Online Adaptive Optimal Control for Continuous-Time Markov Jump Linear Systems Using A Novel Policy Iteration Algorithm*

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Abstract: This paper studies the online adaptive optimal control problems for a class of continuous-time Markov Jump Linear Systems (MJLSs) based on a novel policy iteration algorithm. By utilizing a new decoupling technique named Subsystems Transformation, we re-construct the MJLSs and a set of new coupled systems composed of N subsystems are obtained. The online policy iteration algorithm was used to solve the coupled algebraic matrix Riccati equations with partial knowledge regarding to the system dynamics, and the relevant optimal controllers equivalent to the investigated MJLSs are designed. Moreover, the convergence of the novel policy iteration algorithm is also established. Finally, a simulation example is given to illustrate the effectiveness and applicability of the proposed approach.

Keywords: Markov Jump Linear Systems (MJLSs); Adaptive Optimal Control; Subsystems Transformation; Policy Iteration (PI); Online.

1 Introduction

Over the past few decades, Reinforcement Learning (RL) and Approximate/Adaptive Dynamic Programming (ADP) have been widely applied for solving the optimal control problems for linear/nonlinear systems with unknown or uncertain parametric models [41]. RL means finding an admissible control policy, i.e., learning the parameters of a controller mapping between the system states and the control signal, such that to maximize a numerical reward signal [38]. It is noted that the integral over time of the reward signal can be viewed as the value/cost function to be maximized/minimized in an optimal control framework. From the viewpoint of control engineering, RL algorithms can be viewed as a class of adaptive controllers which solve the optimal control

*This work was supported in part by the National Natural Science Foundation of P. R. China (Grant No. 61203051), the Key Program of Natural Science Foundation of Education Department of Anhui Province (Grant No. KJ2012A014) and the Joint Specialized Research Fund for the Doctoral Program of Higher Education (Grant No. 20123401120010).

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problem based on reward information which gives information on the performance of a given controller [37][42].

One class of RL algorithms is Policy Iteration (PI) which was first introduced in the computational intelligence community in the framework of stochastic decision theory [43]. PI algorithms iterate on two-steps named *policy evaluation*, in which the cost (i.e. predefined *value function*) associated with an admissible control policy is evaluated, and *policy improvement*, when the policy is updated such that will have a lower associated cost. The two-steps are forced to repeat until the policy improvement step no longer changes the present policy, thus convergence to the optimal controller is achieved [39][40].

Recently, a new RL method, namely Integral Reinforcement Learning (IRL), was provided to learn solution online to optimal control problem without requiring knowledge of the system drift dynamics. In [6], the authors employed this approach to solve the optimal control problem for continuous-time linear time-invariant systems using partial knowledge regarding to the system dynamics [6]. Thereafter, the approach was then extended to similar systems with completely unknown system dynamics in [7]. For an overview of contributions for linear systems, see [21][5]. The extension to the nonlinear field was given by [29][28][15][27][26].

In the work of [29], the authors studied an online algorithm based on PI for learning the continuous-time optimal control solution with infinite horizon cost with known dynamics. For the infinite-horizon optimal control problem for continuous-time uncertain nonlinear systems, Bhasin et al. have proposed an online adaptive reinforcement learning-based solution in [27]. The most important advantage of using ADP technique to get the solution of the optimal controller associated with linear/nonlinear systems is that this approach can be implemented online effective with partial system dynamics.

On another research front line, MJLSs have been widely studied by many researchers since the pioneering work of Krasovskii and Lidskii on quadratic control [12]; see, e.g., [13][8][23][22][25]. As an active research topic of control problem for MJLSs, the optimal controller design problems have also regained increasing interest, as evidenced by several publications for related results [24][31]. In fact, the problem to solve the jump linear quadratic (JLQ) optimal controller (i.e. H_2 control [32]) for MJLSs is equivalent to finding the unique positive definite sequence solutions of a set of coupled algebraic Riccati equations [8][10]. In [10], Borno constructed a sequence of Lyapunov algebraic equations whose solutions converge to the solutions of the coupled algebraic Riccati equations of the optimal control problem for MJLSs. The algorithm was named as parallel algorithm, and similar results have been presented in [1][2][4]. Thereafter, Zhou et al. developed the gradient-based algorithm to solve the coupled discrete-time Markovian jump Lyapunov matrix equations [11]; Li et al. proposed a positive operator based iteration algorithms for solving Lyapunov equations for Itô stochastic systems with Markovian jumps [19]. It is easy to find that all of the aforementioned approaches are off-line algorithms for solving the coupled algebraic Riccati equations or coupled algebraic Lyapunov equations. Moreover, they also require the exactly information of the system dynamics.

