Phase-admixed states: Coherence and incoherence

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The standard Glauber coherent states associated with a given boson or oscillator mode have a well-known decomposition in terms of the occupation-number eigenstates. In this paper we study a much more general class of quasicohherent states in which each term of this decomposition acquires its own extra phase factor. These new states are not minimum-uncertainty states, but we show that they may be represented as eigenstates of some new canonical annihilation operator which is obtained from the original one by an appropriate unitary transformation. By choosing particular sets of values for the phases which characterize the quasicohherent states, we show how we are thereby able to study, for example, the effects of (i) optical propagation through a nonlinear medium with an intensity-dependent refractive index, and (ii) phase incoherence caused by the stochasticity typical of an optical source. The quasicohherent states are shown to share many of the same coherence properties as the original Glauber states. Nevertheless, they exhibit a mixture of coherent and incoherent properties. This is particularly illustrated by the important practical example where the phase angles are randomized.

I. INTRODUCTION

It is by now very well known how the (Glauber) coherent states $|A\rangle$, which are eigenstates of the annihilation operator a_0 associated with a given quantum oscillator or boson mode, may be represented in terms of the corresponding states $|n\rangle$ of definite occupation number, with $a_0^\dagger a_0 |n\rangle = n |n\rangle$. Soon after their introduction, it was realized^{1,2} that the much more general class of states $|A; \{\gamma_n\}\rangle$, where

$$|A; \{\gamma_n\}\rangle \equiv \exp(-\frac{1}{2}|A|^2) \sum_{n=0}^{\infty} (n!)^{-1/2} A^n e^{i\gamma_n} |n\rangle$$

have exactly the same correlations $\langle (a_0^\dagger)^n a_0^n \rangle$ as the pure coherent states which have $\gamma_n=0$ for all n . However, the off-diagonal correlations, $\langle (a_0^\dagger)^n a_0^m \rangle$ with $m \neq n$, do depend on the phase angles $\{\gamma_n\}$, and hence differ from the corresponding correlations for the pure coherent states. To date, such off-diagonal correlation terms have been of little practical importance in quantum optics because the vast majority of experiments employ photodetectors that are sensitive to the number of photons but not to the phase of the light field. For this reason there has been only a relatively small amount of work^{3,4} that focuses attention on the off-diagonal correlations, $\langle (a_0^\dagger)^n a_0^m \rangle$ with $m \neq n$.

Nevertheless, it is clear that the above states of indefinite occupation number have very definite coherent properties, but are not eigenstates of the standard annihilation operator connected with that mode. It seems natural to inquire whether these new states $|A; \{\gamma_n\}\rangle$ might,

however, be eigenstates of some other suitable annihilation operator. It is from this viewpoint that the present discussion begins.

We introduce in this paper the unitary operators $U(\Gamma)$, where

$$U(\Gamma) \equiv \sum_{n=0}^{\infty} e^{i\gamma_n} |n\rangle \langle n|, \quad \Gamma \equiv \{\gamma_n; n=0, 1, \dots\}.$$

These operators provide a unitary isomorphism of the Hilbert space onto itself, in which each ket state $|s\rangle$ becomes mapped into the state $U(\Gamma)|s\rangle$ and each operator $\hat{O}$ is mapped into $U(\Gamma)\hat{O}U^\dagger(\Gamma)$. In particular, the creation and annihilation operators $a_0^\dagger$ and a_0 , respectively, are mapped as $a_0^\dagger \rightarrow a^\dagger(\Gamma) = U(\Gamma)a_0^\dagger U^\dagger(\Gamma)$ and $a_0 \rightarrow a(\Gamma) = U(\Gamma)a_0 U^\dagger(\Gamma)$; and the standard coherent state $|A\rangle$ is mapped into the state $|A; \{\gamma_n\}\rangle \equiv |A; \Gamma\rangle$ defined above. We shall further prove that the mapped states $|A; \Gamma\rangle$ are eigenstates of the mapped operators $a(\Gamma)$, and hence that the new states $|A; \Gamma\rangle$ may also be regarded as standard coherent states with respect to the new annihilation operators $a(\Gamma)$.

By passing from a description in terms of the "Cartesian" operators, a_0 and $a_0^\dagger$, to one in terms of the "polar" operators,⁵⁻⁷ $a_0^\dagger a_0$ and $\cos\Theta$, we may easily visualize our mapping in terms of "rotations." Furthermore, our mapping contains an infinite number of phases basically because $\cos\Theta$ is here an operator. It should be clear from the very outset that the parameters $\{\gamma_n\}$ are purely quantum phases. We emphasize that they have no classical counterparts.

