Categories of Modules over Infinite Groups

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In the study of representation theory of groups, different kinds of triangulated categories arise naturally. We investigate some interesting generation properties of a range of derived categories associated to groups in Kropholler's hierarchy that need not be finite. Our investigations take us through dealing with many related cohomological properties of groups that are usually tackled with properties of various cohomological invariants. Connected to these invariants are some longstanding questions on the behaviour of certain classes of modules and certain families of groups that we also deal with. We also develop a theory around some generation operators in the standard module category that we use both to observe how generation properties travel from the module category to various derived categories and also to frame some of the other related questions that we mentioned earlier in terms of those generation operators.

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Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Acknowledgements

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Introduction

In this thesis, our principal motivation is to study various interesting properties (mostly generation properties) of a range of derived categories of modules over groups belonging to some large classes of infinite groups. The main reason behind getting interested in the derived categories associated to these groups is that in a recent paper by Mazza and Symonds [46] it has been shown that there is a way to construct well-defined stable module categories for groups not necessarily finite that satisfy certain properties, and the stable module category is quite ubiquitous in the study of representation theory of groups. For finite groups over fields, the stable module category is very well understood due to the work of Benson, Carlson, Iyengar, Krause, Rickard, etc. (one can check [14], [15], [13] for some references) who have studied numerous properties admitted by the stable module category as triangulated categories.

The infinite groups that we deal with in this thesis mostly come from a hierarchy of groups called Kropholler's hierarchy (see Definition 1.2.1). These groups and groups from some other classes that we deal with admit many interesting cohomological properties that are often expressed best in the language of a number of cohomological invariants. In fact, to figure out what groups can be used for the Mazza-Symonds construction of stable module categories that we mentioned in the previous paragraph, we usually need to check whether certain cohomological properties like admitting complete resolutions are satisfied by our candidate groups.

We start, in Chapter 2 and Chapter 3, with developing a theory of generation operators in the module category. Most of our results in Chapter 2 and Chapter 3 are fairly general in theme, so even though some of them are not made use of later, they deserve to be recorded in their own right. They involve two generation operators - $[]$ and $\langle\rangle$ (in some contexts, especially when we consider compositions of operators and treat them as words on symbols, we denote the generation operators $[]$ and $\langle\rangle$

as B and C respectively, see Definitions 2.1.1, 2.2.1 and 3.1.1). We define words on these symbols by iterations : for example, if we take $W = BCC$, then for any class of modules $\mathscr{T}$, $W(\mathscr{T}) = B(C(C(\mathscr{T})))$; the collection of all words on B and C is denoted by $W(B, C)$. The definition of $[]$ arrives straight out of resolutions, whereas $\langle\rangle$ is defined keeping some analogous notions of generation for triangulated categories in mind [56][58].

This theory of generation operators in the module category helps us lay an abstract module-theoretic framekwork of inspiration for the work that we do on generation properties of derived categories in Chapter 8. In Chapter 8, we derive many interesting generation properties of derived categories of chain complexes of modules of those groups and highlight how they can be used in deriving generation properties of the stable module categories of these groups when those stable module categories can be defined in the sense of [46].

Related to this and also related generally to the topic of cohomological properties of groups in Kropholler's hierarchy, as mentioned earlier, is the study of various invariants and related questions that we take part in mostly in Chapter 4 and to a small extent in Chapter 5 and Chapter 6. For the study of most of the invariants, when we deal with groups from Kropholler's hierarchy, we take a large class of infinite groups called type Φ groups (see Definition 1.2.6) to be the base class, and in some cases we extend results that are already known for groups in Kropholler's hierarchy with finite groups as the base class to groups in the hierarchy with all type Φ groups as the base class. Staying with this theme, in Chapter 7, we provide a miscellaneous collection of results from the literature that are known to be true for groups in Kropholler's hierarchy with the class of finite groups as the base class and extend them to Kropholler's hierarchy with type Φ groups as the base class.

We provide the following figure (see Figure 1) to better illustrate how the different chapters in this thesis are connected to each other. The lines between boxes denote connection of some kind.

- Chapter 1 is connected to all other chapters because Chapter 1 contains introductory remarks and preliminary material that are useful throughout the thesis. Connections between Chapter 1 and other chapters are shown by dashed lines.

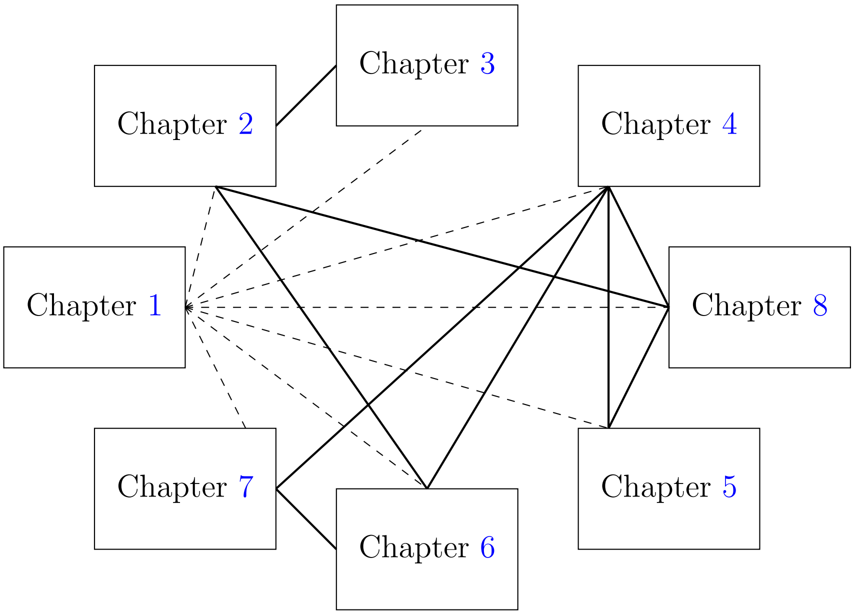

Figure 1: Connections between chapters

- Chapter 2 and Chapter 3 are connected because they are both on generation in the module category.
- Chapter 4 and Chapter 5 are connected because they both deal with cohomological and homological properties of large classes of infinite groups.
- Chapters 4 and 5 are connected to Chapter 8 because Chapter 8 deals with properties of various derived categories of chain complexes of modules over the same large classes of infinite groups.
- Chapter 2 is connected to Chapter 8 because they both deal with notions of generation that are connected to each other.
- Chapter 6 is related to Chapter 2 because the notions of generation developed in Chapter 2 are used in Chapter 6, and to Chapter 4 because thematically, like Chapter 4, Chapter 6 also deals with cohomological properties of infinite groups.
- Chapter 7 is related to Chapters 4 and 6 because both Chapter 4 and Chapter 6, on a number of occasions, prove results for groups in Kropholler's hierarchy with the base class being extended to type Φ groups from finite groups and Chapter 7 does more of that.

We shall end the Introduction with a few highlights of our original results.

Highlights of original results

As mentioned before, a lot of this thesis deals with generation concepts in both module categories and various derived categories. Another area explored in the thesis, especially in Chapters 4, 5 and 6, is the study of various cohomological invariants of groups and related questions. However, to provide highlights of results from these chapters, a lot of context on definitions is absolutely necessary, and that is why most of the main results from these chapters are highlighted in Chapter 1 once some necessary definitions have been made. Nevertheless, we highlight a lot of those results in amalgamated form at the end of the introduction.

Definition. *(this is dealt with in Chapter 2) Let R be a ring and let $\mathscr{T}$ be a class of R-modules. An R-module M is generated in 0 steps from $\mathscr{T}$ iff $M \in \mathscr{T}$, and it is generated in n steps from $\mathscr{T}$ iff there is a short exact sequence $0 \to M_2 \to M_1 \to M \to 0$ where each M_i is generated from $\mathscr{T}$ in a_i steps and $a_1 + a_2 \leq n - 1$. The class of all R-modules that can be generated in finitely many steps from $\mathscr{T}$ is denoted by $\langle \mathscr{T} \rangle$.*

On the other hand, an R-module M is said to be in $[\mathscr{T}]$ iff there is an exact sequence of R-modules $0 \to T_n \to T_{n-1} \to ... \to T_1 \to M \to 0$, for some integer n, where each $T_i \in \mathscr{T}$.

For two classes of R-modules, $\mathscr{T}$ and $\mathscr{U}$, we write $\mathscr{T} \sim \mathscr{U}$ iff $\mathscr{T}\text{-dim}(M) < \infty \Leftrightarrow \mathscr{U}\text{-dim}(M) < \infty$, for all M, i.e. iff $[\mathscr{T}] = [\mathscr{U}]$.

For word notation, as indicated earlier, we denote $B(\mathscr{T}) := [\mathscr{T}]$, and $C(\mathscr{T}) = \langle \mathscr{T} \rangle$.

Remark. (see Section 2.4) On a large number of classes that occur in the study of cohomology of groups, the two generation operators $[]$ and $\langle\rangle$ coincide. Examples of such classes include the class of all projectives or Gorenstein projectives over any ring, the class of all modules admitting complete resolutions over any ring, the class of all modules of type FP_∞ over a group algebra (over a given ring R, an R-module M is said to be of type FP_∞ iff M admits a projective resolution with finitely generated R-projectives), etc. Classes where the two generation operators coincide are defined to be *good classes* (See Definition 2.2.5 and Definition 3.3.3).

Theorem. *(= Theorem 3.3.2) For any class of R-modules, $\mathscr{T}$, the following statements are equivalent.*

a^*) For any short exact sequence of R-modules, $0 \to A \to B \to C \to 0$, if $\mathscr{T}$-dim(A), $\mathscr{T}$-dim$(B) < \infty$, then $\mathscr{T}$-dim$(C) < \infty$.

a) $B^2(\mathscr{T}) = B(\mathscr{T})$.

b) $B(\mathscr{T}) = C(\mathscr{T})$.

c) $C(\mathscr{T}) \sim \mathscr{T}$.

d) $\mathscr{T} \sim B(\mathscr{T})$.

e) For any class of R-modules, $\mathscr{U}$, if $\mathscr{U} \subseteq B(\mathscr{T})$, then $B(\mathscr{U}) \subseteq B(\mathscr{T})$.

f) For any non-empty word $W \in W(B, C)$, $W(\mathscr{T}) = B(\mathscr{T})$.

g) For any non-empty word $W \in W(B, C)$, $W(\mathscr{T}) = C(\mathscr{T})$.

Also related to the behaviour of these words of operators, we derive some results on how they behave in sequences. The following is a highlight of one of these results.

Theorem. *(see Theorem 3.4.2) For any class of R-modules, $\mathscr{T}$, the following are equivalent.*

a^*) Any sequence $\{W_i(\mathscr{T})\}_{i\in\mathbb{N}}$ where, for each i, $W_i \in W(B, C)$, and as words in $W(B, C)$, $W_i \neq W_j$ when $i \neq j$, eventually stabilises.

a) Any sequence $\{W_i(\mathscr{T})\}_{i\in\mathbb{N}}$ where, for each i, $W_i \in W(B, C)$, and as words in $W(B, C)$, $W_i \neq W_j$ when $i \neq j$, eventually stabilises to $C(\mathscr{T})$.

b) The sequence $B^0(\mathscr{T})$, $B(\mathscr{T})$, $B^2(\mathscr{T})$, $B^3(\mathscr{T})$, . . . eventually stabilises.

c) The sequence in (b) stabilises to $C(\mathscr{T})$.

d) $B^n(\mathscr{T})$ is a good class for some n.

One main application of these concepts of generation is for modules over groups in Kropholler's hierarchy (see Definition 1.2.1 for a definition of the class $H\mathscr{X}$ and $H_\alpha\mathscr{X}$, for any ordinal α, for a given class of groups $\mathscr{X}$). Let, for any group Γ, a class of groups $\mathscr{X}$, and any commutative ring A, the class of all $A\Gamma$-modules induced up from $H_\alpha\mathscr{X}$-subgroups of Γ, for any ordinal α, be denoted by $\Lambda_\alpha(\Gamma, \mathscr{X})$, and we use the superscript "$\oplus$" to indicate that we are closing the class of modules under direct sums. Under the assumption of an extra finiteness condition, we give an explicit number of steps that is sufficient for the generation of modules in $\Lambda_n(\Gamma, \mathscr{X})^\oplus$ by modules in $\Lambda_m(\Gamma, \mathscr{X})^\oplus$, for non-negative integers m and n such that $m < n$ - we show that this number of steps can be taken to be a polynomial in degree $n-m$. This result is quite helpful if Γ is an $H_n\mathscr{X}$-subgroup for some integer n itself, because then $\Lambda_n(\Gamma, \mathscr{X})$ contains all modules.

Theorem. *(see Theorem 8.1.5 and Remark 8.1.7) Assume that the finitistic $\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$-dimension, i.e. the supremum over this dimension of all $A\Gamma$-modules for which this dimension is finite, is finite, and denote it by t. Then, for all $m < n$, every module in $\Lambda_n(\Gamma, \mathscr{X})^{\oplus}$ can be generated from $\Lambda_m(\Gamma, \mathscr{X})^{\oplus}$ in $(1+t)^{n-m} - 1$ steps.*

For other related results, see Chapter 8.

Moving into derived categories of chain complexes of modules over groups in Kropholler's hierarchy, we get some interesting results in the language of generation of triangulated categories. We are not going to provide definition of triangulated categories here (one can consult [51]). For any triangulated category $\mathscr{T}$ and for any class of objects $\mathscr{U} \subseteq \mathscr{T}$, we say $\mathscr{U}$ strongly generates $\mathscr{T}$ iff the smallest full triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$ is all of $\mathscr{T}$. If $\mathscr{T}$ admits arbitrary products and coproducts, then we say $\mathscr{U}$ generates (resp. cogenerates) $\mathscr{T}$ if the smallest localising (resp. colocalising) subcategory of $\mathscr{T}$ containing $\mathscr{U}$ is all of $\mathscr{T}$. Using this language, two of our most important results regarding generation of derived categories of groups in Kropholler's hierarchy are stated below.

Theorem. *(see Theorem 8.2.5) Let $\Gamma \in H_n\mathscr{X}$, for some class of groups $\mathscr{X}$ and some positive integer n, and let A be a commutative ring. Then,*

a) $\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$, treated as a class of chain complexes concentrated in degree zero, both generates and cogenerates the derived unbounded category of chain complexes of $A\Gamma$-modules, $\mathcal{D}(Mod\text{-}A\Gamma)$.

b) $\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$, treated as a class of chain complexes concentrated in degree zero, strongly generates the derived bounded category of chain complexes of $A\Gamma$-modules, $\mathcal{D}^b(Mod\text{-}A\Gamma)$

As briefly touched upon earlier, Mazza and Symonds showed in [46] that a large class of not necessarily finite groups called groups of type Φ (see Definition 1.2.6) admit a well-behaved stable module category over rings of finite global dimension that coincides with the stable module category of finite groups if the group of type Φ under consideration is a finite group. We consider this stable category under an added finiteness condition, just like, for finite groups, it is natural to study the stable module category of finitely generated modules. Our extra finiteness condition is that we consider all those modules that are of type FP_∞, i.e. those modules that admit finitely generated projective resolutions, and then we consider the smallest triangulated subcategory of the stable module category containing them. Then, we prove a generation property of this triangulated subcategory with the aid of a class of modules called "basic" modules (see Definition 8.4.20). Before we state this generation result below, we need the following definition.

Definition. *(= Definition 8.4.24) Let $\mathscr{T}$ be a triangulated category. A thick subcategory of $\mathscr{T}$ is defined to be a full triangulated subcategory of $\mathscr{T}$, $\mathscr{S}$, such that given $M, N \in \mathscr{S}$, with $M \oplus N \in \mathscr{S}$, then M, $N \in \mathscr{S}$.*

For any class of objects $\mathscr{U}$ in $\mathscr{T}$ and any object $M \in \mathscr{T}$, we say M is properly generated by $\mathscr{U}$ in $\mathscr{T}$ if M is in the smallest thick subcategory of $\mathscr{T}$ containing $\mathscr{U}$.

Theorem. *(= Theorem 8.4.26) Let A be a commutative ring of finite global dimension. Let Γ be a group locally in Kropholler's hierarchy $H\mathscr{F}$, where $\mathscr{F}$ is the class of all finite groups, such that Γ admits complete resolutions over A. Denote by $stab(A\Gamma)$ the smallest triangulated subcategory of $Stab(A\Gamma)$ containing all $A\Gamma$-modules of type FP_∞. Then, every object in $stab(A\Gamma)$ is thickly generated in $Stab(A\Gamma)$ by the class of basic $A\Gamma$-modules.*

We end with a highlight of one of our major result involving various cohomological and homological invariants of groups that are not necessarily finite. Note that this result cannot be explained without introducing all the invariants first. So, we refer to their definitions as they are made in Chapter 1.

For any group Γ and any commutative ring A (usually, we require A to be of finite global dimension, but that requirement is not necessary for the definitions of the invariants), the invariants we use are the projective dimension of a specific module denoted $B(\Gamma, A)$ (see Definition 1.3.6), the Gorenstein cohomological dimension of Γ with respect to A denoted $Gcd_A(\Gamma)$ (see Definition 1.1.5), (for the rest of the invariants, see Definitions 1.3.1 and 4.6.1) the finitistic dimension of the group ring $A\Gamma$ denoted $\operatorname{fin.dim}(A\Gamma)$, the supremum over the projective dimension of $A\Gamma$-injectives denoted $\operatorname{spli}(A\Gamma)$, the supremum over the injective dimension of $A\Gamma$-projectives denoted silp$(A\Gamma)$, the supremum over the flat dimension of $A\Gamma$-injectives denoted sfli$(A\Gamma)$, the supremum over the injective dimension of $A\Gamma$-flat modules denoted silf$(A\Gamma)$, and an invariant $k(A\Gamma)$. Also, we deal with a large class of (not ncecessarily finite) groups called groups of type Φ over A (see Definition 1.2.6), and we denote this class by $\mathscr{F}_{\phi,A}$. The following is a collection of original results that we prove in this context.