However, to the best of our knowledge, the relevant research focusing on the adaptive control problem for MJLSs by means of ADP technique has not fully been researched in the literature, saying nothing of the online optimal controller design. In fact, different with regular linear/nonlinear systems, MJLSs have two components which are the jumping modes and continuous states, modeled by finite state Markov chains and differential equations. The variables not only associate with continuous-time domain but also depend on the probability space, thus the ADP algorithm proposed in regular linear/nonlinear systems can not be used for MJLSs by a simple extension.

In this paper, our aim is to develop a new iterative algorithm based on ADP for solving the JLQ optimal control problem for continuous-time MJLSs. The novel algorithm solves the optimal control problem for continuous-time

MJLSs only use partial knowledge regarding to the system dynamics. Inspired by the parallel algorithm [10][35][4] and a new decoupling method named Subsystems Transformation, we re-construct a set of new continuous-time Markov jump linear subsystems and separate the coupled relation from the coupled algebraic matrix Riccati equations based on the jumping parameters. We also show that the optimal controllers of the constructed subsystems with the explicit parameters equal to the investigated continuous-time MJLSs. Subsequently, based on the framework of two-step policy iteration, a novel iterative algorithm which can be implementation online are studied and the convergence of the algorithm is also proved. In order to illustrate the proposed results, we provide a fourth-order jump linear models, presented by [2][4]. The obtained simulation results demonstrate the feasibility and validity of the new approach.

This paper is organized as follows: The problem formulation and preliminaries are given in Section 2. A new decoupling technique named Subsystems Transformation is proposed in Subsection 3.1. The novel iterative algorithm is presented in Subsection 3.2 and the proof of convergence is shown in Subsection 3.3. In Section 4, a simulation example is given to demonstrate the capabilities of the proposed novel policy iteration scheme. Conclusions are presented in Section 5.

Notations. Throughout this paper, we use the following notations: $\mathbf{R}^n$ and $\mathbf{R}^{n \times m}$ stand for an n -dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively; $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximal and minimal eigenvalues of a positive-definite matrix P ; $E\{*\}$ denotes the statistical expectation of the stochastic process or vector; $P < 0$ or $P > 0$ means matrix P is negative or positive-definite; I and 0 are respectively the unit and the zero matrices with appropriate dimensions.

2 Problem Formulation and Preliminaries

2.1 Continuous-Time MJLSs and JLQ Optimal Problem

Consider the following continuous-time linear systems with Markovian jumps in the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ [12][9]:

$$\dot{x}(t) = A(r(t))x(t) + B(r(t))u(t), \quad (1)$$

where Ω is the sample space, $\mathcal{F}$ is the algebra of events, and $\mathcal{P}$ is the probability measure defined on $\mathcal{F}$; $x(t) \in \mathbf{R}^n$ is the state vector, and $u(t) \in \mathbf{R}^m$ is the controlled input; $A(r(t))$ and $B(r(t))$ are known mode-dependent matrices with appropriate dimensions; $\{r(t), t \geq 0\}$ is a continuous-time discrete-state Markovian process with left continuous trajectories [13] and taking values from a finite set $\mathbf{S} = \{1, 2, \dots, N\}$ with transition probabilities given by

$$P_r\{r(t + \Delta t) = j | r(t) = i\} = \begin{cases} \pi_{ij}\Delta t + o(\Delta t) & j \neq i, \\ 1 + \pi_{ii}\Delta t + o(\Delta t) & j = i, \end{cases} \quad (2)$$

where $\Delta t > 0$, $\lim_{\Delta t \rightarrow 0} \frac{o(\Delta t)}{\Delta t} = 0$, and $\pi_{ij} \geq 0$ ($j \neq i$) is the transition rate from mode i at time t to mode j at time $t + \Delta t$ and

$$\pi_{ii} = - \sum_{j \in \mathbf{S}, j \neq i} \pi_{ij}. \quad (3)$$

For convenience, we denote $A(r(t))$, $B(r(t))$, $Q(r(t))$, $R(r(t))$ as A_i , B_i , Q_i , R_i respectively with $r(t) = i$, $i \in \mathbf{S}$.

Subject to system (1)-(2) which is assumed to be stochastically stabilizable [8], our objective is to find a set of mode-dependent control policy $\psi(t, x(t), i) \in \Psi$, i.e.

$$u_i(t) \triangleq \psi(t, x(t), i), \psi : [t_0, \infty) \times \mathbf{R}^n \times \mathbf{S} \rightarrow \mathbf{R}^m, \quad (4)$$

which minimizes the following performance criterion

$$V(t, x(t), r(t) = i, u_i) = \mathbb{E} \left\{ \int_{t_0}^{\infty} [x^T(t) Q_i x(t) + u^T(t) R_i u(t)] dt | x(t_0), r(t_0), \forall t_0 \geq 0 \right\} \quad (5)$$

where the mode-dependent weighting matrices Q_i and R_i are real valued with $Q_i^T = Q_i \geq 0$, $R_i^T = R_i > 0$ ($\forall i \in \mathbf{S}$). Note that the mode-dependent control policy $u_i(t)$ must not only stochastic stabilize the MJLSs on $\mathbf{R}^m$ but also guarantee that (5) is finite for all $i \in \mathbf{S}$. That is to say, mode-dependent control policy $u_i(t)$ must be admissible for all $i \in \mathbf{S}$. To this end, the definition of admissible control policy for the regular linear/nonlinear [33][18] is generalized to MJLSs as follows.