After a general exposition of the new (quasicohherent)

states, we consider two particular applications of them. The first of these involves the propagation of light through a nonlinear medium with an intensity-dependent refractive index. Interferometers which employ such media have been studied recently,⁸⁻¹¹ and we find that the special case, $\gamma_n = \frac{1}{2}n(n-1)\phi$ with ϕ constant, of our general theory provides a theoretical framework in which to study them. Whereas the media considered so far can be described in terms of a third-order nonlinear susceptibility, media with a higher-order nonlinearity would require our general formalism.

A second specific case that we study is that in which the phases $\{\gamma_n\}$ take random values, with equal probability for any value in the range $(0, 2\pi)$. Many of the general results are seen to simplify in this case. Furthermore, we are motivated to study this case since in practice most experimental arrangements employ photodetectors that can count the number of photons but which are phase insensitive. The discussion emphasizes the intriguing juxtaposition of both coherent and incoherent properties that is displayed by these quasicohherent states.

II. BASIC FORMALISM

In terms of a pair of Hermitian conjugate creation and annihilation operators, $a_0^\dagger$ and a_0 , respectively, which obey the usual boson commutation relation,

$$[a_0, a_0^\dagger] = 1, \quad (1)$$

the orthonormal states $|n\rangle$ of definite particle number are given as

$$|n\rangle = (n!)^{-1/2} (a_0^\dagger)^n |0\rangle, \quad (2)$$

in terms of the vacuum state $|0\rangle$. They obey the relations,

$$\begin{aligned} a_0 |n\rangle &= n^{1/2} |n-1\rangle, \\ a_0^\dagger |n\rangle &= (n+1)^{1/2} |n+1\rangle. \end{aligned} \quad (3)$$

All of the elements of the coset,

$$I_n \equiv \{e^{i\Theta} |n\rangle; \Theta \in \mathbb{R}\}, \quad (4)$$

are normalized eigenstates of the number operator $a_0^\dagger a_0$ with eigenvalue n .

We now consider sequences of real numbers $\Gamma \equiv \{\gamma_n; n=0, 1, \dots\}$, and the unitary operators $U(\Gamma)$ defined as

$$U(\Gamma) \equiv \sum_{n=0}^{\infty} \exp(i\gamma_n) |n\rangle \langle n|. \quad (5)$$

It is clear that these operators form an Abelian group. Indeed it is trivial to show the following relations:

$$\begin{aligned} U(\Gamma=0) &= 1, \\ U(\Gamma)U(\Delta) &= U(\Gamma+\Delta) = U(\Delta)U(\Gamma), \\ U^{-1}(\Gamma) &= U(-\Gamma) = U^\dagger(\Gamma), \\ [U(B)U(\Gamma)]U(\Delta) &= U(B)[U(\Gamma)U(\Delta)]. \end{aligned} \quad (6)$$

The transformation operators of Eq. (5) define a uni-

tary isomorphism of the Hilbert space onto itself, by mapping each ket $|s\rangle$ into $|s; \Gamma\rangle \equiv U(\Gamma)|s\rangle$, and each operator $\hat{\Theta}_0 = f(a_0^\dagger, a_0)$ into $\hat{\Theta}(\Gamma) \equiv U(\Gamma)f(a_0^\dagger, a_0)U^\dagger(\Gamma)$. If the state $|s\rangle$ is an eigenstate of some such operator $\hat{\Theta}_0$,

$$\hat{\Theta}_0 |s\rangle = s |s\rangle, \quad (7a)$$

then we have

$$\hat{\Theta}(\Gamma) |s; \Gamma\rangle = U(\Gamma)\hat{\Theta}_0 U^\dagger(\Gamma)U(\Gamma) |s\rangle = s |s; \Gamma\rangle, \quad (7b)$$

and hence that the state $|s; \Gamma\rangle$ is the corresponding eigenstate of the transformed operator $\hat{\Theta}(\Gamma)$. It is straightforward to show that the basic creation and annihilation operators, $a_0^\dagger$ and a_0 , respectively, are themselves transformed as

$$\begin{aligned} a(\Gamma) &= U(\Gamma)a_0 U^\dagger(\Gamma) \\ &= \sum_{n=0}^{\infty} \exp[i(\gamma_n - \gamma_{n+1})] (n+1)^{1/2} |n\rangle \langle n+1|, \\ a^\dagger(\Gamma) &= U(\Gamma)a_0^\dagger U^\dagger(\Gamma) \\ &= \sum_{n=0}^{\infty} \exp[i(\gamma_{n+1} - \gamma_n)] (n+1)^{1/2} |n+1\rangle \langle n|, \end{aligned} \quad (8)$$

$$[a(\Gamma), a^\dagger(\Gamma)] = 1.$$

The mode of action of these transformed basic operators on the original number eigenstates is easily seen from Eq. (8) to be