Theorem. *Let Γ be a group and let A be a commutative ring of finite global dimension t. Then,*

a) We conjecture that $Gcd_A(\Gamma) = proj.dim_{A\Gamma}B(\Gamma, A)$ (see Conjecture 4.2.1). We prove that this conjecture is true when $proj.dim_{A\Gamma}B(\Gamma, A) < \infty$ (see Theorem 4.2.6).

b) In connection to a conjecture made by Dembegioti and Talelli (see Conjecture 4.2.7), the class of Gorenstein projective $A\Gamma$-modules (see Definition 1.1.1) coincides with the class of $A\Gamma$-modules M such that $M \otimes_A B(\Gamma, A)$ is $A\Gamma$-projective (see Remark 6.4.7).

c) If Γ is locally in $H\mathscr{F}_{\phi,A}$, i.e. if all of the finitely generated subgroups of Γ are in Kropholler's hierarchy $H\mathscr{F}_{\phi,A}$ (see Definition 1.2.1) with $\mathscr{F}_{\phi,A}$ as the base class, then $proj.dim_{A\Gamma}B(\Gamma, A) = Gcd_A(\Gamma)$, and $\operatorname{fin.dim}(A\Gamma) = \operatorname{silp}(A\Gamma) = \operatorname{silf}(A\Gamma) = \operatorname{sfli}(A\Gamma) = \operatorname{spli}(A\Gamma) = k(A\Gamma)$ with the commong value lying between $proj.dim_{A\Gamma}B(\Gamma, A)$ and $proj.dim_{A\Gamma}B(\Gamma, A) + t$ (see Theorem 4.6.13).

d) (See Remark 4.6.15) It follows from (c) that if Γ is locally in $H\mathscr{F}_{\phi,A}$, then it satisfies a big part of the main conjecture (See Conjecture 4.1.12 and also Conjecture 4.6.14) for $\mathscr{F}_{\phi,A}$-groups.

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Chapter 1

Preliminaries and Some Background

There are three separate themes in this thesis - one of them is dealings with various cohomological and homological invariants related to groups coming from Kropholler's hierarchy and another hierarchy due to Ikenaga which take up Chapters 4, 5 and 7, and the second theme is the development of an independent theory of generation operators in the module category which we do in Chapter 2 and Chapter 3 with some mixed applications in Chapter 6, and the third theme is the work on generation properties of derived categories of modules over groups in Kropholler's hierarchy that is done in Chapter 8.

We start with introducing a very important class of modules called the *Gorenstein projectives* which will play quite an important role across many chapters.

1.1 Complete Resolutions and Gorenstein Projectives

We start this section with the definition of complete resolutions.

Definition 1.1.1. *Let R be a ring and let M be an R-module. A complete resolution admitted by M is an infinite exact sequence of R-projective modules, (F_*, d_*): $.. \overset{d_{n+1}}{\to} F_n \overset{d_n}{\to} F_{n-1} \overset{d_{n-1}}{\to} ... \overset{d_1}{\to} F_0 \overset{d_0}{\to} F_{-1} \to ...$ such that*

a) There exists a projective resolution of M, $(P_*, \delta_*) \twoheadrightarrow M$, such that for some

$n \geq 0$, $(P_i, \delta_i)_{i \geq n} = (F_i, d_i)_{i \geq n}$. The smallest such n is called the coincidence index of the complete resolution with respect to a given projective resolution.

b) $Hom_R(F_*, P)$ is acyclic for any R-projective module P.

If (F_*, d_*) satisfies (a) but does not satisfy (b), we call (F_*, d_*) a weak complete resolution admitted by M.

For any commutative ring A, a group Γ is said to admit a complete resolution over A if the trivial $A\Gamma$-module A admits a complete resolution. The motivation behind such language is made clear, in the case where A has finite global dimension, in Theorem 1.1.12, and later in Theorem 4.1.3 and Remark 4.1.4, which shows that in such cases the trivial module having a complete resolution is equivalent to all modules having complete resolutions.

An R-module is called (weak) Gorenstein projective iff it occurs as a kernel in a (weak) complete resolution. For a detailed survey on the properties of Gorenstein projectives, one can consult [34].

Lemma 1.1.2. *([17]) Let R be a ring, then two complete resolutions admitted by the same R-module are unique up to chain homotopy.*

Complete resolutions are a useful tool in group cohomology because they can be used to compute complete cohomology of groups. In Chapter 8, we see that there is a connection between complete resolutions and stable module categories. This connection stems from the fact that if we know that every module admits a complete resolution, then we can define the inverse syzygy functor for every module in the stable module category. This inverse syzygy functor then acts as the suspension functor to give the stable module category the structure of a triangulated category. For now, we keep our focus on Gorenstein projective modules and record some useful properties of the class of Gorenstein projective modules.

First, note that if we are working over a group algebra over a commutative ring A of finite global dimension, then Gorenstein projectives over the group algebra are projective over A. This follows straightforwardly from the definition of complete resolutions. We record this in the following lemma. We will be making use of this fact in Chapter 6.

Lemma 1.1.3. *Let A be a commutative ring of finite global dimension and Γ some* *group, and let M be a kernel in an exact sequence of projective $A\Gamma$-modules that extends to infinity in both directions. Then, M is A-projective.*

Proof. Let the global dimension of A be r. We have M as a kernel in a doubly infinite exact sequence of projectives (F_*, d_*). Let $M = \mathrm{Ker}(d_t)$. Looking at the exact sequence $0 \to M \hookrightarrow F_t \to F_{t-1} \to ..... \to F_{t-s} \twoheadrightarrow \mathrm{Im}(d_{t-s}) \to 0$ where $s > r$, we see that M is the s-th syzygy of $\mathrm{Im}(d_{t-s})$ and since $s > r$, this implies that M is A-projective. □

Lemma 1.1.4. *(Theorem 2.5 of [34]) The class of Gorenstein projective R-modules, for any ring R, contains the class of projectives and is closed under arbitrary direct sums and direct summands.*

We see in Lemma 1.1.4 that being closed under arbitrary direct sums and direct summands are properties shared by Gorenstein projectives with the projectives. Now note that from Lemma 1.1.4 it is clear that given any R-module M, we can find a Gorenstein projective module G_0 that maps surjectively onto M, we can then take the kernel of this map, K, and find, by Lemma 1.1.4 again, another Gorenstein projective R-module G_1 mapping surjectively onto K and going on like this, we get an exact sequence $... \to G_n \to .. \to G_1 \to G_0 \to M \to 0$. Analogous to the way projective dimension of a module is defined, we define the *Gorenstein projective dimension* of M over R to be the length of the smallest resolution, $G_* \twoheadrightarrow M$, comprising of Gorenstein projective R-modules admitted by M.

Definition 1.1.5. *Let R be a ring and let M be an R-module. The Gorenstein projective dimension of M as an R-module, denoted $Gpd_R(M)$, is defined as min$\{n \in \mathbb{N} : \exists$ an exact sequence $0 \to G_n \to ... \to G_1 \to G_0 \to M \to 0$ where each G_i is a Gorenstein projective R-module$\}$. If M does not admit a finite length resolution by Gorenstein projective R-modules, we say $Gpd_R(M)$ is not finite.*

For any commutative ring A and any group Γ, the Gorenstein cohomological dimension of Γ with respect to A, denoted $Gcd_A(\Gamma)$, is defined as $Gpd_{A\Gamma}(A)$.

Example 1.1.6. *If one takes Γ to be a free abelian group of infinite cardinality, then Γ does not admit complete resolutions (this is easy to show, see [49]), and so in this case the trivial module is an example of a module with infinite Gorenstein projective dimension.*

Any given group Γ achieves its highest Gorenstein cohomological dimension over the ring of integers among all rings:

Proposition 1.1.7. *(Proposition 2.1 of [29]) Let Γ be a group. Then, for any commutative ring A, $Gcd_A(G) \leq Gcd_{\mathbb{Z}}(G)$.*

Analogous to the fundamental properties of projective dimension, we have the following fundamental properties of Gorenstein projective dimension.

Theorem 1.1.8. *(Theorem 2.20 of [34]) Let R be a ring and let M be an R-module such that $Gpd_R(M) < \infty$. The following are equivalent.*

a) $Gpd_R(M) \leq n$.

b) $\mathrm{Ext}^k_R(M, N) = 0$, for all $k > n$, for any R-module N of finite projective dimension.

c) $\mathrm{Ext}^k_R(M, P) = 0$, for all $k > n$, for any projective P.

d) In any exact sequence $0 \to K \to L_{n-1} \to ... \to L_1 \to L_0 \to M \to 0$, where each L_i is Gorenstein projective, K is Gorenstein projective.

It follows quite straightforwardly from Theorem 1.1.8 that the class of modules with an upper bound on their Gorenstein projective dimensions is closed under direct summands. We provide a short proof below.

Lemma 1.1.9. *For any ring R and two R-modules M and N, $Gpd_R(M) \leq Gpd_R(M \oplus N)$.*

Proof. If $Gpd_R(M \oplus N)$ is not finite, we have nothing to prove so we assume that it is n where n is some non-negative integer. As discussed before, using Lemma 1.1.4, we can construct two exact sequences $0 \to K_1 \to G_{n-1} \to ... \to G_1 \to G_0 \to M \to 0$ and $0 \to K_2 \to H_{n-1} \to ... \to H_1 \to H_0 \to N \to 0$ where each G_i and H_i is Gorenstein projective. We thus have an exact sequence $0 \to K_1 \oplus K_2 \to G_{n-1} \oplus H_{n-1} \to ... \to G_1 \oplus H_1 \to G_0 \oplus H_0 \to M \oplus N \to 0$ where each $G_i \oplus H_i$ is Gorenstein projectives because Gorenstein projectives are closed under direct sums (Lemma 1.1.4). By the (a)-(d) equivalence in Theorem 1.1.8, it follows that $K_1 \oplus K_2$ is Gorenstein projective and therefore K_1 is Gorenstein projective by Lemma 1.1.4, and therefore $Gpd_R(M) \leq n$. $\square$

It also follows from the Ext-formulation of Gorenstein projective dimension ((a)-(c) equivalence in Theorem 1.1.8) that we cannot have a situation where there is an exact sequence with one module of infinite Gorenstein projective dimension and every other module of finite Gorenstein projective dimension. We record this fact in terms of short exact sequences in the following way.

Lemma 1.1.10. *Take a short exact sequence of R-modules $0 \to A \to B \to C \to 0$.*

Then $Gpd_R(C) \leq max\{Gpd_R(A), Gpd_R(B)\} + 1$.

Proof. Let $n = \max\{Gpd_R(A), Gpd_R(B)\}$. For any projective module P, we look at the long exact Ext-sequence $... \to \mathrm{Ext}_R^{n+1}(C, P) \to \mathrm{Ext}_R^{n+1}(B, P) \to \mathrm{Ext}_R^{n+1}(A, P) \to \mathrm{Ext}_R^{n+2}(C, P) \to \mathrm{Ext}_R^{n+2}(B, P) \to ...$ where $\mathrm{Ext}_R^{n+1}(A, P) = \mathrm{Ext}_R^{n+2}(B, P) = 0$, and therefore $\mathrm{Ext}_R^{n+2}(C, P) = 0$ for any projective P, therefore by Theorem 1.1.8, $Gpd_R(C) \leq n + 1$. □

We will see later how Lemma 1.1.10 will be useful in proving that the class of Gorenstein projectives is a "good" class, i.e. any module generated in finitely many steps from the class of Gorenstein projectives also admits a finite length resolution with Gorenstein projectives. We discuss this in the next chapter where we formally introduce the definition of generation of a module from a given class of modules.

The following well-known result gives an example of Gorenstein projectives.

Lemma 1.1.11. *(done over $\mathbb{Z}$ in Lemma 2.21 of [4])*

Let Γ be a group and A a commutative ring of finite global dimension. Then, any $A\Gamma$-permutation module with finite stabilisers is Gorenstein projective. In other words, for any arbitrary family of finite subgroups $\{G_\lambda\}_{\lambda \in \Lambda}$ of Γ, $\bigoplus_{\lambda \in \Lambda} A[\Gamma/G_\lambda]$ is Gorenstein projective.

Lemma 1.1.11 gives us a way of getting concrete examples of Gorenstein projectives, that need not be projective, even when we know that the trivial module is not of finite Gorenstein projective dimension.

It is important to note that, when working over group algebras over rings of finite global dimension, whenever we have the trivial module admitting complete resolutions, all other modules admit complete resolutions:

Theorem 1.1.12. *(See Theorem 4.1.3) Let Γ be a group and let A be a ring of finite global dimension. Then, $Gpd_{A\Gamma}(A) < \infty$ iff $Gpd_{A\Gamma}(M) < \infty$ for all $A\Gamma$-modules M.*

Since the class of Gorenstein projectives contains the class of projectives, it is an interesting question as to whether a module can have finite projective dimension and finite Gorenstein projective dimension with those two dimensions being different or whether a module can have finite Gorenstein projective dimension but not finite projective dimension. In this regard, the following result is of note, and we shall make use of this result in the next chapter when we will look at *equivalent* classes of modules in the context of generation.

Proposition 1.1.13. *(Proposition 2.27 of [34]) For any ring R, the projective dimension of Gorenstein projective modules is either 0 or not finite.*

Example 1.1.14. *Take G to be C_p and take k to be a field of characteristic p. Then, the ring kG has infinite global dimension. But, we know that the finitistic dimension of kG, i.e. the supremum over the projective dimensions of all kG-modules that have finite projective dimension, is 0 because any kG-module with finite projective dimension has projective dimension 0 as the global dimension of k is 0 (see Lemma 2.3 of [46]).*

The global dimension of kG is not finite. So, there must be a kG-module M of infinite projective dimension. Let M be such a kG-module that has infinite projective dimension. Again, as G is finite and k is a field, all kG-modules are Gorenstein projective (this is easy to show by constructing a complete resolution of M of coincidence index 0, see [17]; another way of showing this is by noting, with the aid of Theorem 2.28 of [34], that the supremum over the Gorenstein projective dimensions of all kG-modules with finite Gorenstein projective dimension is 0 as the finitistic dimension of kG is 0). So, M is Gorenstein projective.

1.2 Groups in Kropholler's Hierarchy

Peter Kropholler [41], in the nineties, introduced a hierarchical system of groups with the hierarchies determined in terms of certain geometric properties admitted by the groups. In this hierarchy, if we start with the class of all finite groups as our base class, we get an infinite family of both finite and infinite groups that satisfy many fascinating properties like, for example, admitting a finite dimensional classifying space for proper actions as long as they are of type FP_∞ (this follows from a deep result of Kropholler and Mislin, see [43]). We begin by providing a definition of Kropholler's hierarchy.

Throughout this thesis, the term "a class of groups" is used to mean a set theoretical collection of groups such that if Γ is in the collection and Γ_1 is isomorphic to Γ, then Γ_1 is also in the collection.

Definition 1.2.1. *(see [41]) Let $\mathscr{X}$ be a class of groups. We define $H_0\mathscr{X} := \mathscr{X}$, and for any successor ordinal (like a positive integer) α, a group Γ is said to be in $H_\alpha\mathscr{X}$ iff there exists a finite dimensional contractible CW-complex on which Γ acts by permuting the cells with cell stabilizers in $H_\alpha\mathscr{X}$. If α is a limit ordinal, then we define $H_\alpha\mathscr{X}$ as $\bigcup_{\beta<\alpha} H_\beta\mathscr{X}$. Γ is said to be in $H\mathscr{X}$ iff Γ is in $H_\alpha\mathscr{X}$ for some ordinal α (note that α need not be a limit ordinal here).*

For any ordinal α, we denote by $H_{<\alpha}\mathscr{X}$ the class of all groups that are in $H_\beta\mathscr{X}$ for some ordinal $\beta < \alpha$.

The class $LH\mathscr{X}$ is defined as the class of all groups all of whose finitely generated subgroups are in $H\mathscr{X}$.

Throughout the thesis, we denote the class of all finite groups by $\mathscr{F}$.

The following result is easy to see from the above definition.

Lemma 1.2.2. *([41]) Let $\mathscr{X}$ be a class of groups. Then, $H_\alpha\mathscr{X} \subseteq H_\beta\mathscr{X}$ where α and β are any two ordinals such that $\alpha < \beta$.*

It is important to note that in the above definition, if we start with the class of all finite groups, denoted $\mathscr{F}$, then the classes $H_0\mathscr{F}$, $H_1\mathscr{F}$,...,$H_n\mathscr{F}$, and so on are all distinct, by which we mean for each positive integer n, there exists a group that is in $H_n\mathscr{F}$ but not in $H_{n-1}\mathscr{F}$. This is quite non-trivial and is due to a result by Januszkiewicz, Kropholler and Leary [36]. We give the following solid example of groups lying in one class of the hierarchy and not in the one immediately below up to $H_3\mathscr{F}$.

Theorem 1.2.3. *(Thm. 7.10 of [26]) Let ω denote the first infinite ordinal. Then,*

a) The free abelian group of rank t, where $1 \le t < \aleph_0$, is in $H_1\mathscr{F}$ but not in $H_0\mathscr{F}$.

b) The free abelian group of rank t, where $\aleph_0 \le t < \aleph_\omega$, is in $H_2\mathscr{F}$ but not in $H_1\mathscr{F}$.

c) The free abelian group of rank $\aleph_\omega$ is in $H_3\mathscr{F}$ but not in $H_2\mathscr{F}$.

