal and image processing. Since the pioneering work of Gabor and others [26] it has been widely accepted that the usual representations of a signal in the time domain or in the frequency domain are not the most convenient ones. Instead, representations in the time-frequency “phase space” are, for many purposes, more advantageous.

Indeed, there is a plethora of papers in the signal processing context (and see, e.g., Refs. [27–32]) that use the Wigner function method and other allied techniques inspired by the coherent-state formalism. The very topical wavelet approach [33] is also a by-product of these ideas. It is our belief that our own extensions of the coherent-state concept described here might also similarly be of practical use in these areas of signal and image processing.

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