$$\begin{aligned} a(\Gamma) |n\rangle &= \exp[i(\gamma_{n-1} - \gamma_n)] n^{1/2} |n-1\rangle, \\ a^\dagger(\Gamma) |n\rangle &= \exp[i(\gamma_{n+1} - \gamma_n)] (n+1)^{1/2} |n+1\rangle, \end{aligned} \quad (9)$$

which may be compared with the corresponding Eq. (3) for the original operators. It is now also trivial to see that the transformed general operator $\hat{\Theta}(\Gamma)$ of Eq. (7b) is given by

$$\hat{\Theta}(\Gamma) \equiv U(\Gamma)f(a_0^\dagger, a_0)U^\dagger(\Gamma) = f(a^\dagger(\Gamma), a(\Gamma)), \quad (10)$$

and is hence the same function of the transformed operators $a^\dagger(\Gamma)$ and $a(\Gamma)$ that the original operator $\hat{\Theta}_0$ is of the untransformed operators $a_0^\dagger$ and a_0 . One may easily prove either from Eqs. (5) and (10), or directly from Eq. (8), that

$$\hat{n} \equiv a_0^\dagger a_0 = a^\dagger(\Gamma)a(\Gamma), \quad (11)$$

or, more generally, if $\hat{\Theta}_0$ is a function only of the number operator, $\hat{\Theta}_0 = g(\hat{n} = a_0^\dagger a_0)$, then $\hat{\Theta}(\Gamma) = \hat{\Theta}_0$. That is, operators which are functions only of the number operator remain invariant under the unitary transformations induced by the operators of Eq. (5).

The number eigenstates $|n\rangle$ of Eq. (2) are themselves transformed according to

$$|n; \Gamma\rangle = U(\Gamma) |n\rangle = \exp(i\gamma_n) |n\rangle, \quad (12)$$

and hence simply acquire an extra phase factor. Consequently, the cosets of Eq. (4) remain invariant. In this sense, the unitary transformations induced by the operators $U(\Gamma)$ leave the number operator and its eigenstates invariant. We also note that for an arbitrary operator $\hat{\Theta}_0$,

its matrix elements in the number-state representation transform very simply as

$$\langle n | \hat{\Theta}_0 | m \rangle = \exp[i(\gamma_m - \gamma_n)] \langle n | \hat{\Theta}(\Gamma) | m \rangle. \quad (13)$$

We finally introduce the Hermitian “position” and “momentum” operators, $\hat{x}_0$ and $\hat{p}_0$, respectively, as

$$\hat{x}_0 \equiv 2^{-1/2}(a_0^\dagger + a_0), \quad \hat{p}_0 \equiv 2^{-1/2}i(a_0^\dagger - a_0), \quad (14a)$$

and their respective transforms as

$$\begin{aligned} \hat{x}(\Gamma) &\equiv U(\Gamma)\hat{x}_0U^\dagger(\Gamma) = 2^{-1/2}\{a^\dagger(\Gamma) + a(\Gamma)\}, \\ \hat{p}(\Gamma) &\equiv U(\Gamma)\hat{p}_0U^\dagger(\Gamma) = 2^{-1/2}i\{a^\dagger(\Gamma) - a(\Gamma)\}. \end{aligned} \quad (14b)$$

The position eigenstate $|x\rangle$ and its transform $|x; \Gamma\rangle$ are given by

$$\begin{aligned} \hat{x}_0 |x\rangle &= x |x\rangle, \\ |x; \Gamma\rangle &\equiv U(\Gamma) |x\rangle. \end{aligned} \quad (15)$$

The overlap $\langle x_1 | x_2; \Gamma \rangle$ may be evaluated by using both the representation

$$\begin{aligned} |x_2; \Gamma\rangle &= U(\Gamma) |x_2\rangle = \sum_{n=0}^{\infty} U(\Gamma) |n\rangle \langle n | x_2 \rangle \\ &= \sum_{n=0}^{\infty} \exp(i\gamma_n) |n\rangle \langle n | x_2 \rangle \end{aligned} \quad (16)$$

for the state $|x_2; \Gamma\rangle$, where we have inserted a complete set of number eigenstates and have used Eq. (12), and also the representation

$$\langle x | n \rangle = (\pi^{1/2} 2^n n!)^{-1/2} \exp(-\frac{1}{2}x^2) H_n(x) \quad (17)$$

for the state $|n\rangle$ in the coordinate-space basis, in terms of the Hermitian polynomials $H_n(x)$. Use of Eqs. (16) and (17) readily yields the result

$$\begin{aligned} \langle x_1 | x_2; \Gamma \rangle &= (\pi^{1/2} 2^n n!)^{-1} \exp[-\frac{1}{2}(x_1^2 + x_2^2)] \\ &\times \sum_{n=0}^{\infty} e^{i\gamma_n} H_n(x_1) H_n(x_2). \end{aligned} \quad (18)$$

It is readily checked that for the special case in which $\gamma_n = 0$ for all n , the right-hand side of Eq. (18) becomes the singular Dirac δ function, $\delta(x_1 - x_2)$, as indeed is required.