We shall see later in Chapter 4 that trying to emulate results on Kropholler's hierarchy where the base class is the class of all finite groups for the case where the base class is itself a large class of infinite groups (like groups of type Φ, see Definition 1.2.6) can be a very interesting exercise. Among many interesting results proved on this theme is the result that just like with the class of all finite groups as the base class $\mathscr{X}$, as mentioned above, $H_n\mathscr{X} \neq H_{n+1}\mathscr{X}$ for all integers n, the same holds true when $\mathscr{X}$ is the class of all groups of type Φ over a fixed commutative ring of finite global dimension.

Remark 1.2.4. *It is also interesting to see examples of groups that are not in Kropholler's hierarchy for different base classes. It has been shown in [41], and we go through the proof in Section 7.7.1 of Chapter 7 that for $\mathscr{X} = \mathscr{F}$, Richard Thompson's group F is not in $LH\mathscr{X}$. We show that F is also not in $LH\mathscr{F}_{\phi,A}$, where $\mathscr{F}_{\phi,A}$ denotes the class of groups of type Φ over A with A some commutative ring of finite global dimension (see Definition 1.2.6).*

Another example of a group not in $H\mathscr{F}$ that we briefly discuss in Chapter 7 is the first Grigorchuk group. For this group, we show that it cannot be in $H\mathscr{F}_{\phi,\mathbb{Q}}$ (see Theorem 7.1.7, here $\mathscr{F}_{\phi,\mathbb{Q}}$ denotes the class of groups of type Φ over $\mathbb{Q}$).

Generally, irrespective of the base class in the hierarchy, groups in Kropholler's hierarchy are useful because as we will see later in Chapter 8, if a group is in the n-th hierarchy, then all of its modules admit finite length resolutions with modules from the class of all modules induced up from subgroups in the $(n-1)$-th hierarchy, closed under direct sums (see Lemma 8.1.3). This property will be a very useful tool for us in generating derived bounded, unbounded, bounded above and bounded below categories of modules over groups in Kropholler's hierarchy in Chapter 8.

In relation to our last section, it is also noteworthy that groups in $H_1\mathscr{F}$ admit complete resolutions. This fact was first established in [21]. In Chapter 4, we provide a short proof of this fact using the language of Gorenstein projectives:

Theorem 1.2.5. *(= Theorem 4.5.6, different proof in [21]) If $\Gamma \in H_1\mathscr{F}$, then Γ admits complete resolutions over any commutative ring of finite global dimension.*

Staying on the question of groups admitting complete resolutions, we introduce our second class of groups which was first introduced by Olympia Talelli in [62].

Definition 1.2.6. *A group Γ is said to be of type Φ over a commutative ring A if for any $A\Gamma$-module M, $proj.dim_{A\Gamma}M < \infty$ iff $proj.dim_{AG}M < \infty$ for all finite subgroups $G \leq \Gamma$.*

The class of all groups of type Φ over A is denoted $\mathscr{F}_{\phi,A}$. For $A = \mathbb{Z}$, we write $\mathscr{F}_{\phi} := \mathscr{F}_{\phi,\mathbb{Z}}$.

When groups of type Φ were first studied by Talelli in [62], they were studied over the ring of integers. The results proved in [62] are enough to show that groups of type Φ admit complete resolutions over the ring of integers and the same was shown later over (Noetherian) rings of finite global dimension by Mazza and Symonds in [46].

Remark 1.2.7. *a)* ***Groups in $H_1\mathscr{F}$ are of type Φ.*** *This was shown by Mazza and Symonds in [46]. See also Lemma 5.1.7.*

b) ***There are groups in $H\mathscr{F}$ that are not of type Φ over any ring of finite global dimension:*** *Theorem 1.2.3 tells us that the free abelian group on a basis of cardinality $\aleph_0$ is in $H\mathscr{F}$ (the statement of Theorem 1.2.3 tells us that it is in $H_2\mathscr{F}$). But it does not admit complete resolutions over any ring of finite global dimension, so it cannot be of type Φ over any ring of finite global dimension. See Remark 5.1.14.*

Example of groups of finite vcd: "Vcd" stands for "virtual cohomological dimension". A group Γ is said to have finite virtual cohomological dimension n is Γ has a finite-index subgroup Γ_1 which has cohomological dimension n (this definition is well-defined due to a result of Serre, see [17]). Here, we are dealing with $\mathbb{Z}$ as our base ring; recall that the cohomological dimension (over $\mathbb{Z}$) of a group Γ is $proj.dim_{\mathbb{Z}\Gamma}\mathbb{Z}$. Linear groups $SL_n(\mathbb{Z})$ have finite virtual cohomological dimension given by $\frac{n(n-1)}{2}$.

There are groups of type Φ that are not of finite vcd: The group $\mathbb{Q}/\mathbb{Z}$ does not have a finite index subgroup with finite cohomological dimension over $\mathbb{Z}$, but it acts on a tree with finite stabilisers so it is of type Φ (see Proposition 2.5 of [46]).

Groups of finite vcd are in $H_1\mathscr{F}$: It follows from a classical construction of a finite dimensional acyclic CW-complex on which a group with finite *vcd* can act with finite stabilisers (see [17]) that such a group is in $H_1\mathscr{F}$.

1.3 Introduction to some cohomological invariants of groups

We have already introduced one major cohomological invariant of groups - the Gorenstein projective dimension of the trivial module (also called the Gorenstein cohomological dimension of the group). Keeping track of this invariant is useful while studying important cohomological properties of groups like whether or not they admit complete resolutions.

We now introduce a few other invariants.

Definition 1.3.1. *For any ring R, we define* $\operatorname{spli}(R)$ *to be the supremum over the projective dimension of R-injective modules, and* $\operatorname{silp}(R)$ *to be the supremum over the injective dimension of R-projective modules.*

We define the finitistic dimension of R, denoted $\operatorname{fin.dim}(R)$*, to be the supremum over the projective dimensions of all R-modules which have finite projective dimension.*

For any group Γ and any commutative ring A, we define $k(A\Gamma)$ to be $sup\{proj.dim_{A\Gamma}M : M \in Mod\text{-}A\Gamma$ such that $proj.dim_{AG}M < \infty$ for all finite subgroups $G \leq \Gamma\}$.

Remark 1.3.2. *In [31], when the invariants* $\operatorname{spli}(R)$ *and* $\operatorname{silp}(R)$*, for any ring R, were introduced, the authors chose the acronyms "*spli*" and "*silp*" to respectively mean the supremum over the projective lengths of R-injectives and the supremum over the injective lengths of R-projectives. "Length" means "dimension" in this context.*

The invariant $k(A\Gamma)$, for any group Γ and any commutative ring A of finite global dimension, is a useful indicator of whether Γ is of type Φ over A because it is not very difficult to show that Γ is of type Φ over A iff $k(A\Gamma)$ is finite - see Lemma 4.5.1.

The invariants $\operatorname{silp}(R)$ and $\operatorname{spli}(R)$, that we defined above, were first introduced and studied by Gedrich and Gruenberg in [31]. The following was proved in the same paper.

Theorem 1.3.3. *(= Theorem 4.1.1) Let R be a ring such that* $\operatorname{spli}(R) < \infty$*. Then every R-module admits a weak complete resolution.*

An analogous result with the hypothesis $\operatorname{silp}(R) < \infty$ and the existence of weak injective complete resolutions for every module was proved in [31], but that is not important to us as here we are not discussing complete resolutions with injective modules. Before going further, we state some results with these invariants.

Theorem 1.3.4. *([31]) Let R be a ring. Then,*

a) If both $\mathrm{spli}(R)$ and $\mathrm{silp}(R)$ are finite, then $\mathrm{silp}(R) = \mathrm{spli}(R)$.

b) If R has finite global dimension t, then $\mathrm{spli}(R) = \mathrm{silp}(R) = t$.

We shall see in Section 5.3 of Chapter 5 that the finiteness of the silp invariant can be related to a generation property of the derived unbounded category by injectives.

We will mostly be dealing with group rings in this thesis. When $R = A\Gamma$, with Γ being some group and A being a commutative ring of finite global dimension t, the following result by Emmanouil and Talelli [29] gives a very useful bound on the silp and spli invariants in terms of the Gorenstein cohomological dimension.

Lemma 1.3.5. *(= Lemma 4.1.5) Let A be a commutative ring of finite global dimension t and let Γ be a group. Then, $\mathrm{silp}(A\Gamma), \mathrm{spli}(A\Gamma) \leq Gcd_A(\Gamma) + t$.*

Before we state one of our original results involving cohomological invariants, we need to introduce one other invariant.

Definition 1.3.6. *For any commutative ring A and any group Γ, denote by $B(\Gamma, A)$ the module of those functions $\Gamma \to A$ that are only allowed to take finitely many values in A. The $A\Gamma$-module structure on $B(\Gamma, A)$ is given the following way: for any $f \in B(\Gamma, A)$, $(\gamma_1.f)(\gamma) := f(\gamma_1^{-1}\gamma)$, for all $\gamma, \gamma_1 \in \Gamma$.*

Following [11], we define an $A\Gamma$-module M to be a Benson's cofibrant if $M \otimes_A B(\Gamma, A)$ is a projective $A\Gamma$-module.

The following is an important property of the module defined above.

Lemma 1.3.7. *(Lemma 3.4 of [11]) For any group Γ and any commutative ring A, $B(\Gamma, A)$ is A-free and is AG-free, for any finite $G \leq \Gamma$.*

The following is an original result that we prove in Chapter 4.

Theorem 1.3.8. *(= Theorem 4.4.1) Let $\Gamma \in LH\mathscr{F}_{\phi,A}$ with A being a commutative ring of global dimension t. Then,*

$$proj.dim_{A\Gamma} B(\Gamma, A) = Gcd_A(\Gamma)$$

and

$$proj.dim_{A\Gamma}B(\Gamma,A) \leq \operatorname{fin.dim}(A\Gamma) = \operatorname{silp}(A\Gamma) = \operatorname{spli}(A\Gamma) = k(A\Gamma) \leq proj.dim_{A\Gamma}B(\Gamma,A)+t$$

Remark 1.3.9. *For $A = \mathbb{Z}$, it has been conjectured (see Conjecture A of [62]) that a group Γ is of type Φ over $\mathbb{Z} \iff \operatorname{fin.dim}(\mathbb{Z}\Gamma) < \infty \iff \operatorname{spli}(\mathbb{Z}\Gamma) < \infty \iff \operatorname{silp}(\mathbb{Z}\Gamma) < \infty$. We extend this conjecture in the general context with $\mathbb{Z}$ replaced by a commutative ring of finite global dimension in Conjecture 4.1.12 and also in Conjecture 4.6.14 and Conjecture 5.1.10 where we add some more conjectured equivalent statements, and we show in Proposition 4.5.2 that almost all these statements are equivalent if $\Gamma \in LH\mathscr{F}_{\phi,A}$.*

We briefly work with a few homological invariants like the supremum over the injective dimensions of flat modules (denoted silf), the supremum over the flat dimensions of injective modules (denoted sfli) in Section 4.6 of Chapter 4, and there we slightly extend the statement of Theorem 1.3.8 to get the following.

Theorem 1.3.10. *(= Theorem 4.6.13) Let A be a commutative ring of finite global dimension t and $\Gamma \in LH\mathscr{F}_{\phi,A}$. Then,*

$$\operatorname{fin.dim}(A\Gamma) = \operatorname{silp}(A\Gamma) = \operatorname{silf}(A\Gamma) = \operatorname{sfli}(A\Gamma) = \operatorname{spli}(A\Gamma) = k(A\Gamma)$$

and the common value of these invariants lies between $proj.dim_{A\Gamma}B(\Gamma,A)$ and $t+ proj.dim_{A\Gamma}B(\Gamma,A)$.

Another interesting conjectured connection between two invariants to note, for any commutative A of finite global dimension and any group Γ, is the coincidence of the Gorenstein cohomological dimension with the projective dimension of $B(\Gamma,A)$. We made this as a separate conjecture in Chapter 4 - see Conjecture 4.2.1. We show the following:

Theorem 1.3.11. *(= Theorem 4.2.6) Let A be a commutative ring of finite global dimension and let Γ be a group. If $proj.dim_{A\Gamma}B(\Gamma,A)$ is finite, then $proj.dim_{A\Gamma}B(\Gamma,A) = Gcd_A(\Gamma)$.*

It follows that if $Gcd_A(\Gamma)$ is not finite, then $proj.dim_{A\Gamma}B(\Gamma,A)$ is not finite.

The module $B(\Gamma, A)$ plays a crucial role in defining a class of modules called Benson's cofibrants.

We shall see later that it follows from a result of Cornick and Kropholler [21] that Benson's cofibrants are Gorenstein projectives, whether the converse holds or not is open to conjecture (first made over the integers in [25]). We state this conjecture here over commutative rings of finite global dimension.

Conjecture 1.3.12. *(= Conjecture 4.2.7) For any commutative ring A of finite global dimension and any group Γ, the class of Benson's cofibrant $A\Gamma$-modules coincides with the class of Gorenstein projective $A\Gamma$-modules.*

In relation to Conjecture 1.3.12, we managed to prove the following in Chapter 6:

Theorem 1.3.13. *(See Remark 6.4.5) Let A be a commutative ring of finite global dimension and let $\Gamma \in LH\mathscr{F}_{\phi,A}$. Then, the class of Benson's cofibrant $A\Gamma$-modules coincides the class of Gorenstein projective $A\Gamma$-modules.*

This result was previously known for $LH\mathscr{F}$-groups [25].

Conjecture 1.3.12 is closely related to our conjecture on the coincidence of the Gorenstein cohomological dimension with the projective dimension of $B(\Gamma, A)$ - see Theorem 4.2.9.

Chapter 2

Generation of Modules from a Given Class

In this chapter, we provide an abstract framework for the concepts of generation that we use in Chapter 8 to derive many useful and interesting generation properties of the module category and derived categories and stable module categories for infinite groups like groups in Kropholler's hierarchy (for stable categories, we need the additional assumption that the group admits complete resolutions).

Most of the treatment in this chapter is quite straightforward to follow.

2.1 Introduction and definitions

Definition 2.1.1. *Let R be a ring and $\mathscr{T}$ be a class of R-modules. We define generation of modules from $\mathscr{T}$ inductively - we say an R module is generated from $\mathscr{T}$ in n steps iff there exists a short exact sequence $0 \to M_1 \to M_2 \to M \to 0$ where M_1, M_2 are generated from $\mathscr{T}$ in a_1, a_2 steps respectively and $a_1 + a_2 \leq n - 1$; to begin the induction, we say an R-module M is generated in 0 steps from $\mathscr{T}$ iff $M \in \mathscr{T}$. So, if we are given a short exact sequence $0 \to M_1 \to M_2 \to M \to 0$ and we know that each M_i is generated from $\mathscr{T}$ in a_i steps, then M is generated from $\mathscr{T}$ in $a_1 + a_2 + 1$ steps.*

We shall denote the class of all modules that can be generated in n steps from $\mathscr{T}$ by $\langle \mathscr{T} \rangle_n$ and the class of all modules that can be generated in finitely many steps from $\mathscr{T}$ by $\langle \mathscr{T} \rangle$.

For any R-module M, we define the $\mathscr{T}$-generation number, $\alpha_{\mathscr{T}}(M) :=min\{n \in \mathbb{Z} :$

$M \in \langle \mathscr{T} \rangle_n\}$. If $M \notin \langle \mathscr{T} \rangle_n$ for any finite n, we define $\alpha_{\mathscr{T}}(M)$ to be infinite.

The following lemma is clear from the definition of the number of steps of generation in Definition 2.1.1.

Lemma 2.1.2. *For any class of R-modules, $\mathscr{T}$, $\mathscr{T} = \langle \mathscr{T} \rangle_0 \subseteq \langle \mathscr{T} \rangle_1 \subseteq \langle \mathscr{T} \rangle_2 \subseteq \dots$.*

Looking at the position of our generated module in the short exact sequence used for generation in Definition 2.1.1, the following question arises naturally.

Question 2.1.3. *Why are we not putting the generated module in the middle in Definition 2.1.1?*

Answer 2.1.4. *The main reason behind this is that when we put the new module on the right of the short exact sequence like we do in Definition 2.1.1, the generation number that we define coincides with the dimension (see Definition 2.2.1) when the class is well-behaved or "good" as in Definition 2.2.5. Later, in Section 3.3, we derive many interesting properties of these "good" classes, and many examples of such classes can be found, among other places, in Section 2.4.*

In fact, if one does put the new generated module in the middle, then one can construct an infinite family of classes such that for each of those classes, the class generated as per this definition (putting the new module in the middle) differ from the class generated from it as per Definition 2.1.1:

Example 2.1.5. *One can take, for any n, $\mathscr{T}$ to be the class of R-modules of length $\leq n$, where R is a fixed ring. Then, $\mathscr{T}$ generates just itself as per Definition 2.1.1 but, putting the new module in the middle, one gets a strictly bigger class, and both of these results follow from the fact for any short exact sequence, the length of the module in the middle is the sum of the lengths of the other two modules.*

Remark 2.1.6. *If we are generating modules in the stable module category (for group rings of finite groups over fields, stable module categories are well studied - see [15]; and for infinite groups admitting complete resolutions over commutative rings of finite global dimension, a well-behaved stable category can be constructed, see Section 8.4.2) using distinguished triangles instead of short exact sequences, it follows from the axioms used to define triangulated categories that it makes no difference whether the new* *generated module is put in the middle or on the right of the distinguished triangle being used to generate it as long as the starting class of modules is closed under the suspension functor of the stable module category.*

Coming back to our notion of generation as introduced in Definition 2.1.1, we now look at situations where we generate modules from a given class with every module of that class being generated in finitely many steps from another class.