Definition 1. (*Mode-Dependent Admissible Control Policy*) For all $i \in \mathbf{S}$, the mode-dependent control policy $u_i(t) = \psi(t, x(t), i)$ with $\psi(t_0, x(t_0), r(t_0)) \equiv 0$ is said to be mode-dependent admissible with respect to (5) on Ψ , if $\psi(t, x(t), i)$ is continuous on a compact set $\Psi \subseteq \mathbf{R}^m$, stochastic stabilizes the MJLSs (1)-(2), and yields a **finite** mode-dependent value function $V_i(x(t), u_i(t)) = \mathbb{E}\{V(t, x(t), r(t) = i, u_i(t))\} | t_0, x(t_0), r(t_0)\} |_{\psi(t, x(t), i)}$.

Then, the JLQ optimal control problem for continuous-time MJLSs (1)-(2) is equivalent to find the following a set of mode-dependent admissible control policy:

$$u_i(t) = u_i^*(x(t)) \triangleq \arg \min_{u_i \in \Psi} V_i(x(t), u_i(t)), \forall i \in \mathbf{S}. \quad (6)$$

2.2 Preliminaries

Lemma 1. (*Ji & Chizeck [8]; Costa, do Val & Geromel [32]*) For JLQ optimal problem (1)-(6), the mode-dependent optimal steady state control policy is

$$u_i(t) = -K_i x(t) = -R_i^{-1} B_i^T P_i x(t), \quad (7)$$

where P_i ($i \in \mathbf{S}$) is the set of unique, positive solutions of the following coupled algebraic matrix Riccati equations

$$A_i^T P_i + P_i A_i - P_i B_i R_i^{-1} B_i^T P_i + \sum_{j=1}^N \pi_{ij} P_j + Q_i = 0. \quad (8)$$

And the controlled system is mean square stable in the sense that

$$\mathbb{E}\{\|x(t)\|^2 | x_0, r_0\} \rightarrow 0 \quad (\text{as } t \rightarrow \infty). \quad (9)$$

It is seen from Lemma 1 that the optimal control problem for MJLSs is reduced to finding a set of stochastically stabilizing solutions P_i of the coupled algebraic matrix Riccati equations (8). Based on the Newton-Raphson-Kantorovich method [3] or Kleinman's method [14], the solution of (7) can be obtained [4].

In [1], Borno used parallel algorithm to solve continuous-time MJLSs of coupled Lyapunov equations. Thereafter, in [10], Gajic and Borno studied a new computation iteration method to solve the optimal control problem for continuous-time MJLSs based on parallel algorithm (the algorithm is introduced in Lemma 2). In fact, the biggest advantage of utilizing the parallel algorithm to solve the above problems is that, both the coupled algebraic Riccati equations and coupled algebraic Lyapunov equations are reduced to a class of decoupled algebraic Lyapunov equations.

Assumption 1. (Gajic & Borno [2]) The triples $(A_i, B_i, \sqrt{Q_i}), i \in \mathbf{S}$ are stabilizable-detectable, and

$$\max_{i \in \mathbf{S}} \left\{ \inf_{\Gamma_i} \left| \lambda_{\max} \left[\int_0^\infty e^{\Upsilon_i^T t} t \times e^{\Upsilon_i t} dt \right] \right| \right\} < 1, \quad (10)$$

where $\Upsilon_i = A_i + B_i \Gamma_i + \frac{\pi_{ii}}{2} I$, Γ_i are arbitrary real matrices.

Condition (10) is a consequence of the fixed point iterations [34] used to establish the existence and uniqueness of the solutions of the proposed parallel iteration algorithm in the following Lemma 2. Actually, Assumption 1 guarantees stabilizability of the matrices $(A_i + \frac{\pi_{ii}}{2} I)$ by the optimal closed-loop feedback gain. In [35], Gajic and Losada have pointed out that the Assumption 1 is equivalent to the following assumption.

Assumption 2. (Gajic & Losada [35]; Borno [1]) The continuous-time MJLSs (1)-(2) are stochastically stabilizable and the pairs $(A_i, \sqrt{Q_i}), i \in \mathbf{S}$, are deterministically detectable.