We remind the reader that the appearance of such Dirac δ functions is a consequence of the fact that the position eigenstates $|x\rangle$ are not strictly normalizable. For many physical purposes this non-normalizability is largely irrelevant. It can often be ignored or removed, at least in principle, by working with the usual box normalization in which the system is constrained to be within some finite volume Ω , at the surface of which some suitable boundary condition is imposed. However, in this case the explicit evaluation of the eigenfunctions, should they actually be needed, becomes more difficult. For example, the Hermite polynomials appearing in the representation of Eq. (17) become replaced by much more complicated functions $H_{n;\Omega}(x)$, which approach the usual polynomials $H_n(x)$ only in the limit $\Omega \rightarrow \infty$.

Strictly speaking, in the case of infinite volume the

states $|x\rangle$ do not form a Hilbert space, although more general mathematical spaces may still usefully be defined in which to imbed them. Thus, the so-called rigged Hilbert spaces¹² were introduced with just such states in mind. One important consequence of the non-normalizability of the position states $|x\rangle$ in the infinite volume case is that one cannot associate a normal density operator $|x\rangle\langle x|$ with them, since such an operator no longer belongs to the trace class. Such quasi-density-operators can, of course, still be of comparable practical use to that for the states $|x\rangle$ themselves. It is on this basis, therefore, that we consider such quasi-density-operators further in Sec. IV, where we deal with applications of the quasicohherent states that we now formally introduce in Sec. III.

III. QUASICOHERENT STATES

We let $|A\rangle$ be a standard (Glauber) coherent state, defined as an eigenstate of the oscillator annihilation operator a_0 ,

$$\begin{aligned} a_0 |A\rangle &= A |A\rangle, \\ |A\rangle &= \exp(-\frac{1}{2}|A|^2) \sum_{n=0}^{\infty} (n!)^{-1/2} A^n |n\rangle, \end{aligned} \quad (19)$$

in terms of the number eigenstates $|n\rangle$, $a_0^\dagger a_0 |n\rangle \equiv n |n\rangle$. The quasicohherent states $|A; \Gamma\rangle$ are now defined with respect to the unitary operators $U(\Gamma)$ of Eq. (5) and the standard coherent states, by any one of the following four definitions:

$$|A; \Gamma\rangle = U(\Gamma) |A\rangle, \quad (20)$$

$$|A; \Gamma\rangle = \exp(-\frac{1}{2}|A|^2) \sum_{n=0}^{\infty} (n!)^{-1/2} A^n e^{i\gamma_n} |n\rangle, \quad (21)$$

$$a(\Gamma) |A; \Gamma\rangle = A |A; \Gamma\rangle, \quad (22)$$

$$|A; \Gamma\rangle = \exp[Aa^\dagger(\Gamma) - A^*a(\Gamma)] |0\rangle. \quad (23)$$

The equivalence of the four defining relations (20)–(23) is rather straightforward to prove. Thus, use of the definitions of Eqs. (5) and (19) immediately proves the equivalence of Eqs. (20) and (21). Similarly, the equivalence of Eqs. (20) and (22) is a consequence of our general theorem of Eq. (7). Finally, the equivalence of Eqs. (22) and (23) is apparent from the standard theory of Glauber coherent states.

The definitions of Eqs. (22) and (23) show clearly that our quasicohherent states are standard (Glauber) coherent states with respect to the unitarily transformed operators $a^\dagger(\Gamma)$ and $a(\Gamma)$. Conversely, the term quasicohherent refers to their properties with respect to the original operators $a_0^\dagger$ and a_0 . We shall show in this regard that although the quasicohherent states are *not* standard coherent states, they do share many of the same or similar coherence properties with the standard coherent states. It is for this reason that we call them quasicohherent states. For example, by considering a particular case, we show in the Appendix that $\Delta x_0 \Delta p_0 > \frac{1}{2}$. Thus the states $|A; \Gamma\rangle$ clearly cannot be minimum-uncertainty or standard coherent states in general, with

respect to the original operators a_0 and $a_0^\dagger$. Of course, it is clear from their coherence properties with respect to the transformed operators $a(\Gamma)$ and $a^\dagger(\Gamma)$ that $\Delta x(\Gamma)\Delta p(\Gamma) = \frac{1}{2}$.