Lemma 2.1.7. *Let $\mathscr{T}$ and $\mathscr{U}$ be 2 classes of R-modules.*

a) If $\mathscr{T} \subseteq \langle \mathscr{U} \rangle$, then $\langle \mathscr{T} \rangle \subseteq \langle \mathscr{U} \rangle$. In other words, any module that can be generated in finitely many steps from $\mathscr{T}$ can also be generated in finitely many steps from $\mathscr{U}$ if every module in $\mathscr{T}$ is generated in finitely many steps from $\mathscr{U}$.

b) If $\mathscr{T} \subseteq \langle \mathscr{U} \rangle_m$, then $\langle \mathscr{T} \rangle_n \subseteq \langle \mathscr{U} \rangle_{mn+m+n}$.

Proof. a) We proceed by strong induction on the $\mathscr{T}$-generation number of modules. First, we check our base case. Note that our lemma holds true for modules in $\mathscr{T}$, i.e. for all modules whose $\mathscr{T}$-generation number is zero.

Now, let us assume that all modules of $\mathscr{T}$-generation number $\leq n$ are in $\mathscr{U}$, this is our induction hypothesis. If $\alpha_{\mathscr{T}}(M) = n+1$, then by definition, M admits a generation sequence $0 \to D_2 \to D_1 \to M \to 0$ where $D_1, D_2 \in \langle \mathscr{T} \rangle_n$. This means $\alpha_{\mathscr{T}}(D_1), \alpha_{\mathscr{T}}(D_2) \leq n$. It follows from our induction hypothesis that $D_1, D_2 \in \langle \mathscr{U} \rangle$. That means D_1 and D_2 can be generated from $\mathscr{U}$ in d_1 and d_2 steps respectively for some non-negative integers d_1, d_2, and from that it follows that M can be generated from $\mathscr{U}$ in $d_1 + d_2 + 1$ steps. Thus, $M \in \langle \mathscr{U} \rangle$, and that ends our induction.

b) We proceed by strong induction on n. If $M \in \langle \mathscr{T} \rangle_0$, then $M \in \mathscr{T} \subseteq \langle \mathscr{U} \rangle_m = \langle \mathscr{U} \rangle_{m.0+m+0}$. Let us assume that the result is true when $n \leq k$. If $M \in \langle \mathscr{T} \rangle_{k+1}$, there exists a short exact sequence $0 \to C_2 \to C_1 \to M \to 0$ where C_1, C_2 are generated from $\mathscr{T}$ in a_1, a_2 steps respectively where $a_1 + a_2 \leq k$, so $a_1, a_2 \leq k$. By the induction hypothesis, C_i is generated from $\mathscr{U}$ in $ma_i + m + a_i$ steps, for $i = 1, 2$. So, M is generated from $\mathscr{U}$ in $(ma_1+a_1+m)+(ma_2+a_2+m)+1 = m(a_1+a_2+2)+a_1+a_2+1 \leq m(k+2)+k+1 = m(k+1)+m+(k+1)$ steps. This ends our induction. $\square$

The following results follow directly from Lemma 2.1.7.

Corollary 2.1.8. *Let $\mathscr{T}$ be a class of R-modules. Then,* *a)* $\langle\langle \mathscr{T} \rangle\rangle = \langle \mathscr{T} \rangle$.

b) Let $\mathscr{U}$ *be a class of* R*-modules such that* $\mathscr{T} \subseteq \mathscr{U}$*, then* $\langle \mathscr{T} \rangle \subseteq \langle \mathscr{U} \rangle$ *and* $\langle \mathscr{T} \rangle_n \subseteq \langle \mathscr{U} \rangle_n$ *for all* n*.*

Proof. a) $\langle \mathscr{T} \rangle \subseteq \langle \mathscr{T} \rangle$. So, by Lemma 2.1.7.a., $\langle\langle \mathscr{T} \rangle\rangle \subseteq \langle \mathscr{T} \rangle$. Note that by the definition of generation from the class $\langle \mathscr{T} \rangle$, $\langle \mathscr{T} \rangle \subseteq \langle\langle \mathscr{T} \rangle\rangle$. Therefore, $\langle\langle \mathscr{T} \rangle\rangle = \langle \mathscr{T} \rangle$.

b) If we take $m = 0$ in Lemma 2.1.7.b., we get $\langle \mathscr{T} \rangle_n \subseteq \langle \mathscr{U} \rangle_n$. And, $\langle \mathscr{T} \rangle = \bigcup_{n \geq 0} \langle \mathscr{T} \rangle_n \subseteq \bigcup_{n \geq 0} \langle \mathscr{U} \rangle_n = \langle \mathscr{U} \rangle$. □

The following result is quite important and it arises out of a natural query as to what can one say about the number of steps needed to generate a module from a given class when we have a finite-length resolution of that module by modules all of which can be generated in finitely many steps from the same class.

Lemma 2.1.9. *Let* $\mathscr{T}$ *be a class of* R*-modules. If there exists an exact sequence* $0 \to M_n \to ... \to M_1 \to M \to 0$*, for some* $n > 1$*, where each* M_i *is generated in* a_i *steps from* $\mathscr{T}$*, then* M *can be generated from* $\mathscr{T}$ *in* $n - 1 + \sum_{i=1}^{n} a_i$ *steps.*

Proof. We will provide a proof by induction on n. Note that when $n = 2$, this result holds true by definition of the number of steps of generation. Now, let us assume that for all $n \leq k$, if there exists an exact sequence $0 \to M_n \to ... \to M_1 \to M \to 0$ where each M_i is generated in a_i steps from $\mathscr{T}$, then M can be generated from $\mathscr{T}$ in $n - 1 + \sum_{i=1}^{n} a_i$ steps - this is our induction hypothesis.

Now let $n = k + 1$. If we have an exact sequence $0 \to M_{k+1} \to M_k \to ... \to M_1 \to M \to 0$, we can split it into two exact sequences:

S1) $0 \to M_{k+1} \to M_k \to \mathrm{Im}(M_k \to M_{k-1}) \to 0$

S2) $0 \to \mathrm{Im}(M_k \to M_{k-1}) \to M_{k-1} \to ... \to M_1 \to M \to 0$.

Since M_{k+1} is generated in a_{k+1} steps and M_k is generated in a_k steps from $\mathscr{T}$, looking at $(S1)$ we can say that $\mathrm{Im}(M_k \to M_{k-1})$ can be generated from $\mathscr{T}$ in $a_{k+1} + a_k + 1$ steps. Since $\mathrm{Im}(M_k \to M_{k-1})$, M_{k-1}, . . ., M_2, M_1 can be generated from $\mathscr{T}$ in $a_{k+1} + a_k + 1, a_{k-1}, ..., a_2, a_1$ steps respectively, looking at $(S2)$, we can say using the induction hypothesis that M can be generated from $\mathscr{T}$ in

$$(k-1) + (a_{k+1} + a_k + 1) + a_{k-1} + ... + a_2 + a_1 = k + \sum_{i=1}^{k+1} a_i = ((k+1) - 1) + \sum_{i=1}^{k+1} a_i$$

steps. This completes our induction.

□

2.2 Resolutions of modules by a given class

Definition 2.2.1. *For any class of R-modules, $\mathscr{T}$, we define the $\mathscr{T}$-dimension of a R-module M, denoted $\mathscr{T}$-dim(M), to be min$\{i \in \mathbb{Z} : \exists$ a exact sequence $0 \to T_i \to T_{i-1} \to ... \to T_0 \to M \to 0$ where each $T_i \in \mathscr{T}\}$. If, for an R-module M, no such exact sequence exists for any i, we say $\mathscr{T}$-dim(M) is infinite.*

$[\mathscr{T}]_n$ is the class of all R-modules M such that $\mathscr{T}$-dim$(M) \leq n\}$.

$[\mathscr{T}]_\infty$ is the class of all R-modules that admit resolutions by modules in $\mathscr{T}$ of possibly infinite length.

$[\mathscr{T}]$ is the class of all R-modules with finite $\mathscr{T}$-dimension.

The following result is obvious.

Lemma 2.2.2. *For any class of R-modules, $\mathscr{T}$,*

a) $[\mathscr{T}]_0 \subseteq [\mathscr{T}]_1 \subseteq [\mathscr{T}]_2 \subseteq ...$

b) $[\mathscr{T}]_n \subseteq \langle\mathscr{T}\rangle_n$ for any $n \in \mathbb{Z}_{\geq 0}$.

c) $[\mathscr{T}] \subseteq \langle\mathscr{T}\rangle$.

d) If $\mathscr{U}$ is a class of R-modules such that $\mathscr{T} \subseteq \mathscr{U}$, then $\mathscr{U}$-dim$(M) \leq \mathscr{T}$-dim(M) for all R-modules M.

Proof. (a) follows from the definition of $[\mathscr{T}]_n$. (b) and (c) follow directly from Lemma 2.1.9. To prove (d), we can start with assuming that $\mathscr{T}$-$dim(M) = n < \infty$ and then note that if $0 \to T_n \to ... \to T_0 \to M \to 0$ is an exact sequence where all the T_i's are in $\mathscr{T}$, then they are also in $\mathscr{U}$ as $\mathscr{T} \subseteq \mathscr{U}$ and therefore $\mathscr{U}$-$dim(M) \leq n$. □

The following result is important, in light of what we said in answer to Question 2.1.3, to explain a major usefulness of putting the new module on the right of the generating exact sequence. Also, the "generation number" invariant that we defined in Definition 2.1.1 comes into good use in this result.

Lemma 2.2.3. *Let $\mathscr{T}$ be a class of R-modules. Then, in the following statements, $(a) \Rightarrow (b) \Rightarrow (c)$.*

a) For any short exact sequence of R-modules, $0 \to M_2 \to M_1 \to M \to 0$, $\mathscr{T}$*-dim*$(M) \leq 1 + max\{\mathscr{T}\text{-}dim(M_1), \mathscr{T}\text{-}dim(M_2)\}$*.*

b) $\mathscr{T}$*-dim*$(M) = \alpha_{\mathscr{T}}(M)$*, for all R-modules M.*

c) For any short exact sequence of R-modules, $0 \to M_2 \to M_1 \to M \to 0$, $\mathscr{T}$*-dim*$(M) \leq 1 + \mathscr{T}\text{-}dim(M_1) + \mathscr{T}\text{-}dim(M_2)$*.*

Proof. $(a) \Rightarrow (b)$: For any R-module M, it is clear from the definition of $\mathscr{T}\text{-}dim(M)$ and Lemma 2.1.9 that $\alpha_{\mathscr{T}}(M) \leq \mathscr{T}\text{-}dim(M)$. Assuming the conditions in the hypothesis of the statement of the lemma hold, we will prove by induction on $\mathscr{T}\text{-}dim(M) \leq \alpha_{\mathscr{T}}(M)$. If $\alpha_{\mathscr{T}}(M) = 0$, then $M \in \mathscr{T}$, and therefore $\mathscr{T}\text{-}dim(M) = 0$. Assume that for all modules M such that $\alpha_{\mathscr{T}}(M) \leq n$, if the hypothesis of our lemma is satisfied, then $\mathscr{T}\text{-}dim(M) \leq \alpha_{\mathscr{T}}(M)$ - this is our induction hypothesis. Now, let $\alpha_{\mathscr{T}}(M) = n + 1$, then by definition, we have an exact sequence $0 \to M_2 \to M_1 \to M \to 0$, where $\alpha_{\mathscr{T}}(M_1), \alpha_{\mathscr{T}}(M_2) \leq n$. By the induction hypothesis, it follows that $\mathscr{T}\text{-}dim(M_1), \mathscr{T}\text{-}dim(M_2) \leq n$, and therefore from the hypothesis of the statement of the lemma, it follows that $\mathscr{T}\text{-}dim(M) \leq n + 1 = \alpha_{\mathscr{T}}(M)$. This ends our induction.

$(b) \Rightarrow (c)$: Assume that we have a short exact sequence $0 \to M_2 \to M_1 \to M \to 0$ where $\mathscr{T}\text{-}dim(M_1), \mathscr{T}\text{-}dim(M_2) < \infty$ and $\mathscr{T}\text{-}dim(M) > 1 + \mathscr{T}\text{-}dim(M_1) + \mathscr{T}\text{-}dim(M_2)$. M_1, M_2 can be generated from $\mathscr{T}$ in $\mathscr{T}\text{-}dim(M_1)$ and $\mathscr{T}\text{-}dim(M_2)$ steps respectively by Lemma 2.2.2.b. and therefore M can be generated from $\mathscr{T}$ in $1 + \mathscr{T}\text{-}dim(M_1) + \mathscr{T}\text{-}dim(M_2)$ steps which is strictly smaller than $\mathscr{T}\text{-}dim(M) = \alpha_{\mathscr{T}}(M)$ which is not possible. $\square$

Remark 2.2.4. *Many standard classes of modules like the projectives, the Gorenstein projectives, etc satisfy (a) of Lemma 2.2.3, which means that for those classes, the generation number and the dimension coincide. See Section 2.4 for other examples of classes satisfying (a) of Lemma 2.2.3 or variants of it.*

Lemma 2.2.3 makes it clear why it makes sense to compare the class $[\mathscr{T}]$ with $\langle \mathscr{T} \rangle$ for any class $\mathscr{T}$. There are some results, like Lemma 2.1.7.b., which if true with $[]$ instead of $\langle\rangle$, we end up with an extra condition on $\mathscr{T}$ - see Lemma 3.4.3. Still, in most cases, it is sensible to expect that the $\mathscr{T}$-dimension of a module be finite if its $\mathscr{T}$-generation number is finite. This motivates our next definition which we will revisit in Proposition 3.3.2 and Definition 3.3.3.

Definition 2.2.5. *For any ring R, a class of R-modules $\mathscr{T}$ is called good if $[\mathscr{T}] = \langle \mathscr{T} \rangle$, i.e. if $\mathscr{T}$-dim$(M) < \infty \Leftrightarrow \alpha_{\mathscr{T}}(M) < \infty$ for all R-modules M.*

We also introduce a definition for two classes of modules to be equivalent in terms of how those classes behave under $[]$.

Definition 2.2.6. *For any ring R, two classes of R-modules, $\mathscr{T}$ and $\mathscr{U}$, are said to be equivalent, written $\mathscr{T} \sim \mathscr{U}$, if $[\mathscr{T}] = [\mathscr{U}]$, i.e. $\mathscr{T}$-dim$(M) < \infty \Leftrightarrow \mathscr{U}$-dim$(M) < \infty$ for all R-modules M.*

The following lemma is easy to see.

Lemma 2.2.7. *Let $\mathscr{T}$ and $\mathscr{U}$ be two classes of R-modules that are equivalent. Then $\mathscr{T}$ is good iff $\mathscr{U}$ is good.*

Proof. Let $\mathscr{U}$ be a good class, then $\langle \mathscr{U} \rangle = [\mathscr{U}] = [\mathscr{T}]$ (the last equality follows from $\mathscr{T} \sim \mathscr{U}$). Now, $\langle [\mathscr{T}] \rangle = \langle \langle \mathscr{U} \rangle \rangle = \langle \mathscr{U} \rangle$ (the last equality follows from Corollary 2.1.8.a.). By Lemma 2.2.2.c., $[\mathscr{T}] \subseteq \langle \mathscr{T} \rangle$, so by Lemma 2.1.7.a., $\langle [\mathscr{T}] \rangle \subseteq \langle \mathscr{T} \rangle$. Since $\mathscr{T} \subseteq [\mathscr{T}]$, we have $\langle \mathscr{T} \rangle \subseteq \langle [\mathscr{T}] \rangle$ by Corollary 2.1.8.b. Thus, $\langle [\mathscr{T}] \rangle = \langle \mathscr{T} \rangle$. Thus, $\langle \mathscr{T} \rangle = \langle \mathscr{U} \rangle$. So, $\langle \mathscr{T} \rangle = [\mathscr{T}] = \langle \mathscr{U} \rangle$, so $\mathscr{T}$ is good. Similarly we can prove that $\mathscr{U}$ is good if $\mathscr{T}$ is good. □

Remark 2.2.8. *Note that although for any ring R, the class of Gorenstein projective R-modules, $GProj(R)$, and the class of projective R-modules, $Proj(R)$, are good classes because they satisfy condition (a) of Lemma 2.2.3, they need not be equivalent. For an example, take R to be kG where k is a field of characteristic p and G is C_p, then the global dimension of kG is not finite. As discussed in Example 1.1.14, in this case there exists a Gorenstein projective kG-module of infinite projective dimension. So, we have that $f(kG) \nsim Proj(kG)$.*

The two main examples of good classes that we have seen in this section are both classes that contain all projectives and are closed under syzygies. Below, we provide an example of a class of modules which generate all modules and is good but need not contain the class of all projectives.

Lemma 2.2.9. *Let $\mathscr{U}$ be a class of R-modules that satisfies the following conditions.*

a) $\mathscr{U}$ is closed under direct summands and finite direct sums.

b) For every R-module M, there is a surjective map of R-modules, $\phi : \mathscr{U}_0 \twoheadrightarrow M$, where U_0 is a module in $\mathscr{U}$.

c) $\mathscr{U}$-dim$(M) \leq n \Leftrightarrow$ For every exact sequence $0 \to K \to U_{n-1} \to U_{n-2} \to \ldots \to U_0 \to M \to 0$, where all the $U_i's$ are in $\mathscr{U}$, K is in $\mathscr{U}$.

Then, for the class of all R-modules that do not have finite $\mathscr{U}$-dimension, $\mathscr{T}$, assuming $\mathscr{T}$ is non-empty, $[\mathscr{T}]$ and $\langle \mathscr{T} \rangle$ coincide with the class of all R-modules.