Lemma 2. (Gajic & Borno [2]) Let Assumption 1 (or Assumption 2) hold. Then, the solution of the following constructed sequence of Lyapunov algebraic equations converges to the solutions of the coupled algebraic Riccati equations (8):

$$(\mathbf{A}_i - B_i R_i^{-1} B_i^T P_i(k))^T P_i(k+1) + P_i(k+1) (\mathbf{A}_i - B_i R_i^{-1} B_i^T P_i(k)) = -P_i(k) B_i R_i^{-1} B_i^T P_i(k) - Q_i(k) \quad (11)$$

with stabilizing $P_i(0) \geq 0$, $i \in \mathbf{S}$ and where

$$\begin{cases} \mathbf{A}_i = A_i + \frac{\pi_{ii}}{2} I, \\ Q_i(k) = Q_i + \sum_{j=1, j \neq i}^N \pi_{ij} P_j(k) \geq 0. \end{cases} \quad (12)$$

Remark 1. It is easy to see that at every iterative step of the algorithm (11), N represents the continuous-time Lyapunov equations with the form

$$AX + XA^T = -Q, \quad Q > 0, \quad (13)$$

to be solved. At present, many numerically algorithms for solving the aforementioned equations have been imbedded in the MATLAB functions, such as, `lyap` and `are` [16][17].

As having been pointed out in [18][6][7][5], all of the above mentioned computational iteration algorithms are offline methods for obtaining the solution of the coupled algebraic matrix Riccati equations (8). In these algorithms, the exact knowledge of system coefficient matrices A_i ($\forall i \in \mathbf{S}$) are required.

However, in many practical applications, the real system dynamics are always unknown, that is, the system is completely model-free or partially model-free. To overcome this limitation, some adaptive control techniques have been proposed by many researchers. For a continuous-time linear time-invariant dynamical system, a new policy iteration algorithm was presented for online finding the solutions of the optimal control problem without using the system matrix by Vrabie et al. [6]. This new policy iteration algorithm is in the following.

Lemma 3. (Vrabie et al. [6]) For a linear time-invariant system described by: $\dot{x}(t) = Ax(t) + Bu(t)$, the solution of the infinite horizon quadratic optimal control problem is given by $u(t) = -Kx(t)$ with

$$K = R^{-1} B^T P, \quad (14)$$

where the matrix P is the unique real symmetric positive definite solution of the Lyapunov matrix equation

$$[A - BK]^T P + P[A - BK] = -[K^T R K + Q]. \quad (15)$$

Then, denoting $x(t)$ with x_t , the following two-step policy iteration algorithm can efficiently solve the above Lyapunov matrix equation online with an initial stabilizing controller gain $K(0)$:

Policy Evaluation:

$$x_t^T P(k) x_t = \int_t^{t+T} x_\tau^T [Q + K(k)^T R K(k)] x_\tau d\tau + x_{t+T}^T P(k) x_{t+T}, \quad (16)$$

Policy Improvement:

$$K(k+1) = R^{-1} B^T P(k). \quad (17)$$

3 Main Results

In this paper, we propose a novel policy iteration algorithm that will online solve the optimal control problem for a class of continuous-time MJLSs only using partial knowledge regarding to the system dynamics. The new algorithm is based on parallel algorithm [1][10] and subsystems transformation [30]. At each iteration step of the new algorithm, N reconstructed parallel subsystems all required to execute a two-step iteration namely the Critic update and the Actor update. Proof of convergence of the algorithm is also provided in this section.

3.1 Subsystems Transformation

In fact, the exact information of the MJLSs coefficient matrices A_i , B_i and the transition matrix elements π_{ij} are all required in solving the coupled algebraic matrix Riccati equations (8). When we directly observe the states online, the information regarding π_{ij} are not contained in the observed states. Therefore, the existing online policy iteration schemes cannot be applied for solving the JLQ optimal control problems of the MJLSs. To overcome this shortcoming, we give the following Theorem 1. By reconstructing the continuous-time MJLSs, we separate the information of π_{ij} from the MJLSs. And the information regarding the system coefficient matrices A_i are embedded in the reconstructed subsystems states which can be observed online.

Theorem 1. (Subsystems Transformation) Let $P_i(k)$ be the solutions of the following i th continuous-time linear reconstructed subsystems

$$\dot{x}_{i,t} = \mathbf{A}_i x_{i,t} + B_i u_{i,t} \quad (i = 1, 2, \dots, N), \quad (18)$$

at k th Kleinman iteration step with the parameter Q_i that satisfy:

$$Q_i(k) = Q_i + \sum_{j=1, j \neq i}^N \pi_{ij} P_j(k-1). \quad (19)$$

Then, the following property hold:

$$\lim_{k \rightarrow \infty} P_i(k) = P_i^*, \quad \forall i \in \mathbf{S} \quad (20)$$

where the P_i^* are the solutions of (8).