By writing $\hat{\Theta}_0 \rightarrow |A\rangle\langle A|$ in Eq. (13) we easily find the relation

$$|\langle n | A; \Gamma \rangle|^2 = |\langle n | A \rangle|^2 = \frac{|A|^{2n}}{n!} e^{-|A|^2}. \quad (24)$$

Thus, the same Poisson distribution is obtained for the quasicohherent states as for the standard coherent states. The essential difference between the two classes of states lies rather in a comparison of their off-diagonal elements. Equation (13) shows how these differ by the expected phase factor

$$\langle n | A \rangle \langle A | m \rangle = e^{i(\gamma_m - \gamma_n)} \langle n | A; \Gamma \rangle \langle A; \Gamma | m \rangle. \quad (25)$$

An equivalent way of expressing the result of Eq. (24) is in terms of the second-order correlation, $g^{(2)}$. Thus, Eq. (11) implies that $(a_0^\dagger)^2 a_0^2 = [a^\dagger(\Gamma)]^2 [a(\Gamma)]^2$, and hence

$$\begin{aligned} g^{(2)} &\equiv \frac{\langle A; \Gamma | (a_0^\dagger)^2 a_0^2 | A; \Gamma \rangle}{\langle A; \Gamma | a_0^\dagger a_0 | A; \Gamma \rangle^2} \\ &= \frac{\langle A; \Gamma | [a^\dagger(\Gamma)]^2 [a(\Gamma)]^2 | A; \Gamma \rangle}{\langle A; \Gamma | a^\dagger(\Gamma) a(\Gamma) | A; \Gamma \rangle^2} \\ &= 1, \end{aligned} \quad (26)$$

where the last equality follows from Eq. (22). The second-order correlations are thus the same for both the coherent and quasicohherent states. The same is true for higher-order correlations which involve the corresponding expectation values of operators which involve equal powers of the creation and annihilation operators, $a_0^\dagger$ and a_0 , respectively. Correlations involving unequal powers of the creation and annihilation operators are different in the two cases, but are less amenable to observation in the context of most physical experiments.

We finally note that since, from Eqs. (22) and (23), our quasicohherent states may be viewed as ordinary coherent states with respect to the operators $a(\Gamma)$ and $a^\dagger(\Gamma)$, they obey the standard completeness relation,

$$\pi^{-1} \int d^2 A |A; \Gamma\rangle \langle A; \Gamma| = 1, \quad (27)$$

and the overlap between two quasicohherent states is given by

$$\langle A; \Gamma | B; \Gamma \rangle = \exp(-\frac{1}{2}|A|^2 - \frac{1}{2}|B|^2 + A^* B). \quad (28)$$

IV. APPLICATIONS

A. Nonlinear media with intensity-dependent refractive indexes

Particular values of the phases $\{\gamma_n\}$ considered above appear when light propagates through a nonlinear medium with an intensity-dependent refractive index. Such media have been discussed by Stolen and Lin¹³ and by

Drummond and Walls,¹⁴ for example. A special case of our general theory is relevant to the study of interferometers with a Kerr medium, which has received some recent attention.⁸⁻¹¹

Typically, in such devices a coherent input signal is divided into two parts by a beam splitter. One part is allowed to propagate through the Kerr medium, whereby its various number-eigenstate components $|n\rangle$, in the decomposition of Eq. (19), are changed in phase by an n -dependent amount. The other part propagates undisturbed, and is finally recombined with the output of the Kerr medium. Within the rotating-wave approximation, the Hamiltonian for a single mode, described by the creation and destruction operators $a_0^\dagger$ and a_0 , in the Kerr medium is specified as

$$H = a_0^\dagger a_0 + \chi a_0^\dagger a_0 (a_0^\dagger a_0 - 1), \quad (29)$$

where χ is a coefficient proportional to the (third-order) nonlinear susceptibility. The state evolution during the propagation of the light through the Kerr medium may be described in the interaction picture by the unitary operator,

$$U = \exp[\frac{1}{2} i \phi a_0^\dagger a_0 (a_0^\dagger a_0 - 1)], \quad (30)$$

in which the coefficient ϕ is proportional to the nonlinear susceptibility coefficient χ . It is clear that this operator is just a special case of the general operator of Eq. (5), in which the phases $\{\gamma_n\}$ take the particular values

$$\gamma_n \rightarrow \frac{1}{2} \phi n(n-1). \quad (31)$$

In this case the operators $a(\Gamma)$ and $a^\dagger(\Gamma)$ of Eq. (8) reduce to the particular forms,

$$\begin{aligned} a(\Gamma) &\rightarrow a_1 = e^{-i\phi\hat{n}} a_0, \\ a^\dagger(\Gamma) &\rightarrow a_1^\dagger = a_0^\dagger e^{i\phi\hat{n}}, \end{aligned} \quad (32)$$

with the number operator $\hat{n}$ given as

$$\hat{n} \equiv a_0^\dagger a_0 = a_1^\dagger a_1, \quad (33)$$

and the study of such media is considerably simplified within our formalism. Only nonlinear media that can be described in terms of the third-order nonlinear susceptibility coefficient χ have been studied in the references cited above.⁸⁻¹¹ Media with higher-order nonlinearity may also be similarly studied within the framework of our general formalism described here.