Proof. We start by noting that since $\mathscr{U}$ satisfies hypothesis (b), every R-module admits a resolution with modules in $\mathscr{U}$. This can be easily seen by taking an arbitrary R-module M and taking a surjective map onto M from some module in $\mathscr{U}$, say U_0. We call this map $\phi : U_0 \twoheadrightarrow M$. Then, (b) guarantees the existence of an onto map of R-modules between some module in $\mathscr{U}$, say U_1, and $\text{Ker}(\phi)$, $\phi_1 : U_1 \twoheadrightarrow \text{Ker}(\phi)$. We thus have two short exact sequences: $0 \to \text{Ker}(\phi_1) \hookrightarrow U_1 \twoheadrightarrow \text{Ker}(\phi) \to 0$ and $0 \to \text{Ker}(\phi) \hookrightarrow U_0 \twoheadrightarrow M \to 0$. We can stitch those exact sequences together and get $0 \to \text{Ker}(\phi_1) \hookrightarrow U_1 \to S_0 \to M \to 0$. Going on like this, we get a resolution of M with modules in $\mathscr{U}$ (of possibly infinite length).

Now, let P be a module in $\mathscr{T}$, i.e. $\mathscr{U}$-*dim*$(P) \not< \infty$. Take an arbitrary R-module Q and form the exact sequence $0 \to P \to P \oplus Q \to Q \to 0$. We now claim $\mathscr{U}$-*dim*$(P \oplus Q) \not< \infty$. To see this, assume to the contrary that $\mathscr{U}$-*dim*$(P \oplus Q) = t < \infty$. Take 2 exact sequences $0 \to K_t \to U_{t-1} \to U_{t-2} \to \ldots \to U_0 \to P \to 0$ and $0 \to K'_t \to U'_{t-1} \to U'_{t-2} \to \ldots \to U'_0 \to Q \to 0$ where each U_i and each U'_i is in $\mathscr{U}$ (we have explained in the previous paragraph why taking such exact sequences is always possible). We can form an exact sequence $0 \to K_t \oplus K'_t \to U_{t-1} \oplus U'_{t-1} \to \ldots \to U_0 \oplus U'_0 \to P \oplus Q \to 0$, where each $U_i \oplus U'_i$ is in $\mathscr{U}$ because $\mathscr{U}$ is closed under finite direct sums by (a). Now, as $\mathscr{U}$-*dim*$(P \oplus Q) = t$ and as $\mathscr{U}$ satisfies hypothesis (c), we have that $K_t \oplus K'_t$ is in $\mathscr{U}$. Since $\mathscr{U}$ is closed under direct summands by (a), it implies that K_t is in $\mathscr{U}$. So, $0 \to K_t \to S_{t-1} \to \ldots \to S_0 \to P \to 0$ is a resolution of P with modules in $\mathscr{U}$ of length t, which gives us that $\mathscr{U}$-*dim*$(P) \leq t < \infty$, which is absurd as $\mathscr{U}$-*dim*$(P) \not< \infty$. So, $P \oplus Q$ is in $\mathscr{T}$.

Thus, from the exact sequence $0 \to P \to P \oplus Q \to Q \to 0$, we can say that Q is in $[\mathscr{T}]$. Since Q was taken to be an arbitrary R-module, this implies that $[\mathscr{T}]$ contains all R-modules. From Lemma 2.2.2.c., we know that $[\mathscr{T}]$ is a subclass of $\langle \mathscr{T} \rangle$. So, $\langle \mathscr{T} \rangle$ contains all R-modules as well, and we are done.

$\square$

The result in the following corollary is quite trivial but we are recording it because it is a direct application of Lemma 2.2.9.

Corollary 2.2.10. *Let R be a ring such that there exist R-modules of infinite projective dimension (see Remark 2.2.8.a. for an example) or such that there exist R-modules of infinite Gorenstein projective dimension (for this, we can take R to be the integral group ring of a group that does not admit complete resolutions, like a free abelian group of infinite rank for example). If $\mathscr{T}$ is the class of all R-modules of infinite projective dimension or the class of all R-modules of infinite Gorenstein projective dimension, then $[\mathscr{T}]$ and $\langle\mathscr{T}\rangle$ coincide with the class of all R-modules.*

Proof. This follows directly from Lemma 2.2.9 by noting that the class of projective R-modules satisfies hypotheses $(a), (b)$ and (c) of Lemma 2.2.9 and that the class of Gorenstein projective R-modules satisfies (a) by Lemma 1.1.4, (b) on account of the fact that projective modules are Gorenstein projective (this again follows from Lemma 1.1.4, and (c) by Theorem 1.1.8. $\square$

2.3 Closure results in generation and resolution

In this and the subsequent section, we look at two types of closures - in this section, we record some results where we show that if a given class is closed under certain operations, then the classes generated by them are closed under the same properties and in the next section we record some results where we show that certain special classes of modules that satisfy nice conditions are closed under $\langle\rangle$.

We often have to close our classes under finite direct sums or arbitrary direct sums while using this theory in generating modules over groups in Kropholler's hierarchy (see Section 8.1).

Remark 2.3.1. *Although the class of all projective R-modules, $Proj(R)$, is closed under arbitrary direct sums, $\langle Proj(R)\rangle$ need not be. This follows from Lemma 2.2.3 because if R is of infinite finitistic dimension, for every positive integer n, we can take an R-module M_n satisfying $n < proj.dim_R M_n < \infty$, and then $\bigoplus_{n\in\mathbb{N}} M_n$ does not have finite projective dimension.*

If we move from closing our class of modules under arbitrary direct sums to finite direct sums, we get the following result.

Lemma 2.3.2. *If a class of R-modules, $\mathscr{T}$, is closed under finite direct sums, so is $\langle \mathscr{T} \rangle$.*

Proof. It suffices to prove that if two R-modules are in $\langle \mathscr{T} \rangle$, then so is their direct sum. We shall proceed by strong induction on the sums of the $\mathscr{T}$-generation numbers of the two modules. If this sum is 0, then each module is in $\mathscr{T}$, and so is their direct sum as $\mathscr{T}$ is closed under finite direct sums.

Now let us assume that our claim is true for all pairs of R-modules, M and N, if $\alpha_{\mathscr{T}}(M) + \alpha_{\mathscr{T}}(N) \leq t$. Now, let $\alpha_{\mathscr{T}}(M) + \alpha_{\mathscr{T}}(N) = t + 1$. M admits a $\mathscr{T}$-generation sequence $0 \to A \to B \to M \to 0$ where A, B are generated from $\mathscr{T}$ in a, b steps respectively and $a + b \leq \alpha_{\mathscr{T}}(M) - 1$. This means $a, b < \alpha_{\mathscr{T}}(M)$. Similarly, N admits a $\mathscr{T}$-generation sequence $0 \to C \to D \to N \to 0$ where C, D are generated from $\mathscr{T}$ in c, d steps respectively and $c + d \leq \alpha_{\mathscr{T}}(N) - 1$. This means $c, d < \alpha_{\mathscr{T}}(N)$. Taking a direct sum of these two sequences, we get the following short exact sequence

$$0 \to A \oplus C \to B \oplus D \to M \oplus N \to 0$$

As, $\alpha_{\mathscr{T}}(A) + \alpha_{\mathscr{T}}(C) \leq a + c < \alpha_{\mathscr{T}}(M) + \alpha_{\mathscr{T}}(N) = t + 1$, by our induction hypothesis, $A \oplus C \in \langle \mathscr{T} \rangle$. Similarly, $B \oplus D \in \langle \mathscr{T} \rangle$. Thus, the above sequence is a $\mathscr{T}$-generation sequence admitted by $M \oplus N$, therefore $M \oplus N \in \langle \mathscr{T} \rangle$. That ends our induction. $\square$

The following lemma is very easy to see, we record it nevertheless because it is made repeated use of in Chapter 8.

Lemma 2.3.3. *For any class of R-modules, $\mathscr{T}$, and any non-negative integer n, $[\mathscr{T}]_n$ is closed under arbitrary direct sums if $\mathscr{T}$ is closed under arbitrary direct sums.*

Proof. Let I be an indexing set such that $M_\alpha \in [\mathscr{T}]_n$ for all $\alpha \in I$. For each $\alpha \in I$, let $0 \to P_{\alpha,n} \to P_{\alpha,n-1} \to .. \to P_{\alpha,0} \to M_\alpha \to 0$ be an exact sequence where each $P_{\alpha,i}$ is in $\mathscr{T}$ or is the zero module (if $\mathscr{T}$-$dim(M_\alpha) = t < n$, take $P_{\alpha,i} = 0$ for $i = t{+}1, t{+}2, .., n$.). We now look at the exact sequence $0 \to \bigoplus_{\alpha \in I} P_{\alpha,n} \to ... \to \bigoplus_{\alpha \in I} P_{\alpha,0} \to \bigoplus_{\alpha \in I} M_\alpha \to 0$. For each i, $\bigoplus_{\alpha \in I} P_{\alpha,i} \in \mathscr{T}$ as $\mathscr{T}$ is closed under arbitrary direct sums. Thus, $\bigoplus_{\alpha \in I} M_\alpha \in [\mathscr{T}]_n$. $\square$

Remark 2.3.4. *There are other results of this sort that one can prove - for example, if $\mathscr{T}$ is a class of R-modules containing all projectives and closed under syzygies, then $\langle \mathscr{T} \rangle$ is closed under syzygies or that if $\mathscr{T}$ is closed under direct sums and summands, then $\langle \mathscr{T} \rangle$ is closed under direct summands. But we are choosing to not include these results as lemmas here because we will not making use of these results later.*

We end this short section with the following easy formula on how far the generation numbers of two modules with respect to a given class differ (we do not make any use of this lemma later however).

Lemma 2.3.5. *Let $\mathscr{T}$ be a class of R-modules. Then, for any $M, N \in \langle \mathscr{T} \rangle$, $|\alpha_{\mathscr{T}}(M) - \alpha_{\mathscr{T}}(N)| \leq \alpha_{\mathscr{T}}(M \oplus N) + 1$.*

Proof. We can assume that $\alpha_{\mathscr{T}}(M \oplus N)$ is finite because if it is not we have nothing to prove. Looking at the short exact sequence $0 \to M \to M \oplus N \to N \to 0$, we can say that N can be generated from $\mathscr{T}$ in $\alpha_{\mathscr{T}}(M) + \alpha_{\mathscr{T}}(M \oplus N) + 1$ steps, so $\alpha_{\mathscr{T}}(N) \leq \alpha_{\mathscr{T}}(M) + \alpha_{\mathscr{T}}(M \oplus N) + 1$, i.e. $\alpha_{\mathscr{T}}(N) - \alpha_{\mathscr{T}}(M) \leq 1 + \alpha_{\mathscr{T}}(M \oplus N)$. Similarly, looking at the short exact sequence $0 \to N \to M \oplus N \to M \to 0$, we can say that M can be generated from $\mathscr{T}$ in $\alpha_{\mathscr{T}}(N) + \alpha_{\mathscr{T}}(M \oplus N) + 1$ steps, so $\alpha_{\mathscr{T}}(M) \leq \alpha_{\mathscr{T}}(N) + \alpha_{\mathscr{T}}(M \oplus N) + 1$, i.e. $\alpha_{\mathscr{T}}(M) - \alpha_{\mathscr{T}}(N) \leq \alpha_{\mathscr{T}}(M \oplus N) + 1$. Therefore, $|\alpha_{\mathscr{T}}(M) - \alpha_{\mathscr{T}}(N)| \leq \alpha_{\mathscr{T}}(M \oplus N) + 1$. □

2.4 ⟨⟩-invariant classes of modules

In this section, we give examples of many standard classes of R-modules for any ring R that remain invariant under $\langle\rangle$, i.e. we do not get any new modules when we consider the classes generated by them, we call such classes $\langle\rangle$-invariant. First, we record the following general result.

Lemma 2.4.1. *$\langle\rangle$-invariant classes are good.*

Proof. This is easy to see as, if for some class of R-modules $\mathscr{T}$, $\mathscr{T} = \langle \mathscr{T} \rangle$, then $[\mathscr{T}] = [\langle \mathscr{T} \rangle] \subseteq \langle \langle \mathscr{T} \rangle \rangle = \langle \mathscr{T} \rangle = \mathscr{T}$ using Corollary 2.1.8.a. and Lemma 2.2.2.c.. □

The following lemma is easy to see.

Lemma 2.4.2. *Let $\mathscr{T}$ be the class of all R-modules that satisfy property P. If for any short exact sequence of R-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ where M_1 and M_2 satisfy property P, M_3 does as well, then $\mathscr{T} = \langle \mathscr{T} \rangle$.*

Proof. Since we know that $\mathscr{T} \subseteq \langle \mathscr{T} \rangle$, we just need to prove $\langle \mathscr{T} \rangle \subseteq \mathscr{T}$.

If $\alpha_{\mathscr{T}}(M) = 0$, then $M \in \mathscr{T}$. Assume as our induction hypothesis that for all M satisfying $\alpha_{\mathscr{T}}(M) \leq n$, $M \in \mathscr{T}$. Now if $\alpha_{\mathscr{T}}(M) = n+1$, then we have a short exact sequence $0 \to M_2 \to M_1 \to M \to 0$ where $\alpha_{\mathscr{T}}(M_i) < n+1$, for $i = 1, 2$, and so by the induction hypothesis each M_i satisfies property P. Therefore, M satisfies property P. $\square$

Our first example of a $\langle\rangle$-invariant class is the class of all modules that admit projective resolutions of eventually finite type, i.e modules M which has a projective resolution $P_* \twoheadrightarrow M$ where for some $n \in \mathbb{N}$, P_k is finitely generated for all $k > n$.

Lemma 2.4.3. *Let $\mathscr{T}$ be the class of all R-modules that admit projective resolutions which are of eventually finite type. Then, $\langle \mathscr{T} \rangle = \mathscr{T}$.*

Proof. Take a short exact sequence $0 \to M_2 \to M_1 \to M \to 0$, where M_1 and M_2 admit projective resolutions that are of finite type after k_1 and k_2 steps respectively. By Lemma 3.5 of [20], M admits a projective resolution that is of finite type after $max\{k_1, k_2+1\}$ steps. Thus, we are done by Lemma 2.4.2. $\square$

A class of modules that arises quite often in the cohomology theory of infinite groups is the class of modules of type FP_∞. In [21], it was shown that that if A is a commutative ring and if $\Gamma \in H\mathscr{F}$, then any $A\Gamma$-module of type FP_∞ admits a complete resolution. There was a related result in [21] that for any commutative ring A, if M is an $A\Gamma$-module such that $M \otimes_A B(\Gamma, A)$ (see Definition 1.3.6) has finite projective dimension as an $A\Gamma$-module, then M admits a complete resolution (we discuss this result later in Theorem 4.2.3). The module $B(\Gamma, A)$ is known to be A-free (see Lemma 1.3.7). In the following lemma, we consider three classes of modules that occur in this treatment and show that all these classes are $\langle\rangle$-invariant.

Lemma 2.4.4. *Let A be a commutative ring and let Γ be a group. Take F to be a fixed A-flat $A\Gamma$-module. Then, the following classes are $\langle\rangle$-invariant.*

a) The class of all $A\Gamma$-modules that are of type FP_∞.

b) The class of all $A\Gamma$-modules M such $proj.dim_{A\Gamma} M \otimes_A F < \infty$.

c) The class of all $A\Gamma$-modules that admit complete resolutions

Proof. (a) and (b) are obvious in light of Lemma 2.4.2.

c) M admits complete resolutions iff $Gpd_{A\Gamma}(M) < \infty$ (this follows from Remark 4.1.6 which we handle later). The lemma now follows from Lemma 1.1.10 and Lemma 2.4.2. □

Following the proof of Lemma 3.5 in [20], the following result, which mirrors Lemma 2.4.3, follows. Note that complete resolutions of eventually finite type are defined in the same way as projective resolutions of eventually finite type - M is said to admit a complete resolution of eventually finite type iff M admits a complete resolution $(F_i, d_i)_{i\in\mathbb{Z}}$ such that there exists an integer n such that F_k is finitely generated for all $k > n$.

Lemma 2.4.5. *Let $\mathscr{T}$ be the class of all R-modules that admit complete resolutions which are of eventually finite type. Then, $\langle\mathscr{T}\rangle = \mathscr{T}$.*

If we have a functor or a sequence of functors that translates short exact sequences of modules into long exact sequences, like the Ext-functor, then it becomes relatively easy to prove results on the $\langle\rangle$-invariant behaviour of classes of modules where this sequence of functors vanish eventually. One comes across such functors in relative homological algebra a lot.

Definition 2.4.6. *(see Section 3 of [41]) Let R_1 and R_2 be two rings. A sequence of additive functors from the category of R_1-modules to the category of R_2-modules, $(S^i : i \in \mathbb{Z})$, denoted by S, is called a $(-\infty, \infty)$-cohomological functor from R_1-modules to R_2-modules if for any short exact sequence of R_1-modules $0 \to A \to B \to C \to 0$, there exists a natural long exact sequence of R_2-modules $\ldots \to S^{n-1}(C) \to S^n(A) \to S^n(B) \to S^n(C) \to S^{n+1}(A) \to \ldots$. We define $\underline{S}(M) := sup\{i \in \mathbb{Z} : S^i(M) \neq 0\}$.*

The following result is obvious in light of Lemma 2.4.2.

Lemma 2.4.7. *Let S be a $(-\infty, \infty)$-cohomological functor from R_1-modules to R_2-modules, and let $\mathscr{T}$ be the class of all modules M such that $\underline{S}(M) < \infty$. Then, $\langle\mathscr{T}\rangle = \mathscr{T}$.*

As noted in [41], the cohomology functors $(H^i(\Gamma, ?) : i \in \mathbb{Z})$, where $H^i(\Gamma, ?)$ is defined to be zero for negative i, and the complete cohomology functors $(\hat{H}^i(\Gamma, ?) : i \in \mathbb{Z})$ are examples of $(-\infty, \infty)$-cohomological functors from $\mathbb{Z}\Gamma$-modules to $\mathbb{Z}$-modules.

We end this section with the following question.