Proof. For the constructed continuous-time linear system (18), the solution of the Kleinman iteration with (19) equals the solution of the following equation at each iteration step [14][7]:

$$[\mathbf{A}_i - B_i K_i(k)]^T P_i(k) + P_i(k) [\mathbf{A}_i - B_i K_i(k)] + Q_i(k) + K_i^T(k) R_i K_i(k) = 0, \quad (21)$$

where $K_i(k) = R_i^{-1} B_i^T P_i(k-1)$.

It is easy to see that (21) is equivalent to (11). Hence, according to Lemma 2, we can obtain (20). This completes the proof. $\square$

Remark 2. By reconstructing the continuous-time MJLSs, the exact information regarding A_i ($\forall i \in \mathbf{S}$) are contained in system matrices $\mathbf{A}_i$ of N parallel continuous-time linear subsystems in form of (18).

3.2 A novel policy iteration algorithm

By utilizing the parallel algorithm and subsystems transformation, the continuous-time MJLSs (1)-(3) are decomposed into N interrelated continuous-time linear time-invariant subsystems in form of (18), and each optimal controller can be obtained by the iteration algorithm (11)-(12). In Remark 1, we have pointed out that, at every iteration step, the iteration solution can be obtained from a standard Lyapunov equation in form of (13). Combining Lemma 3, we devise the following new policy iteration algorithm:

Policy Evaluation:

$$\left\{ \begin{array}{l} x_{1,t}^T P_1(k) x_{1,t} \\ \quad = \int_t^{t+T} x_{1,\tau}^T [Q_1(k) + K_1^T(k) R_1 K_1(k)] x_{1,\tau} d\tau \\ \quad \quad + x_{1,t+T}^T P_1(k) x_{1,t+T}, \\ \quad \quad \quad \vdots \\ x_{i,t}^T P_i(k) x_{i,t} \\ \quad = \int_t^{t+T} x_{i,\tau}^T [Q_i(k) + K_i^T(k) R_i K_i(k)] x_{i,\tau} d\tau \\ \quad \quad + x_{i,t+T}^T P_i(k) x_{i,t+T}, \\ \quad \quad \quad \vdots \\ x_{N,t}^T P_N(k) x_{N,t} \\ \quad = \int_t^{t+T} x_{N,\tau}^T [Q_N(k) + K_N^T(k) R_N K_N(k)] x_{N,\tau} d\tau \\ \quad \quad + x_{N,t+T}^T P_N(k) x_{N,t+T}, \end{array} \right. \quad (22)$$

Policy Improvement:

$$Q_i(k+1) = Q_i + \sum_{j=1, j \neq i}^N \pi_{ij} P_j(k), \quad (23)$$

$$K_i(k+1) = R_i^{-1} B_i^T P_i(k). \quad (24)$$

with stabilizing sequences $\{P_i(0)\} \geq 0$ ($\forall i \in \mathbf{S}$).

Note that the system matrices A_i and the transfer matrix elements π_{ii} are not included in implementing this algorithm. But in the policy improvement equations (23)-(24), the exact knowledge of the system matrices B_i and the transfer matrix elements π_{ij} are still required for the iterations. Therefore, this proposed policy iteration algorithm is a partially mode-free adaptive dynamic programming algorithm.

Remark 3. The information regarding to the system matrices A_i are embedded in the subsystems state trajectories $x_{i,t}$ which can be observed online. According to (3), it is easy to see that the information of π_{ii} is also included in π_{ij} by the proposed policy iteration scheme.

Remark 4. In the policy update equation (23), it contains all the of coupling relation information of the coupled algebraic matrix Riccati equations (8). Thus, the exact knowledge of state transfer matrix $\{\pi_{ij}\}$ is required in either off-line algorithms (such as, [1][2][19]) or online algorithm (22)-(23).

3.3 Proof of convergence and online implementation

Theorem 2. Under the assumptions of the system matrices A_i are stabilizable, the condition equation (10) is hold for arbitrary real matrices Γ_i , and the new policy iteration (22)-(24) converges to the optimal controller given by (7), wherein the matrices P_i is the unique real symmetric positive definite solution of the coupled algebraic matrix Riccati equations (8).