B. Random phases

As a second application of our formalism, we turn to a much more general phenomenon. Thus, all light fields used in practice are subject to certain stochastic processes which depend upon the details of the particular experiment. For example, any vibrations in the various optical components can cause uncontrollable variations in the amplitude and frequency of the waves. More fundamentally, the radiating atoms responsible for the emission of the light are subjected to random collisions with other atoms. These collisions therefore lead to a random element in the phases of the emitted light waves, and it is to

this particular stochastic uncertainty, which is almost always present in practice, that we now turn our attention. We also recall from the outset in this context that most experiments simply use photodetectors as analyzing devices for the light waves. These are sensitive to the numbers of photons involved, but are otherwise phase insensitive.

In such cases where the relative phases are subject to an underlying stochastic indeterminacy, we are led to describe our system by some suitable phase-averaged density matrix. To be specific, we consider the two examples of a system in the quasicohherent state $|A; \Gamma\rangle$ and its associated pure-state density operator,

$$\rho_1(\Gamma) \equiv |A; \Gamma\rangle\langle A; \Gamma|, \quad (34)$$

and a system in the state $|x; \Gamma\rangle$ of Eqs. (15) and (16) and its associated pure-state quasi-density-operator,

$$\rho_2(\Gamma) \equiv |x; \Gamma\rangle\langle x; \Gamma|. \quad (35)$$

We have already discussed in Sec. II the fact that the operator $\rho_2(\Gamma)$ does not belong to the trace class, and hence that it is not, strictly speaking, a density operator. For this reason, the normalizable states $|A; \Gamma\rangle$ are, for many purposes, more convenient than the unnormalizable states $|x; \Gamma\rangle$. We now "randomize" the two operators of Eqs. (34) and (35) by assuming a uniform random distribution of phases $\{\gamma_n\} \equiv \Gamma$,

$$\begin{aligned} \rho_1(\Gamma) &\rightarrow \bar{\rho}_1 \equiv \int d\Gamma \rho_1(\Gamma), \\ \rho_2(\Gamma) &\rightarrow \bar{\rho}_2 \equiv \int d\Gamma \rho_2(\Gamma), \end{aligned} \quad (36)$$

with the notation

$$d\Gamma = \frac{d\gamma_1}{2\pi} \frac{d\gamma_2}{2\pi} \dots \quad (37)$$

Using Eq. (21) we have the representation,

$$\begin{aligned} \rho_1(\Gamma) &= e^{-|A|^2} \sum_{m,n=0}^{\infty} e^{i(\gamma_n - \gamma_m)} A^n (A^*)^m \\ &\quad \times (m!n!)^{-1/2} |n\rangle\langle m|, \end{aligned} \quad (38)$$

which, when inserted into Eq. (36), gives its phase-averaged counterpart as

$$\bar{\rho}_1 = e^{-|A|^2} \sum_{n=0}^{\infty} (n!)^{-1} |A|^2 |n\rangle\langle n|. \quad (39)$$

We note that whereas the density operator ρ_1 describes a pure state, its phase-randomized equivalent $\bar{\rho}_1$ describes a mixed state. We also point out that in practical cases where the phase, $\arg(A)$, of the complex coherent-state parameter A is not known, the density operator $|A\rangle\langle A|$ associated with the ordinary (Glauber) pure coherent state $|A\rangle$, is known¹⁵ to reduce after randomization of this phase to exactly the same mixed-state density operator $\bar{\rho}_1$ as given by Eq. (39). It is interesting to note that even if $\arg(A)$ is known exactly, the imprecision associated with the fluctuations in the phases $\{\gamma_n\}$ still reduces the pure coherent-state density operator to the mixed-state operator $\bar{\rho}_1$. We also note that the diagonal

matrix elements of the operators $\rho_1(\Gamma)$ and $\bar{\rho}_1$ are equal in the number-state representation, and that in this representation $\bar{\rho}_1$ is diagonal, with elements which are just the usual Poisson distribution.