Question 2.4.8. *a) We have seen examples of classes $\mathscr{T}$ where $\alpha_{\mathscr{T}}$ and $\mathscr{T}$-dimension coincide for all modules and are finite. Is there a class $\mathscr{T}$ of R-modules that is good but $\alpha_{\mathscr{T}}(M) > \mathscr{T}$-dim$(M)$ for some R-module M?*

b) Given a class of R-modules $\mathscr{T}$ and an R-module M such that $\alpha_{\mathscr{T}}(M) = m$, with what precision can we find a module N such that $\alpha_{\mathscr{T}}(N) = n$ for a given $n < m$? This question makes sense because mostly when we work with generation numbers, we find bounds rather than prove that some generation number is exactly equal to some number.

Chapter 3

Compositions of Generation Operators

This chapter is a little technical and abstract. Although most of the abstract results of this chapter are not made use of later, this chapter is important as a continuation of the development of the theory of generation operators started in Chapter 2.

Note that if we take a class of R-modules and apply any of the brackets [] or $\langle\rangle$ on it, and then apply some other bracket again and so on, what we get is a string of brackets applied to the class, so it makes sense to study these brackets as operators denoted by symbols and their iterations as concatenated words. This motivates us to introduce the following definition.

3.1 Definitions and a fundamental property

For this section, we introduce two new notations for the brackets [] and $\langle\rangle$.

Definition 3.1.1. *For any class of R-modules $\mathscr{T}$, we denote $B(\mathscr{T}) = [\mathscr{T}]$ and $C(\mathscr{T}) = \langle\mathscr{T}\rangle$. For any non-empty word in B and C, $W = B_1B_2..B_n$ for some n where each B_i is either B or C, we define $W(\mathscr{T}) := B_1(B_2(..(B_n(\mathscr{T}))..)$. The empty word is denoted by B^0 or C^0, and, as operators, $B^0(\mathscr{T}) = C^0(\mathscr{T}) := \mathscr{T}$.*

The notation $B^{\geq n}$ or $B^{>n}$ means any operator B^k where $k \geq n$ or $k > n$ respectively.

We denote the set of all finite words in B and C by $W(B, C)$.

We start by providing a summary of the results that we prove in this chapter.

It seems a natural question to ask whether defining the operator B^ω where ω is the first infinite ordinal as $B^\omega(\mathscr{T}) := \bigcup_{n\in\mathbb{N}} B^n(\mathscr{T})$ for any class $\mathscr{T}$, gives us new operators. Unfortunately, that is not the case as our next result shows.

Theorem 3.1.2. *For any class of R-modules, $\mathscr{T}$, $C(\mathscr{T}) = B^\omega(\mathscr{T})$ where ω is the first infinite ordinal.*

Proof. First we show that $B^n(\mathscr{T}) \subseteq C(\mathscr{T})$ for all n. This can be shown easily by induction on n. If $n = 1$, it follows from Lemma 2.1.9. Let $B^k(\mathscr{T}) \subseteq C(\mathscr{T})$ for all $k \leq n$-this is our induction hypothesis. Then, if we take $M \in B^{n+1}(\mathscr{T}) = [B^n(\mathscr{T})]$, we see that we have an exact sequence $0 \to M_t \to M_{t-1} \to ... \to M_0 \to M \to 0$ for some t where each $M_i \in B^n(\mathscr{T})$ and therefore by our induction hypothesis, each M_i is generated from $\mathscr{T}$ in, say, a_i steps where a_i is a finite integer. Thus, by Lemma 2.1.9 again, M can be generated from $\mathscr{T}$ in $t + \sum_{i=0}^{t} a_i$ steps and is therefore in $\langle \mathscr{T} \rangle = C(\mathscr{T})$.

Now to prove that if $M \in \langle \mathscr{T} \rangle$, then $M \in \bigcup_{n\in\mathbb{N}} B^n(\mathscr{T})$, we proceed by induction on $\alpha_{\mathscr{T}}(M)$. If $\alpha_{\mathscr{T}}(M) = 0$, then $M \in \mathscr{T} \subseteq [\mathscr{T}] = B(\mathscr{T})$. Let us assume that for all $k \leq n$, if $\alpha_{\mathscr{T}}(M) = k$, then $M \in \bigcup_{n\in\mathbb{N}} B^n(\mathscr{T})$ - this is our induction hypothesis. Now let $\alpha_{\mathscr{T}}(M) = n + 1$, then by definition, we have an exact sequence $0 \to A \to B \to M \to 0$ for some A and B where $\alpha_{\mathscr{T}}(A), \alpha_{\mathscr{T}}(B) < n$. Therefore, $A, B \in \bigcup_{n\in\mathbb{N}} B^n(\mathscr{T})$, by our induction hypothesis. Let $A \in B^a(\mathscr{T})$, and $B \in B^b(\mathscr{T})$. Then, for any $d > max\{a, b\}$, both $A, B \in B^d(\mathscr{T})$, and that implies $M \in [B^d(\mathscr{T})] = B^{d+1}(\mathscr{T})$. This ends our induction.

□

3.2 Classification of all words as operators

Summary of this section's results:

a) We show that, as operators, $BC = CB = C = C^2$. So, to describe all operators, we can say that they are given by the set of all finite words on B and C such that $BC = CB = C = C^2$. This lets us figure out what all possible words can look like as operators.

b) We show that, as operators on a given class $\mathscr{T}$, $B^2 = B \Leftrightarrow B = C$, and that when either of these conditions holds, $\mathscr{T}$ satisfies many other nice conditions. In this case, any module that is generated from $\mathscr{T}$ in finitely many steps admits a finite-length resolution of modules from $\mathscr{T}$.

c) We prove some results on the eventual stability of sequences words in B and C as operators on a given class $\mathscr{T}$. We find what precise word such sequences stabilise to and what conclusions we can then make about $\mathscr{T}$ in terms of its 'good'ness.

It makes sense to wonder if choosing W to be a large word in $W(B, C)$ gives rise to a long expression for $W(\mathscr{T})$. In our next result, we see how things get simplified if we have C in our word.

Lemma 3.2.1. *Let $W \in W(B, C)$ be a non-empty word.*

a) If W is of the form UC for some $U \in W(B, C)$, then $W(\mathscr{T}) = C(\mathscr{T})$ for all classes of modules $\mathscr{T}$.

b) If W is of the form CV for some $V \in W(B, C)$, then $W(\mathscr{T}) = C(\mathscr{T})$ for all classes of modules $\mathscr{T}$.

Proof. a) It suffices to prove that $C^2(\mathscr{T}) = C(\mathscr{T})$ and $BC(\mathscr{T}) = C(\mathscr{T})$. $C^2(\mathscr{T}) = C(\mathscr{T})$, by Corollary 2.1.8. Now, it is easy to see that $C(\mathscr{T}) \subseteq BC(\mathscr{T})$. If $M \in BC(\mathscr{T})$, by definition of B, there exists an exact sequence $0 \to C_n \to \ldots \to C_1 \to M \to 0$, for some n, where each $C_i \in C(\mathscr{T})$. By Lemma 2.1.9, therefore, $M \in C(\mathscr{T})$. So, $BC(\mathscr{T}) = C(\mathscr{T})$.

b) We first prove that $C(\mathscr{T}) = CB(\mathscr{T})$. By Lemma 2.2.2.c., $B(\mathscr{T}) \subseteq C(\mathscr{T})$. Therefore, by Lemma 2.1.7, $CB(\mathscr{T}) \subseteq C(\mathscr{T})$. Also, as $\mathscr{T} \subseteq B(\mathscr{T})$, $C(\mathscr{T}) \subseteq CB(\mathscr{T})$ by Corollary 2.1.8. Thus, $C(\mathscr{T}) = CB(\mathscr{T})$, for all $\mathscr{T}$. Putting $B(\mathscr{T})$ in place of $\mathscr{T}$ in $C(\mathscr{T}) = CB(\mathscr{T})$, we get $CB^2(\mathscr{T}) = CB(\mathscr{T}) = C(\mathscr{T})$. Thus, we have shown that for any non-negative integer n, $CB^n(\mathscr{T}) = C(\mathscr{T})$ for all classes of modules $\mathscr{T}$. Now, for any $V \in W(B, C)$, it follows that $CV = C$ (as operators) because $CC = C$ and $CB^n = C$ for any n (as operators).

$\square$

Corollary 3.2.2. *For any word $W \in W(B, C)$, if W contains C, then $W(\mathscr{T}) = C(\mathscr{T})$, for all classes of modules $\mathscr{T}$.*

Proof. If C is in a word W, then W is of the form UCV for some $U, V \in W(B, C)$. Now, by Lemma 3.2.1, as operators, $W = (UC)V = CV = C$. So, we are done. $\square$

The following result, which follows as a corollary to Lemma 3.2.1, shows that for any word in $W(B, C)$ and for any class of modules $\mathscr{T}$, $W(\mathscr{T})$ can be one of two things.

Corollary 3.2.3. *Take an arbitrary non-empty $W \in W(B, C)$. Then, for any class of modules $\mathscr{T}$, $W(\mathscr{T})$ is one of the following*

a) $C(\mathscr{T})$.

b) Of the form $B^n(\mathscr{T})$, for some positive integer n.

Proof. If W contains C, then, by Lemma 3.2.1, $W = C$ as operators. If there is no C in W, then $W = B^n$, for some n. $\square$

Corollary 3.2.4. *$BC = CB$ as operators, i.e. the brackets $[]$ and $\langle\rangle$ commute with each other.*

Proof. This is obvious from Corollary 3.2.2 $\square$

3.3 Characterisation of good classes through word problems

We start this section with a basic lemma which will be used in the proof of Proposition 3.3.2.

Lemma 3.3.1. *Let $\mathscr{T}$ be a class of R-modules. Then, the following two statements are equivalent.*

a) For any short exact sequence of R-modules $0 \to A \to B \to C \to 0$, if $A, B \in [\mathscr{T}]$, then $C \in [\mathscr{T}]$.

b) For any exact sequence of R-modules $0 \to C_n \to C_{n-1} \to ... \to C_1 \to C \to 0$, for any $n > 1$, if each $C_i \in [\mathscr{T}]$, then $C \in [\mathscr{T}]$.

Proof. $(b) \Rightarrow (a)$ is obvious.

$(a) \Rightarrow (b)$: we shall proceed by induction on n. If $n = 2$, (b) holds true, by (a). Let the statement of (b) hold true for $n = k$, this is our induction hypothesis. Now let

$0 \to C_{k+1} \to C_k \to ... \to C_1 \to C \to 0$ be an exact sequence where each C_i is in $[\mathscr{T}]$. We split this into two exact sequences.

S0) $0 \to C_{k+1} \to C_k \to \mathrm{Im}(C_k \to C_{k-1}) \to 0$.

S1) $0 \to \mathrm{Im}(C_k \to C_{k-1}) \hookrightarrow C_{k-1} \to ... \to C_1 \to C \to 0$.

From (a), it follows that in $(S0)$, $\mathrm{Im}(C_k \to C_{k-1}) \in [\mathscr{T}]$. Therefore, in $(S1)$, every module other than C is in $[\mathscr{T}]$. So, by our induction hypothesis, $C \in [\mathscr{T}]$. □

The following proposition gives us some equivalent properties of a class being good in terms of words of operators from $W(B, C)$ acting on it.

Proposition 3.3.2. *For any class of R-modules, $\mathscr{T}$, the following statements are equivalent.*

a^*) For any short exact sequence of R-modules, $0 \to A \to B \to C \to 0$, if $\mathscr{T}$-dim(A), $\mathscr{T}$-dim$(B) < \infty$, then $\mathscr{T}$-dim$(C) < \infty$.

a) $B^2(\mathscr{T}) = B(\mathscr{T})$.

b^*) For all R-modules M, $\mathscr{T}$-dim$(M) < \infty$ iff $\alpha_{\mathscr{T}}(M) < \infty$.

b) $B(\mathscr{T}) = C(\mathscr{T})$.

c) $C(\mathscr{T}) \sim \mathscr{T}$.

d) $\mathscr{T} \sim B(\mathscr{T})$.

e) For any class of R-modules, $\mathscr{U}$, if $\mathscr{U} \subseteq B(\mathscr{T})$, then $B(\mathscr{U}) \subseteq B(\mathscr{T})$.

f) For any non-empty word $W \in W(B, C)$, $W(\mathscr{T}) = B(\mathscr{T})$.

g) For any non-empty word $W \in W(B, C)$, $W(\mathscr{T}) = C(\mathscr{T})$.

Proof. We start by noting that $(a^*) \Leftrightarrow B^2(\mathscr{T}) = B(\mathscr{T})$ (by Lemma 3.3.1). Also, as, by definition, $B(\mathscr{T}) = \{M \in Mod(R) : \mathscr{T}\text{-}dim(M) < \infty\}$ and $C(\mathscr{T}) = \{M \in Mod(R) : \alpha_{\mathscr{T}}(M) < \infty\}$, $(b^*) \Leftrightarrow B(\mathscr{T}) = C(\mathscr{T})$.

Now, for any two classes of modules, $\mathscr{U}$ and $\mathscr{V}$, $\mathscr{U} \sim \mathscr{V} \Leftrightarrow B(\mathscr{U}) = B(\mathscr{V})$. So, $(c) \Leftrightarrow BC(\mathscr{T}) = B(\mathscr{T}) \Leftrightarrow C(\mathscr{T}) = B(\mathscr{T}) \Leftrightarrow (b)$ (here we used Corollary 3.2.2 to say that $BC(\mathscr{T}) = C(\mathscr{T})$). And, $(d) \Leftrightarrow B(\mathscr{T}) = B^2(\mathscr{T}) \Leftrightarrow (a)$. We thus have $(b) \Leftrightarrow (c)$, $(a) \Leftrightarrow (d)$.

$(b) \Rightarrow (a) : B(\mathscr{T}) = C(\mathscr{T}) \Rightarrow B^2(\mathscr{T}) = BC(\mathscr{T}) = C(\mathscr{T}) = B(\mathscr{T})$ ($BC = C$, by Corollary 3.2.2).

$(a) \Rightarrow (b)$: As $B(\mathscr{T}) \subseteq C(\mathscr{T})$ by Lemma 2.2.2.c, for all R-modules M, $\alpha_{\mathscr{T}}(M) < \infty$ if $\mathscr{T}$-$dim(M) < \infty$. We now proceed by strong induction on $\alpha_{\mathscr{T}}(M)$ to show that

$\mathscr{T}$-*dim*$(M) < \infty$ if $\alpha_{\mathscr{T}}(M) < \infty$. If $\alpha_{\mathscr{T}}(M) = 0$, $M \in \mathscr{T}$, and $\mathscr{T}$-*dim*$(M) < \infty$. Let us assume that any module with $\mathscr{T}$-generation number $\leq n$ has finite $\mathscr{T}$-dimension, this is our induction hypothesis. Now if $\alpha_{\mathscr{T}}(M) = n+1$, we have a short exact sequence $0 \to C_2 \to C_1 \to M \to 0$ such that $\alpha_{\mathscr{T}}(C_1), \alpha_{\mathscr{T}}(C_2) \leq n$. By the induction hypothesis, $\mathscr{T}$-*dim*(C_1), $\mathscr{T}$-*dim*$(C_2) < \infty$. By (a), $\mathscr{T}$-*dim*$(M) < \infty$. So, if $\alpha_{\mathscr{T}}(M) < \infty$, $\mathscr{T}$-*dim*$(M) < \infty$.

We have thus shown $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d)$.

$(d) \Rightarrow (e)$: Let $\mathscr{U} \subseteq B(\mathscr{T})$. Take $M \in B(\mathscr{U})$. There exists an exact sequence $0 \to C_n \to C_{n-1} \to \dots \to C_1 \to M \to 0$, for some n, where each $C_i \in \mathscr{U} \subseteq B(\mathscr{T})$. Thus, $B(\mathscr{T})$-*dim*$(M) < \infty$, which implies that $\mathscr{T}$-*dim*$(M) < \infty$ as $\mathscr{T} \sim B(\mathscr{T})$. Thus, $M \in B(\mathscr{T})$. So, $B(\mathscr{U}) \subseteq B(\mathscr{T})$.

$(e) \Rightarrow (a)$: Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules where $\mathscr{T}$-*dim*(A), $\mathscr{T}$-*dim*$(B) < \infty$, i.e., $A, B \in B(\mathscr{T})$. Let $\mathscr{U}$ be a class of R-modules that consists of just A and B. Then, $C \in B(\mathscr{U})$ and $\mathscr{U} \subseteq B(\mathscr{T})$. From (e), it follows that, $C \in B(\mathscr{U}) \subseteq B(\mathscr{T})$. Thus, $\mathscr{T}$-*dim*$(C) < \infty$.

Thus, $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e)$.

$(f) \Rightarrow (a)$: Take $W = BB$, then $W(\mathscr{T}) = B^2(\mathscr{T}) = B(\mathscr{T}) \Leftrightarrow (a)$.

$(a) \Rightarrow (f)$: Let $W \in W(B, C)$ be an arbitrary non-empty word. Then $W(\mathscr{T})$ is either $C(\mathscr{T})$ or $B^n(\mathscr{T})$ for some n by Corollary 3.2.3. But $(a) \Rightarrow B^2(\mathscr{T}) = B(\mathscr{T}) \Rightarrow B^n(\mathscr{T}) = B(\mathscr{T})$, for all n, and $(a) \Rightarrow (b) \Rightarrow B(\mathscr{T}) = C(\mathscr{T})$. So, we are done.

$(b) \Rightarrow (g)$: Let $W \in W(B, C)$ be an arbitrary non-empty word. Then $(b) \Rightarrow B(\mathscr{T}) = C(\mathscr{T}) \Rightarrow W(\mathscr{T}) = C(\mathscr{T})$ (by Corollary 3.2.2, if W contains C, then $W(\mathscr{T}) = C(\mathscr{T})$).