Proof. For any i th subsystem described as

$$\dot{x}_{i,t} = \mathbf{A}_i x_{i,t} + B_i u_{i,t}. \quad (25)$$

Obviously, the system is a continuous-time linear time-variant dynamical one. Along with the state trajectories of the dynamical system (25), we operate the new adaptive iteration algorithm (22)-(24). Letting $k = 0$ and giving initial stabilizing sequences $\{P_i(0)\}$ ($i = 1, 2, \dots, N$), we start the policy iteration scheme by using policy improvement (23)-(24), then one can obtain the following solutions:

$$Q_i(1) = Q_i + \sum_{j=1, j \neq i}^N \pi_{ij} P_j(0), \quad (26)$$

$$K_i(1) = R_i^{-1} B_i^T P_i(0). \quad (27)$$

Substituting (26)-(27) into the policy evaluation (22), then it yields the solutions of $\{P_i(1)\}$.

According to Lemma 3, the sequence $\{P_i(1)\}$ is also the solution of the underlying Lyapunov equation

$$[\mathbf{A}_i - B_i K_i(1)]^T P_i(1) + P_i(1) [\mathbf{A}_i - B_i K_i(1)] = -[K_i^T(1) R_i K_i(1) + Q_i(1)] \quad (28)$$

It is easy to see that (28) equals to (11) while $k = 0$. Performing the same procedure, we can get the each iteration step solution $P_i(k)$ of the iteration algorithm (22)-(24), which is equivalent to the k th iteration solution of the iteration algorithm (11).

From Lemma 2, we know that the solutions of computational algorithm (11) converge to the solutions of coupled algebraic matrix Riccati equations (8). This implies that the solutions of iteration algorithm (22)-(24) converge to the solutions of (8). This completes the proof. $\square$

Remark 5. In [4], the author has pointed out that the iteration algorithm (10) equals to the standard Newton's algorithm [14][3][20] with the form:

$$P_i(k+1) = P_i(k) - (\mathcal{R}'_{P_i(k)})^{-1} \mathcal{R}(P_i(k)), \quad (29)$$

where

$$\mathcal{R}(P_i(k)) = A_i^T P_i(k) + P_i(k) A_i - P_i(k) B_i R_i^{-1} B_i^T P_i(k) + \sum_{j=1}^N \pi_{ij} P_j(k) + Q_i, \quad (30)$$

and the notation $\mathcal{R}'_{P_i(k)}$ means the Fréchet derivative of $\mathcal{R}$ at point $P_i(k)$. Therefore, according to Theorem 2, the proposed new iteration algorithm (22)-(24) is also equivalent to the Newton's method (29)-(30).

Next, we derive the implement approach [6][5] for the new online iteration algorithm (22)-(24). It is necessary to point that the states of N parallel subsystems are simultaneously online measured at each iteration step. To this end, we rewrite the term $x_{i,t}^T P_i(k) x_{i,t}$ in (22) as

$$x_{i,t}^T P_i(k) x_{i,t} = [\hat{p}_i(k)]^T \hat{x}_i(t), \quad (31)$$

where $\hat{x}_i(t)$ denotes the Kronecker product quadratic polynomial basis vector with the elements $\{x_{i,h}(t)x_{i,l}(t)\}$ ($h = 1, 2, \dots, n; l = h, h+1, \dots, n$), that is,

$$\hat{x}_i = [x_{i,1}^2, x_{i,1}x_{i,2}, \dots, x_{i,1}x_{i,n}, x_{i,2}^2, x_{i,2}x_{i,3}, \dots, x_{i,n}^2]; \quad (32)$$

and $\hat{p}_i(k)$ denotes the a column vector by stacking the elements of the diagonal and upper triangular part of the symmetric matrix P_i into a vector where the off-diagonal elements are taken as $2p_{i,hl}$, that is,

$$\hat{p}_i = [p_{i,11}, 2p_{i,12}, \dots, 2p_{i,1n}, p_{i,22}, 2p_{i,23}, \dots, p_{i,nn}]^T. \quad (33)$$

Then, (22) can be written as

$$[\hat{p}_i(k)]^T [\hat{x}_{i,t} - \hat{x}_{i,t+T}] = \int_t^{t+T} x_{i,\tau}^T [Q_i(k) + K_i^T(k) R_i K_i(k)] x_{i,\tau} d\tau, \quad (34)$$

In (34), for the i th subsystem, at k th iteration step, the regression vector $\hat{x}_{i,t} - \hat{x}_{i,t+T}$ and $\mathcal{R}(P_i(k))$ can be obtained by online measuring the trajectories of the i th subsystem. And $\hat{p}_i(k)$ is a vector of unknown parameters which can be obtained from (34). Notice that the symmetric matrix P_i has $n(n+1)/2$ unknown independent elements. Hence, we need at least M ($\geq n(n+1)/2$) independent equations to get $P_i(k)$. Therefore, for any i th system, we should sample M state vector in each time interval T . Then, the sequences $\{P_i(k)\}$ can be found in the following parallel least-square sense:

$$\begin{cases} \hat{p}_1(k) = [X_1 X_1^T]^{-1} X_1 Y_1, \\ \vdots \\ \hat{p}_i(k) = [X_i X_i^T]^{-1} X_i Y_i, \\ \vdots \\ \hat{p}_N(k) = [X_N X_N^T]^{-1} X_N Y_N, \end{cases} \quad (35)$$

where

$$X_i = \begin{bmatrix} \hat{x}_i(t) - \hat{x}_i(t + \delta t) & \hat{x}_i(t + \delta t) - \hat{x}_i(t + 2\delta t) & \dots & \hat{x}_i(t + (M-1)\delta t) - \hat{x}_i(t + M\delta t) \end{bmatrix}, \quad (36)$$

$$Y_i = \begin{bmatrix} y_i^{(0)}(k) & y_i^{(1)}(k) & \dots & y_i^{(M-1)}(k) \end{bmatrix}^T, \quad (37)$$

with $\delta t = T/M$ and $y_i^{(r)}(k) = \int_{t+r\delta t}^{t+(r+1)\delta t} x_{i,\tau}^T [Q_i(k) + K_i^T(k) R_i K_i(k)] x_{i,\tau} d\tau$ ($r = 0, 1, \dots, M-1$).

Remark 6. In order to have persistent excitation property of the data necessary to obtain the solution given by (35), one has to continue reading the states information from overall N subsystems at each iterative step until the solution of the parallel least-squares sense is feasible. Here, we further point out that, before finding the ultimate optimal controller, the controller solutions obtained at each step by (24) are all suboptimal state-feedback controllers for the concerned continuous-time MJLSs. Namely, these obtained suboptimal state-feedback controllers be able to control the states of the MJLSs reach to zeros in finite time interval but the performance criterions computed by using the information of the closed-loop systems that are controlled by these suboptimal controllers are not the minimal one.

Figure 1: Subsystems Transformation and Policy Iteration.

Remark 7. It is necessary to point out that there is no significant relationship between the each iteration time interval T in (22) and the jumping time constant Δt of the continuous-time MJLSs (1)-(3). We need to point out that the continuous-time MJLSs (1)-(3) can be regarded as a sequence of subsystems in form of (18). And the set of decoupled algebraic Lyapunov equations (11) converges to the coupled algebraic matrix Riccati equations (8). Using parallel algorithm and subsystems transformation, the coupled relation of (8) is expressed in the **explicit formula** (12) or (23). Subsystems Transformation and Policy Iteration are presented in Figure 1.

Remark 8. We define the following judging condition which need to be calculated at each iteration step of the implementation of the novel policy iteration algorithm (22)-(24):

$$\max_{\forall i \in \mathbf{S}} (\|P_i(k+1) - P_i(k)\|) \leq \varepsilon, \quad (38)$$

where ε is the given error accuracy. The proposed novel policy iteration algorithm always execute until the stopping criterion (38) is satisfied. A flow chart of the novel policy iteration algorithm (22)-(24) is shown in Figure 2.

4 Simulation Example

In this section, we consider a fourth-order jump linear control problem [10][4] to illustrate the feasibility and validity of the proposed novel policy iteration algorithm. Using the proposed algorithm, the optimal control problem of MJLSs can be solved efficiently while the system matrices A_i ($i \in \mathbf{S}$) are unknown or not be utilized.

Figure 2: The flowchart of policy iteration algorithm (22)-(24).

The matrix coefficients of the Markov jump linear system (1)-(3) are

$$\begin{aligned} \Pi &= \begin{bmatrix} -2 & 2 \\ 1.5 & -1.5 \end{bmatrix}, \\ A_1 &= \begin{bmatrix} -2.1051 & -1.1648 & 0.9347 & 0.5194 \\ -0.0807 & -2.8949 & 0.3835 & 0.8310 \\ 0.6914 & 10.5940 & -36.8199 & 3.8560 \\ 1.0692 & 13.4230 & 22.1185 & -13.1801 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.7564 \\ 0.9910 \\ 9.8255 \\ 7.2266 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} -2.6430 & -1.2497 & 0.5269 & 0.6539 \\ -0.7910 & -2.8570 & 0.0920 & 0.4160 \\ 21.0357 & 22.8659 & -26.4655 & -1.7214 \\ 27.3096 & 7.8736 & -3.8604 & -29.5345 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.3653 \\ 0.2470 \\ 7.5336 \\ 6.5152 \end{bmatrix}, \\ Q_1 = Q_2 &= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R_1 = R_2 = 1. \end{aligned}$$

We set that the optimal controller is updated every 0.3sec and the two subsystems states information are conducted over each interval of 0.03sec . One thus have 10 sample data for every subsystem in each update interval. It should be noted that there are 10 independent elements in the symmetric matrices P_1 and P_2 . Hence, at each update interval 0.3sec and for every reconstructed subsystem, the 10 sample data are enough for exciting to solving parallel least-square equations (35). In addition, we set the initial states of the two subsystems as $x_1(0) = x_2(0) = \begin{bmatrix} 0 & -0.1 & 0.1 & 0 \end{bmatrix}^T$ and the value of ε in (38) is selected as 10^{-15} .