By using the representation of Eqs. (16) and (17), we may also express the quasidensity operator $\rho_2(\Gamma)$ in the form

$$\begin{aligned} \rho_2(\Gamma) &= e^{-x^2} \sum_{m,n=0}^{\infty} e^{i(\gamma_n - \gamma_m)} (\pi 2^m + n m! n!)^{-1/2} \\ &\quad \times H_n(x) H_m(x) |n\rangle\langle m|. \end{aligned} \quad (40)$$

Randomization of the phases, as in Eq. (36), leads to the phase-averaged version,

$$\bar{\rho}_2 = e^{-x^2} \sum_{n=0}^{\infty} (\pi^{1/2} 2^n n!)^{-1} H_n^2(x) |n\rangle\langle n|. \quad (41)$$

It is easy to see from Eq. (41) that the trace, $\text{Tr} \bar{\rho}_2$, diverges as a Dirac δ function, $\delta(\epsilon)$, with $\epsilon \rightarrow 0$. We note again that whereas $\rho_2(\Gamma)$ describes a pure state, its phase-randomized counterpart $\bar{\rho}_2$ describes a mixed state.

Finally, we note that for any function $f(\hat{n})$ of the number operator, it is easy to prove that its average value is the same whether calculated via the density operator $\rho_1(\Gamma)$ or $\bar{\rho}_1$,

$$\text{Tr}[f(\hat{n})\rho_1(\Gamma)] = \text{Tr}[f(\hat{n})\bar{\rho}_1]. \quad (42)$$

Equation (42) is perhaps most easily proven by evaluating the trace in the number-state representation, and using the explicit representations of Eqs. (38) and (39) for the density operators. A comparable relation to Eq. (42) also holds for the quasi-density operator $\rho_2(\Gamma)$,

$$\text{Tr}[f(\hat{n})\rho_2(\Gamma)] = \text{Tr}[f(\hat{n})\bar{\rho}_2], \quad (43)$$

where we have to assume that the function $f(x)$ converges fast enough to zero as $x \rightarrow \infty$, so that the trace exists. Thus, once again, if we are only interested in operators that depend solely on photon number, the pure-state density operators that describe the states with definite phases $\{\gamma_n\}$ cannot be distinguished from their mixed-state counterparts which arise from a randomization of these phases.

V. CONCLUDING REMARKS

The so-called quasicohherent states of Eq. (21) have been studied in the context of states that exhibit definite coherence properties, but which are not eigenstates of the annihilation operator corresponding to the original oscillator mode. The quasicohherent states are obtained by a unitary transformation of the ordinary (Glauber) coherent states, which themselves are eigenstates of this annihilation operator. The corresponding unitary operator, given by Eq. (5), introduces a definite set of relative phases between the number eigenstates with reference to which it is constructed. We have shown in this paper how the quasicohherent states may be represented as eigenstates of a new annihilation operator, which obeys the usual oscillator (or bosonic) canonical commutation relations, and which is correspondingly found by a uni-

tary transformation of the original annihilation operator with the same transformation operator. More generally, this unitary transformation may be usefully viewed as defining a unitary isomorphism of the Hilbert space onto itself. Two examples of this mapping have been studied here. They have been chosen to illustrate both phase coherence and phase incoherence.

Of particular interest in this regard is the case where the phases $\{\gamma_n\}$ are randomized. In many practical situations, the phase of the radiation emitted from the source is subject to such stochastic fluctuations. Furthermore, in many experiments the signal is received by photodetectors that are phase insensitive. Thus, we have seen that such detectors, which are only able to measure the diagonal correlations $\langle (a_0^\dagger)^n a_0^n \rangle$, are quite insensitive to the phases $\{\gamma_n\}$ of our quasicohherent states. Their phase coherence is sampled only by measurements on the off-diagonal correlations which change particle number. In particular, the random-phase quasicohherent states are quite indistinguishable from the standard Glauber coherent states by such detectors.

For most present experimental situations, such random-phase quasicohherent states would appear to possess all of the usual properties of coherent states. Yet, as we have seen, they also possess distinctly incoherent properties as mixed states with a Poissonian distribution in terms of the occupation-number representation. It is this simultaneous exhibition of both coherent and incoherent properties by the most general quasicohherent states discussed here which endows them with such interesting properties. We have already discussed some of their practical uses here. It is our belief that many other similar applications will be found.