$(g) \Rightarrow (b)$. Take $W = C$. Then, $W(\mathscr{T}) = B(\mathscr{T}) \Rightarrow C(\mathscr{T}) = B(\mathscr{T}) \Rightarrow (b)$. □

We defined good classes in Definition 2.2.5 and here we provide an equivalent definition of good classes in light of Proposition 3.3.2.

Definition 3.3.3. *For any ring R, a class of modules $\mathscr{T}$ is called good iff $\mathscr{T}$ satisfies the equivalent conditions of Proposition 3.3.2.*

From Proposition 3.3.2, we see that on good classes all words from $W(B, C)$ as operators give the same result. This is one of the motivations behind using the adjective "good".

Corollary 3.3.4. *If $\mathscr{T}$ is a good class, then so is $B^n(\mathscr{T})$, $\forall n \in \mathbb{N}$.*

Proof. $\mathscr{T}$ is a good class $\Leftrightarrow B^2(\mathscr{T}) = B(\mathscr{T}) \Rightarrow B^{n+2}(\mathscr{T}) = B^{n+1}(\mathscr{T}) \Leftrightarrow B^2(B^n(\mathscr{T})) = B(B^n(\mathscr{T}))$, for any $n \in \mathbb{N}$. Thus condition (a) of Proposition 3.3.2 holds for $B^n(\mathscr{T})$. □

Corollary 3.3.5. *For any class of modules $\mathscr{T}$, $B(\mathscr{T})$ is a good class iff $B(\mathscr{T}) \sim C(\mathscr{T})$.*

Proof. $B(\mathscr{T})$ satisfies the condition (b) of Proposition 3.3.2
$\Leftrightarrow B^2(\mathscr{T}) = CB(\mathscr{T}) = BC(\mathscr{T})$ (By Corollary 3.2.2, $BC(\mathscr{T}) = CB(\mathscr{T}) = C(\mathscr{T})$.)
$\Leftrightarrow B(\mathscr{T}) \sim C(\mathscr{T})$.

□

Remark 3.3.6. *We have the conditions "$C(\mathscr{T}) \sim \mathscr{T}$" and "$\mathscr{T} \sim B(\mathscr{T})$" in Proposition 3.3.2. Whether "$B(\mathscr{T}) \sim C(\mathscr{T})$" can be added as an equivalent condition in Proposition 3.3.2 is a natural question to ask. From Corollary 3.3.5, "$B(\mathscr{T}) \sim C(\mathscr{T})$" can be added as an equivalent condition in Proposition 3.3.2 iff $\mathscr{T}$ being a good class and $[\mathscr{T}]$ being a good class are equivalent, for all $\mathscr{T}$.*

Corollary 3.3.4 and Corollary 3.3.5 deal with cases where $B^n(\mathscr{T})$ is a good class for some n as a result of some conditions on $\mathscr{T}$. Also, note that if $B^n(\mathscr{T})$ is good for some n, then words with more than n many B's collapse to simpler expressions as operators on $\mathscr{T}$ - Theorem 3.4.2, which we prove later, details this. This motivates our next definition.

Definition 3.3.7. *For any class of R-modules, $\mathscr{T}$, $g(\mathscr{T})$:=min$\{n \in \mathbb{Z} : B^n(\mathscr{T})$ is a good class $\}$.*

Lemma 3.3.8. *For any class of R-modules $\mathscr{T}$, $g(\mathscr{T}) > n$ iff $B^n(\mathscr{T})$ is not a good class.*

Proof. The "only if" part is easy to see. If $g(\mathscr{T}) > n$, then by definition of $g(\mathscr{T})$, $B^n(\mathscr{T})$ cannot be a good class.

The "if" part follows from the fact that if $B^n(\mathscr{T})$ is not a good class, then $B^k(\mathscr{T})$, for any non-negative integer $k \leq n$, cannot be good - this is because if $B^k(\mathscr{T})$ is good for some $k < n$, then, by repeated applications of Corollary 3.3.4, $B^n(\mathscr{T})$ is good. □

3.4 On sequences of words and other results

We start this section with the following lemma which later helps is determining some crucial properties about the behaviour of sequences of words in $W(B,C)$ as operators using Proposition 3.3.2.

Lemma 3.4.1. *Let W_1,$W_2 \in W(B,C)$, not necessarily non-empty such that $W_1(\mathscr{T}) = W_2(\mathscr{T})$, for some class of R-modules, $\mathscr{T}$. Then, one of the following is true:*

a) $W_1 = W_2$, as operators.

b) $B^n(\mathscr{T})$ is a good class for some non-negative integer n, i.e. $g(\mathscr{T}) < \infty$.

Proof. If $W_1 = W_2$ as words in $W(B,C)$, then (a) holds true, so we can assume that $W_1 \neq W_2$ in $W(B,C)$.

Case 1 : One of W_1 and W_2 is empty. Say, W_1 is empty. Then, by Corollary 3.2.3, one of the following is true:

Case 1.i : $\mathscr{T} = B^n(\mathscr{T})$, for some n. Note that $\mathscr{T} \subseteq B(\mathscr{T}) \subseteq B^2(\mathscr{T}) \subseteq ....$ So, $B^2(\mathscr{T}) = B(\mathscr{T})$, i.e. $\mathscr{T}$ is a good class by (a) of Proposition 3.3.2.

Case 1.ii : $\mathscr{T} = C(\mathscr{T})$. Then, $B(\mathscr{T}) = BC(\mathscr{T}) = C(\mathscr{T})$ (by Lemma 3.2.1).

Case 2 : Neither of W_1 and W_2 is empty. There are three possibilities here.

Case 2.i : Neither of W_1 and W_2 contain C. Then, the two sequences are just comprised of B"'s. If both the words contain the same number of B's, then $W_1 = W_2$ as words in $W(B,C)$.

If W_1 is comprised of n B's and W_2 is comprised of $n+t$ B's for some $t > 0$. Again, as $B^n(\mathscr{T}) \subseteq B(B^n(\mathscr{T})) \subseteq B^2(B^n(\mathscr{T})) \subseteq ...$, this means $B^2(B^n(\mathscr{T})) = B(B^n(\mathscr{T}))$, thus $B^n(\mathscr{T})$ is a good class by (a) of Proposition 3.3.2.

Case 2.ii : One of W_1 and W_2, say W_1, contains C, but W_2 does not. Then, $C(\mathscr{T}) = B^n(\mathscr{T})$, for some n. Then, $C.B^{n-1} = B.B^{n-1}$, as operators on $\mathscr{T}$, because $C.B^{n-1} = C$ as operators on $\mathscr{T}$, by Corollary 3.2.2. Thus, $B^{n-1}(\mathscr{T})$ is a good class.

Case 2.iii : Both W_1 and W_2 contain C. Then, by Corollary 3.2.2, $W_1 = W_2 = C$ as operators. □

We are now in a position to prove the following theorem about sequences of words in $W(B,C)$ as operators on a given class of modules.

Theorem 3.4.2. *For any class of R-modules, $\mathscr{T}$, the following are equivalent.*

a^*) Any sequence $\{W_i(\mathscr{T})\}_{i\in\mathbb{N}}$ where, for each i, $W_i \in W(B,C)$, and as words in $W(B,C)$, $W_i \neq W_j$ when $i \neq j$, eventually stabilises.

a) Any sequence $\{W_i(\mathscr{T})\}_{i\in\mathbb{N}}$ where, for each i, $W_i \in W(B,C)$, and as words in $W(B,C)$, $W_i \neq W_j$ when $i \neq j$, eventually stabilises to $C(\mathscr{T})$.

b) The sequence $B^0(\mathscr{T})$, $B(\mathscr{T})$, $B^2(\mathscr{T})$, $B^3(\mathscr{T})$, . . . eventually stabilises.

c) The sequence in (b) stabilises to $C(\mathscr{T})$.

d) $g(\mathscr{T}) < \infty$.

Proof. Note that $(c) \Rightarrow (b)$ trivially and also $(a) \Rightarrow (b)$ trivially because we can choose W_i to be B^i in the statement of (a).

$(b) \Rightarrow (c)$, $(b) \Rightarrow (d)$: Let n be the smallest integer such that $B^n(\mathscr{T}) = B^m(\mathscr{T})$ for all integers $m \geq n$. We then have $B(B^n(\mathscr{T})) = B^2(B^n(\mathscr{T}))$. So, $B^n(\mathscr{T})$ satisfies the condition on $\mathscr{T}$ in (a) of Proposition 3.3.2, and is therefore good. This proves that $g(\mathscr{T}) \leq n$, which establishes that $(b) \Rightarrow (d)$. $\mathscr{U} := B^n(\mathscr{T})$ should also satisfy (b) of Proposition 3.3.2, so $B(\mathscr{U}) = C(\mathscr{U})$. We therefore have $B(B^n(\mathscr{T})) = C(B^n(\mathscr{T})) = C(\mathscr{T})$ (as operators on $\mathscr{T}$, $CB^n = C$, by Corollary 3.2.2). Thus, $B^{\geq n}(\mathscr{T}) = B^{n+1}(\mathscr{T}) = C(\mathscr{T})$. This establishes that $(b) \Rightarrow (c)$.

$(d) \Rightarrow (b)$: If $g(\mathscr{T}) = k < \infty$, then $B^k(\mathscr{T})$ is a good class. $B^k(\mathscr{T})$ satisfies the conditions on $\mathscr{T}$ in Proposition 3.3.2. Condition (f) of Proposition 3.3.2 says that $B^{\geq 0}(B^k(\mathscr{T})) = B^{k+1}(\mathscr{T})$, for all $t \geq 0$. Thus, (b) holds.

$(c) \Rightarrow (a)$: Let n be the smallest integer such that $C(\mathscr{T}) = B^m(\mathscr{T})$ for all integers $m \geq n$. We can assume that $n > 0$ because if $n = 0$, then $C(\mathscr{T}) = \mathscr{T}$, which means that $C(\mathscr{T}) = BC(\mathscr{T}) = B(\mathscr{T})$ ($C = BC$, as operators, follows from Corollary 3.2.2), which in turn means that every term in any sequence of the form described in (a) will be equal to $C(\mathscr{T})$.

For any $W \neq B^i$, where $i \leq n-1$, $W(\mathscr{T}) = C(\mathscr{T})$. This is because if we take a $W \neq B^i$ for some $i \leq n-1$, then either W contains C, in which case $W(\mathscr{T}) = C(\mathscr{T})$ by Corollary 3.2.2, or W does not contain C, in which case it must have at least n B's, and then $W(\mathscr{T}) = C(\mathscr{T})$ by our definition of n. Let us now take an arbitrary sequence $\{W_i(\mathscr{T})\}_{i\in\mathbb{N}}$ where, as words in $W(B,C)$, $W_i \neq W_j$ if $i \neq j$. Any term in this sequence that is not of the form $B^m(\mathscr{T})$ for some $m \leq n-1$ is equal to $C(\mathscr{T})$. Thus, the sequence $\{W_i(\mathscr{T})\}_{i\in\mathbb{N}}$ eventually stabilises to $C(\mathscr{T})$ because all of its terms that are not equal to $C(\mathscr{T})$ can occur only finitely many times (this is where we are using the fact that the exact same word cannot occur more than once which is guaranteed by the assumption that $W_i \neq W_j$ when $i \neq j$).

We thus have $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d)$. Now note that $(a^*) \Rightarrow (a)$ trivially. Now assume that (a) holds. Then, there exists some n such that $W_n(\mathscr{T}) = W_{n+1}(\mathscr{T})$, and from the proof of Lemma 3.4.1, we have that either $W_n = W_{n+1} = C$ as operators on $\mathscr{T}$ or $W_n = W_{n+1}$ as words in $W(B, C)$ which is not possible as $W_i \neq W_j$ as words in $W(B, C)$ when $i \neq j$, or $g(\mathscr{T}) < \infty$. If, for all $m \geq n$, $W_m = C$ as operators on $\mathscr{T}$, then (a) follows. Otherwise, from the proof of Lemma 3.4.1 as mentioned, (d) follows. $\square$

In Lemma 2.1.7.b., we saw that for any two classes of R-modules $\mathscr{T}$ and $\mathscr{U}$, if $\mathscr{U} \subseteq \langle \mathscr{T} \rangle_m$, for some m, then $\langle \mathscr{U} \rangle_n \subseteq \langle \mathscr{T} \rangle_{mn+m+n}$ for any n. A similar result involving a different kind of brackets implies stronger conditions on $\mathscr{T}$:

Lemma 3.4.3. *Let $\mathscr{T}$ be a class of R-modules. In the following sequence of statements, $(a) \Rightarrow (b)$.*

(a) There exists a function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ such that, for any class of R-modules, $\mathscr{U}$, if $\mathscr{U} \subseteq [\mathscr{T}]_m$ for some m, then, for any n, $[\mathscr{U}]_n \subseteq [\mathscr{T}]_{f(m,n)}$.

(b) $\mathscr{T}$ is a good class.

Proof. Take $\mathscr{X} \subseteq [\mathscr{T}]$. Pick an arbitrary module M in $[\mathscr{X}]$. Let $\mathscr{X}\text{-}dim(M) = k$, this means we have an exact sequence $0 \to X_k \to X_{k-1} \to \ldots \to X_0 \to M \to 0$ where $X_i \in \mathscr{X} \subseteq [\mathscr{T}]$, for all $i \in \{0, 1, .., k\}$.

Let $p := max\{\mathscr{T}\text{-}dim(X_i) : 0 \leq i \leq k\}$. Now, if $\mathscr{M} := \{X_0, X_1, ..., X_k\}$, then $\mathscr{M} \subseteq [\mathscr{T}]_p$. Note that $M \in [\mathscr{M}]_k$, and therefore by (a), $M \in [\mathscr{M}]_k \subseteq [\mathscr{T}]_{f(p,k)} \subseteq [\mathscr{T}]$. Thus, $[\mathscr{X}] \subseteq [\mathscr{T}]$, and by (e) of Proposition 3.3.2, $\mathscr{T}$ is a good class. $\square$

In Lemma 3.4.1, we investigated the question of the different words as operators yielding the same result upon being applied to a given class of modules. It is an interesting question to ponder as to what one can tell about one operator giving the same result upon being applied to two different classes if we have a different operator that gives the same result upon being applied to the same two classes. From Corollary 3.2.2 it follows that if for any two classes of R-modules, $\mathscr{T}$ and $\mathscr{U}$, $B(\mathscr{T}) = B(\mathscr{U})$, then $C(\mathscr{T}) = C(\mathscr{U})$. Note that whether it can be concluded from $C(\mathscr{T}) = C(\mathscr{U})$

that $B(\mathscr{T}) = B(\mathscr{U})$ is not clear. The following result deals with the case when that implication is true.

Lemma 3.4.4. *Let Λ be a collection of classes of R-modules that is closed under B or under C - Λ closed under B (resp. closed under C) means that if $\mathscr{T}$ is in Λ, then $B(\mathscr{T})$ (resp. $C(\mathscr{T})$) is in Λ. Then, the following statements are equivalent.*

a) $B(\mathscr{T}) = B(\mathscr{U})$ iff $C(\mathscr{T}) = C(\mathscr{U})$, for all $\mathscr{T}, \mathscr{U}$ in Λ.

b) All classes in Λ are good classes.

Proof. $(b) \Rightarrow (a)$ is easy to see - if all classes in Λ are good classes, then $B(\mathscr{T}) = C(\mathscr{T})$ for all $\mathscr{T}$ in Λ, and therefore, for any $\mathscr{T}$ and $\mathscr{U}$ in Λ, the statements $B(\mathscr{T}) = B(\mathscr{U})$ and $C(\mathscr{T}) = C(\mathscr{U})$ are equivalent.

$(a) \Rightarrow (b)$: First let us assume that Λ is closed under the operation B. Now, assume that, for all $\mathscr{T}, \mathscr{U}$ in Λ, $B(\mathscr{T}) = B(\mathscr{U})$ iff $C(\mathscr{T}) = C(\mathscr{U})$. Note that it is always true that $C(\mathscr{T}) = C(\mathscr{U})$ if $B(\mathscr{T}) = B(\mathscr{U})$ (by Corollary 3.2.2). As Λ is closed under B, $B(\mathscr{T})$ is in Λ if $\mathscr{T}$ is in Λ. Now note that $\mathscr{U} = B(\mathscr{T})$ satisfies $C(\mathscr{T}) = C(\mathscr{U})$ for all $\mathscr{T}$ in Λ (as $CB = C$ as operators, by Lemma 3.2.1). Therefore, $B(\mathscr{T}) = B^2(\mathscr{T})$ for all $\mathscr{T}$ in Λ. By condition (a) of Proposition 3.3.2, it now follows that all $\mathscr{T}$ in Λ are good.

Now let us assume that Λ is closed under the operation C. Again, assume $B(\mathscr{T}) = B(\mathscr{U})$ iff $C(\mathscr{T}) = C(\mathscr{U})$, for all $\mathscr{T}, \mathscr{U}$ in Λ. Note that, for any $\mathscr{T}$, $\mathscr{U} = C(\mathscr{T})$ satisfies $C(\mathscr{T}) = C(\mathscr{U})$ because $C^2 = C$ as operations by Corollary 3.2.2. Therefore, $B(\mathscr{T}) = BC(\mathscr{T}) = C(\mathscr{T})$ (by Lemma 3.2.1), so $\mathscr{T}$ is good. □

We end this chapter with a few questions.

Question 3.4.5. *a) For any given n, can we find a class of R-modules $\mathscr{T}$, for some ring R, such that $g(\mathscr{T}) \geq n$?*

b) We saw in Theorem 3.1.2 that for any class of modules $\mathscr{T}$, $C(\mathscr{T}) = B^{\omega}(\mathscr{T})$. Are the following two statements equivalent for any 2 classes of R-modules, $\mathscr{U}$ and $\mathscr{T}$: (i).$C(\mathscr{U}) = C(\mathscr{T})$, (ii). $g(\mathscr{U}) < \infty \Leftrightarrow g(\mathscr{T}) < \infty$?