The initial matrices $P_1(0)$ and $P_2(0)$ are chosen to be $I_{4 \times 4}$. At first, by solving policy improvement (23)-(24), the corresponding matrices $Q_i(1)$ and $K_i(1)$ are obtained. Thereafter, executing the proposed novel policy iteration algorithm (22)-(24), the following P_1 and P_2 of the JLQ optimal control problem, without knowing the plant internal dynamics A_i , are found after 13 iterations:

$$P_1(13) = \begin{bmatrix} 0.2426 & 0.0675 & 0.0393 & 0.0180 \\ 0.0675 & 0.0365 & 0.0086 & 0.0067 \\ 0.0393 & 0.0086 & 0.0157 & 0.0025 \\ 0.0180 & 0.0067 & 0.0025 & 0.0017 \end{bmatrix}, \quad P_2(13) = \begin{bmatrix} 0.5054 & 0.1312 & 0.0516 & 0.0097 \\ 0.1312 & 0.0555 & 0.0135 & 0.0025 \\ 0.0516 & 0.0135 & 0.0191 & 0.0004 \\ 0.0097 & 0.0025 & 0.0004 & 0.0002 \end{bmatrix}.$$

Under the computational error 10^{-15} , the following solutions of the coupled algebraic matrix Riccati equation for the fourth-order jump linear system have been solved by using Newton's iteration approach shown in [4] and the parallel iteration algorithm proposed in [2]:

$$P_1^* = \begin{bmatrix} 0.2408 & 0.0705 & 0.0393 & 0.0182 \\ 0.0705 & 0.0308 & 0.0085 & 0.0064 \\ 0.0393 & 0.0085 & 0.0157 & 0.0025 \\ 0.0182 & 0.0064 & 0.0025 & 0.0016 \end{bmatrix}, \quad P_2^* = \begin{bmatrix} 0.5026 & 0.1343 & 0.0518 & 0.0097 \\ 0.1343 & 0.0485 & 0.0138 & 0.0026 \\ 0.0518 & 0.0138 & 0.0193 & 0.0002 \\ 0.0097 & 0.0026 & 0.0002 & 0.0003 \end{bmatrix}.$$

The simulation results are shown in Figures 3-5. In Figures 3-4, the evolution of the complete parameters of the $P_1(k)$ and $P_2(k)$ matrices are presented, respectively. It is observed that the convergence of $P_1(k)$ and

Figure 3: Evolution of the complete parameters for the P_1 matrix.

Figure 4: Evolution of the complete parameters for the $P_2(k)$ matrix.

Figure 5: The state error of $x(t) - x^*(t)$ during the simulation.

$P_2(k)$ to their optimal values in the 10^{-2} accuracy at time 3.9sec (i.e. 13th iteration). With the simulation time $t \in [0 \text{ } 10\text{sec}]$ and the jumping time $\Delta t = 0.5\text{sec}$, the responses of the state errors $x(t) - x^*(t)$ between the system controlled by our obtained controller and the system controlled by the actual optimal controller are depicted in Figure 5. One can see from Figure 5 that the state errors are in the range of 10^{-4} . This verifies that, by using the novel policy iteration algorithm in this paper, the JLQ optimal control problem for continuous-time MJLSs can be solved online efficiently without using the exact information of the continuous-time MJLSs matrices A_i .

Remark 9. *It is observed that there are some obvious irregular perturbations in the convergence processes of $P_1(k)$ (see 2.7sec $\sim$ 3.3sec in Figure 3) and $P_2(k)$ (see 2.1sec $\sim$ 3sec in Figure 4) matrices. This is because the persistently exciting condition required for solving parallel least-square sense (35) becomes very weak (perhaps even lost) along with the system state converges to almost zero. An effective method to prevent the potential perturbations is to add some small excitation signal (or some additive exploration noise) in the control input $u_i(t)$. For more details to this technique, we refer the readers to [36][7][29], and the references therein.*

5 Conclusions

In this paper, we proposed a novel online policy iteration algorithm to solve the JLQ optimal control problem for the continuous-time MJLSs without the exact knowledge of the system matrices. The algorithm was obtained by a new decoupling technique named Subsystems Transformation with a set of new continuous-time Markov jump linear subsystems. Actually, this method is equivalent to online solve the coupled algebraic matrix Riccati

equations by utilizing the sampling information of the subsystems states. The convergence of the novel policy iteration algorithm is proved. Simulation results also illustrate the effectiveness and applicability of the proposed approach.

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