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APPENDIX

In this appendix we consider the particular case of the unitary transformation operator of Eq. (5) where the phases are given by

$$\gamma_n \rightarrow \phi_n \equiv \pi \delta_{n0}. \quad (\text{A1})$$

Completeness of the number eigenstates then yields the corresponding relation

$$U(\Phi) = 1 - 2|0\rangle\langle 0|. \quad (\text{A2})$$

The expectation value of the position operator $\hat{x}_0$ of Eq. (14a) in the quasicohherent state $|A; \Phi\rangle$,

$$\begin{aligned} \langle \hat{x}_0 \rangle_\Phi &\equiv \langle A; \Phi | \hat{x}_0 | A; \Phi \rangle \\ &= \langle A | U^\dagger(\Phi) \hat{x}_0 U(\Phi) | A \rangle, \end{aligned} \quad (\text{A3})$$

may now readily be evaluated using the specific representations of Eqs. (14a) and (A2) and the defining relation of Eq. (19). We find

$$\langle \hat{x}_0 \rangle_\Phi = 2^{-1/2} (A + A^*) (1 - 2e^{-|A|^2}). \quad (\text{A4})$$

We similarly find that the expectation value of the square of the position operator, $\hat{x}_0^2$, may be expressed as

$$\langle \hat{x}_0^2 \rangle_\Phi = \frac{1}{2} + \frac{1}{2} (A + A^*)^2 - (A^2 + A^{*2}) e^{-|A|^2}. \quad (\text{A5})$$

Comparable expressions to those in Eqs. (A4) and (A5) for the position operator $\hat{x}_0$ are readily also derived for the momentum operator $\hat{p}_0$ given in Eq. (14a). We find

$$\langle \hat{p}_0 \rangle_\Phi = 2^{-1/2} i (A^* - A) (1 - 2e^{-|A|^2}), \quad (\text{A6})$$

and

$$\langle \hat{p}_0^2 \rangle_\Phi = \frac{1}{2} - \frac{1}{2} (A^* - A)^2 + (A^2 + A^{*2}) e^{-|A|^2}. \quad (\text{A7})$$

If we define Δx_0 and Δp_0 in the usual way as

$$\begin{aligned} (\Delta x_0)^2 &= [(\Delta x_0)_\Phi]^2 \equiv \langle \hat{x}_0^2 \rangle_\Phi - (\langle \hat{x}_0 \rangle_\Phi)^2, \\ (\Delta p_0)^2 &= [(\Delta p_0)_\Phi]^2 \equiv \langle \hat{p}_0^2 \rangle_\Phi - (\langle \hat{p}_0 \rangle_\Phi)^2, \end{aligned} \quad (\text{A8})$$

then Eqs. (A4)–(A7) immediately yield the expressions

$$\begin{aligned} (\Delta x_0)^2 &= \frac{1}{2} + [2(A + A^*)^2 - A^2 - (A^*)^2] e^{-|A|^2} \\ &\quad - 2(A + A^*)^2 e^{-2|A|^2}, \\ (\Delta p_0)^2 &= \frac{1}{2} + [A^2 + (A^*)^2 - 2(A - A^*)^2] e^{-|A|^2} \\ &\quad + 2(A - A^*)^2 e^{-2|A|^2}. \end{aligned} \quad (\text{A9})$$

If we write $A = u + iv$, with

$$u \equiv \text{Re } A, \quad v \equiv \text{Im } A, \quad (\text{A10})$$

it is trivial to rewrite Eq. (A9) in the form

$$\begin{aligned} (\Delta x_0)^2 &= \frac{1}{2} + (6u^2 + 2v^2) e^{-|A|^2} - 8u^2 e^{-2|A|^2}, \\ (\Delta p_0)^2 &= \frac{1}{2} + (2u^2 + 6v^2) e^{-|A|^2} - 8v^2 e^{-2|A|^2}. \end{aligned} \quad (\text{A11})$$

It is then a relatively easy matter to write their product in the form,

$$\begin{aligned} (\Delta x_0 \Delta p_0)^2 &= \frac{1}{4} \\ &= +4|A|^2 e^{-|A|^2} (1 - e^{-|A|^2} - |A|^2 e^{-|A|^2}) \\ &\quad + 16|A|^4 e^{-2|A|^2} (1 - e^{-|A|^2}) \\ &\quad + 16u^2 v^2 e^{-2|A|^2} (1 - 2e^{-|A|^2})^2. \end{aligned} \quad (\text{A12})$$

The second and third terms on the right-hand side of Eq. (A12) are clearly positive for all nonzero values of A . If we define

$$f(x) \equiv 1 - e^{-x} - x e^{-x}, \quad (\text{A13})$$

it is trivial to prove that

$$\begin{aligned} f(0) &= 0, \\ f'(x) &= x e^{-x} > 0, \quad x > 0, \end{aligned} \quad (\text{A14})$$

and hence that $f(x) > 0$ for all positive values of x . Thus the first term on the right-hand side of Eq. (A12) is also positive for all nonzero values of A . We thus find the desired result,

$$(\Delta x_0 \Delta p_0)^2 \geq \frac{1}{4}, \quad (\text{A15})$$

where the equality holds only for $A = 0$.

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