Chapter 4

Cohomological and Homological Invariants of Groups Not Necessarily Finite

In this chapter, we will be dealing with two classes of groups - one called groups of type Φ (the type Φ property is defined over various commutative rings) which were introduced over the ring of integers by Talelli in [62] (see Definition 1.2.6), and our other class is derived from Kropholler's hierarchy (see Definition 1.2.1) - we will also be forming a class of groups mixing ideas behind the formation of both these classes. One of our aims is to prove a collection of equalities of a bunch of cohomological invariants extending some results by Cornick and Kropholler. We prove a large part of a conjecture for type Φ groups proposed by Talelli (Conjecture 4.1.12) with the extra assumption that the groups in question are in our large mixed class mentioned earlier.

For clarity, we provide some background in Section 4.1 on the cohomological invariants that we shall be using and the classes of groups, because cohomological invariants often need to be accompanied with a lot of context and significance. Our original results are mostly collected in Section 4.2, Section 4.4 and Section 4.5.

We deal with some homological invariants shortly in Section 4.6.

4.1 Background on cohomological invariants

For any ring R, as briefly mentioned in Chapter 1, the finiteness of either $\operatorname{spli}(R)$ or $\operatorname{silp}(R)$ (see Definition 1.3.1) is connected to the question of whether R-modules admit complete projective resolutions (usually called just "complete resolutions") or complete injective resolutions.

The following result was proved in [31].

Theorem 4.1.1. *(Result 4.1 of [31]) Let R be a ring. If $\operatorname{spli}(R) < \infty$, then every R-module admits a weak complete resolution.*

Remark 4.1.2. *Whether a group or a module admitting weak complete resolutions over a ring is equivalent to the same admitting complete resolutions over the same ring is an interesting question (see Theorem 4.4.2).*

Like the spli invariant defined earlier, the Gorenstein cohomological dimension of a group is a good indicator of whether the group admits complete resolutions. Here, it helps if the base ring is of finite global dimension:

Theorem 4.1.3. *(Theorem 1.7 of [29]) For any commutative ring A of finite global dimension and any group Γ, the following are equivalent.*

a) $Gcd_A(\Gamma) < \infty$, i.e. the trivial module A admits complete resolutions as an $A\Gamma$-module.

b) $\operatorname{silp}(A\Gamma) = \operatorname{spli}(A\Gamma) < \infty$.

c) $Gpd_{A\Gamma}(M) < \infty$, for all $A\Gamma$-modules M, i.e. all $A\Gamma$-modules admit complete resolutions.

Remark 4.1.4. *Using the notations of Theorem 4.1.3, it is easy to see that M admits a complete resolution iff it has finite Gorenstein projective dimension as an $A\Gamma$-module: if M admits a complete resolution F_* which has coincidence index say n with respect to a projective resolution $P_* \twoheadrightarrow M$, then the n-th kernel in P_*, which we can denote by $\Omega^n(M)$, is a kernel in the complete resolution F_*, which means $\Omega^n(M)$ is Gorenstein projective. We now have an exact sequence $0 \to \Omega^n(M) \to P_{n-1} \to .. \to P_0 \to M \to 0$, where each term other than M is Gorenstein projective (projectives are Gorenstein projective), and so $Gpd_{A\Gamma}(M) \leq n$.*

Now, let M satisfy $Gpd_{A\Gamma}(M) \leq n$. Take $P_* \twoheadrightarrow M$ be a projective resolution of M. Then by Theorem 1.1.8, $\Omega^n(M)$, the n-the kernel in P_*, is Gorenstein projective. So, $\Omega^n(M)$ admits a complete resolution of coincidence index 0, and M admits a complete resolution of coincidence index $\leq n$.

Also of use is the fact that one can put an upper bound on the spli and silp invariants using the Gorenstein cohomological dimension if the base ring is of finite global dimension:

Lemma 4.1.5. *(Corollary 1.6 of [29]) For any commutative ring A of global dimension t and any group Γ, $\mathrm{silp}(A\Gamma), \mathrm{spli}(A\Gamma) \leq Gcd_A(\Gamma) + t$.*

Remark 4.1.6. *There are no known examples of group rings where the* silp *and* spli *invariants differ, and they are known to coincide when they are both finite (See Theorem 1.3.4). Result 2.4 of [31] showed that if A is a Noetherian commutative ring of global dimension t, then $\mathrm{silp}(A\Gamma) \leq \mathrm{spli}(A\Gamma) + t$. It is possible that one might be able to prove this result without the Noetherian condition.*

The following is an important result regarding the silp and spli invariants:

Theorem 4.1.7. *(Theorem 4.4 of [28]) For any group Γ and any commutative Noetherian ring A of finite global dimension, $\mathrm{silp}(A\Gamma) = \mathrm{spli}(A\Gamma)$.*

Very similar in use and purpose to the Gorenstein cohomological dimension is the invariant "generalized cohomological dimension" which was introduced by Ikenaga in [35] over the integers:

Definition 4.1.8. *For any commutative ring A and any group Γ, define the generalized cohomological dimension of Γ with respect to A, denoted $\underline{cd}_A(\Gamma)$, to be $\sup\{n \in \mathbb{Z}_{\geq 0} : \mathrm{Ext}^n_{A\Gamma}(M, F) \neq 0$, for some A-free M and some $A\Gamma$-free $F\}$.*

The Gorenstein cohomological dimension and the generalized cohomological dimension over rings of finite global dimension are very closely related:

Theorem 4.1.9. *For any group Γ,*

a) $Gcd_{\mathbb{Z}}(\Gamma) = \underline{cd}_{\mathbb{Z}}(\Gamma)$.

b) For any commutative ring A of finite global dimension, if $Gcd_A(\Gamma)$ is finite, $Gcd_A(\Gamma) = \underline{cd}_A(\Gamma)$.

Proof. (a) has been shown in Theorem 2.5 of [7].

(b) follows from the proof of Theorem 2.5 of [7]. The proof of (a) in [7] makes use of Emmanouil's result that $\mathrm{silp}(\mathbb{Z}\Gamma) = \mathrm{spli}(\mathbb{Z}\Gamma)$ (see Theorem 4.1.7) and we do not know if this equality holds if $\mathbb{Z}$ is replaced by any commutative ring of finite global dimension. □

Remark 4.1.10. *It also follows from the proof of Theorem* 2.5 *of [7] that if A is a commutative ring of finite global dimension and it is known that* $\mathrm{silp}(A\Gamma) = \mathrm{spli}(A\Gamma)$ *(we do not know this for all such A, see Theorem 4.1.7), then $Gcd_A(\Gamma) = \underline{cd}_A(\Gamma)$.*

The following is an interesting characterisation of the finiteness of a group in terms of cohomological invariants - see Theorem 4.6.6 for an additional characterisation in terms of a homological invariant. It is a collection of some results from the literature.

Theorem 4.1.11. *For any group Γ and any commutative ring A, the following are equivalent.*

a) Γ *is finite.*

b) $proj.dim_{A\Gamma} B(\Gamma, A) = 0$.

c) $Gcd_A(\Gamma) = 0$.

d) $\mathrm{spli}(\mathbb{Z}\Gamma) = 1$.

e) $\underline{cd}_{\mathbb{Z}}(\Gamma) = 0$.

Proof. $(a) \iff (b)$: if Γ is finite, then $proj.dim_{A\Gamma} B(\Gamma, A) = 0$ by Lemma 1.3.7. Conversely, if $proj.dim_{A\Gamma} B(\Gamma, A) = 0$, then the inclusion of all constant functions $\Gamma \to A$ in $B(\Gamma, A)$ gives an embedding of the trivial module A in the projective module $B(\Gamma, A)$, so Γ is finite.

$(a) \iff (c)$: follows from Corollary 2.3 of [29].

$(a) \iff (d)$: follows from the main theorem of [23].

$(a) \iff (e)$: follows from Theorem 4.1.9.a. and Corollary 2.3 of [29].

□

In Section 4.2, we shall see how in some cases to achieve bounds in calculations with different cohomological invariants, it is more helpful to work with the generalized cohomological dimension instead of the Gorenstein cohomological dimension.

We can make the following conjecture mixing a part of Conjecture A of [62] (where the base ring was the ring of integers) and Conjecture 43.1 of [19] and adding a few extra conditions.

Conjecture 4.1.12. *For any group Γ and any commutative ring A of finite global dimension, the following are equivalent.*

a) Γ *is of type* Φ *over* A.

b) $\mathrm{silp}(A\Gamma) < \infty$.

c) $\mathrm{spli}(A\Gamma) < \infty$.

d) $proj.dim_{A\Gamma} B(\Gamma, A) < \infty$.

e) $Gcd_A(\Gamma) < \infty$.

f) $\mathrm{fin.\,dim}(A\Gamma) < \infty$.

g) $k(A\Gamma) < \infty$.

When $A = \mathbb{Z}$, we can add the condition

h) $\Gamma \in H_1\mathscr{F}$.

In Section 4.5, we prove that statements (a) to (g) are equivalent if $\Gamma \in LH\mathscr{F}_{\phi,A}$.

4.2 Main results, I

We start this section by making the following conjecture.

Conjecture 4.2.1. *Let A be a commutative ring of finite global dimension and let Γ be a group. Then, $Gcd_A(\Gamma) = proj.dim_{A\Gamma} B(\Gamma, A)$.*

We are now in a position to move towards proving our first result involving the comparison of different invariants (Theorem 4.2.6). We start with the following proposition.

Proposition 4.2.2. *Let Γ be a group and let A be a commutative ring. Assume that $proj.dim_{A\Gamma} B(\Gamma, A)$ is finite. Then $proj.dim_{A\Gamma} B(\Gamma, A) \leq \underline{cd}_A(\Gamma)$.*

Proof. We can assume that $\underline{cd}_A(\Gamma)$ is finite because if it is not, we have nothing to prove.

Assume that $proj.dim_{A\Gamma} B(\Gamma, A) = k > \underline{cd}_A(\Gamma)$. There exists an $A\Gamma$-module M such that $\mathrm{Ext}^k_{A\Gamma}(B(\Gamma, A), M) \neq 0$ because otherwise $proj.dim_{A\Gamma} B(\Gamma, A) \leq k - 1$. Let

F be the $A\Gamma$-free module on M. We have a short exact sequence $0 \to \Omega(M) \to F \to M \to 0$. We now look at the following long exact Ext-sequence associated to this short exact sequence and get $.. \to \mathrm{Ext}^k_{A\Gamma}(B(\Gamma, A), \Omega(M)) \to \mathrm{Ext}^k_{A\Gamma}(B(\Gamma, A), F) \to \mathrm{Ext}^k_{A\Gamma}(B(\Gamma, A), M) \to \mathrm{Ext}^{k+1}_{A\Gamma}(B(\Gamma, A), \Omega(M)) \to ...$ Here, $\mathrm{Ext}^k_{A\Gamma}(B(\Gamma, A), F) = 0$ because $k > \underline{cd}_A(\Gamma)$ (see Definition 4.1.8) and $B(\Gamma, A)$ is A-free by Lemma 1.3.7 and F is $A\Gamma$-free. Also, $\mathrm{Ext}^{k+1}_{A\Gamma}(B(\Gamma, A), \Omega(M)) = 0$ because $proj.dim_{A\Gamma} B(\Gamma, A) = k$. So, $\mathrm{Ext}^k_{A\Gamma}(B(\Gamma, A), M) = 0$ which gives us a contradiction. □

Before we state our next result regarding comparison of the invariants that we have introduced, we state the following result which gives a sufficient condition on a module for it to admit complete resolutions.

Theorem 4.2.3. *(Theorem 3.5 of [21]) Let A be a commutative ring and Γ a group. If $M \otimes_A B(\Gamma, A)$ is projective, then M is Gorenstein projective, i.e. it admits a complete resolution of coincidence index 0.*

Remark 4.2.4. *Note that in [21] when Theorem 4.2.3 was proved, it was stated in different language. What we state in Theorem 4.2.3 is exactly what was proved in proving Theorem 3.5 in [21].*

Proposition 4.2.5. *For any commutative ring A and any group Γ, $Gcd_A(\Gamma) \leq proj.dim_{A\Gamma} B(\Gamma, A)$*

Proof. We can assume that $proj.dim_{A\Gamma} B(\Gamma, A)$ is finite because otherwise we have nothing to prove. Let M be an $A\Gamma$-module satisfying $proj.dim_{A\Gamma} M \otimes_A B(\Gamma, A) = n$. Since $B(\Gamma, A)$ is A-free by Lemma 1.3.7, if we take a projective resolution $(P_*, d_*) \twoheadrightarrow M$ of $A\Gamma$-projective modules P_i with the kernels given by $\Omega^*(M)$, we get a projective resolution $(P_* \otimes_A B(\Gamma, A), d_* \otimes id) \twoheadrightarrow M \otimes_A B(\Gamma, A)$ where the kernels are given by $\Omega^*(M) \otimes_A B(\Gamma, A)$. So, $\Omega^n(M) \otimes_A B(\Gamma, A)$ is projective as an $A\Gamma$-module. Now we can use Theorem 4.2.3 to deduce that $\Omega^n(M)$ is Gorenstein projective; it therefore follows from Theorem 1.1.8 that $Gpd_{A\Gamma}(M) \leq n$. If we replace M by the trivial module A, the hypothesis $proj.dim_{A\Gamma} M \otimes_A B(\Gamma, A) = n$ becomes $proj.dim_{A\Gamma} B(\Gamma, A) = n$, and we get that $n \geq Gpd_{A\Gamma}(A) = Gcd_A(\Gamma)$.

□

Collecting the results Theorems 4.1.9, Proposition 4.2.2 and Proposition 4.2.5, we get the following result which proves Conjecture 4.2.1 under the assumption of an extra finiteness condition and also in the absence of a finiteness condition.

Theorem 4.2.6. *Let A be a commutative ring of finite global dimension and let Γ be a group. If $proj.dim_{A\Gamma}B(\Gamma, A)$ is finite, then $proj.dim_{A\Gamma}B(\Gamma, A) = Gcd_A(\Gamma)$. Also, if $Gcd_A(\Gamma)$ is not finite, then $proj.dim_{A\Gamma}B(\Gamma, A)$ is not finite.*

Another conjecture related very closely to Conjecture 4.2.1 involves two classes of modules - the class of Benson's cofibrants (see Definition 1.3.6) and the class of Gorenstein projectives.

Conjecture 4.2.7. *(stated over $\mathbb{Z}$ in [25]) For any commutative ring A of finite global dimension and any group Γ, the class of Benson's cofibrants coincides with the class of Gorenstein projectives.*

We will work more on Conjecture 4.2.7 in Chapter 6. We end this section showing how Conjecture 4.2.1 relates to Conjecture 4.2.7.

We need to state a lemma first which is standard knowledge.

Lemma 4.2.8. *(standard knowledge, see Lemma 2.1.c. of [25]) Let R be a ring and let $(F_i, d_i)_{i\in\mathbb{Z}}$ be a doubly infinite exact complex of R-projective modules with a uniform finite bound on the projective dimensions of the kernels as R-modules. Then, each kernel is R-projective, and $(F_i, d_i)_{i\in\mathbb{Z}}$ is completely split.*

Proof. Let m be the bound on the projective dimensions of the kernels, and let us denote the kernels as $K_i = \mathrm{Ker}(d_i)$, for all $i \in \mathbb{Z}$. Let, for some integer t, $K_t := \mathrm{Ker}(d_t)$ be of projective dimension $n > 0$. Then, from the short exact sequence $0 \to K_t \hookrightarrow F_t \twoheadrightarrow K_{t-1} \to 0$, it follows that $proj.dim_R K_{t-1} = n + 1$. Going on like this, we get that $proj.dim_R K_{t-m} = n + m > m$, which is not possible.

Thus, we get that every kernel in $(F_i, d_i)_{i\in\mathbb{Z}}$ is R-projective and therefore $(F_i, d_i)_{i\in\mathbb{Z}}$ is completely split. □

Theorem 4.2.9. *For any group Γ and any commutative ring A of finite global dimension,*

a) Conjecture 4.2.7 implies Conjecture 4.2.1.

b) Conjecture 4.2.1 implies Conjecture 4.2.7 if the invariants in the statement of Conjecture 4.2.1 are finite.

Proof. a) To show that Conjecture 4.2.7 implies Conjecture 4.2.1, we can assume that $proj.dim_{A\Gamma}B(\Gamma, A)$ is not finite, because if it is finite, then from Theorem 4.2.6 it follows that Conjecture 4.2.1 holds. Now, let us assume that $Gcd_A(\Gamma) = n < \infty$. Then, $\Omega^n(A)$ is Gorenstein projective, and from Conjecture 4.2.7, it follows that $\Omega^n(A) \otimes_A B(\Gamma, A)$ is projective as an $A\Gamma$-module, and since $B(\Gamma, A)$ is A-free, we get that $\Omega^n(A \otimes_A B(\Gamma, A)) = \Omega^n(B(\Gamma, A))$ is projective as an $A\Gamma$-module. Therefore, $proj.dim_{A\Gamma}B(\Gamma, A)$ is finite, and we have a contradiction.

b) Theorem 4.2.3 tells us that the class of Benson's cofibrants is a subclass of the class of Gorenstein projectives. So, we need to focus on going the other direction with the assumption $proj.dim_{A\Gamma}B(\Gamma, A) = Gcd_A(\Gamma) < \infty$. Let M be a Gorenstein projective $A\Gamma$-module. So, M occurs as a kernel in a complete resolution $(F_i, d_i)_{i\in\mathbb{Z}}$. Since the global dimension of A is t, $\Omega^t(N)$ is A-projective, for any $A\Gamma$-module N. Since $proj.dim_{A\Gamma}B(\Gamma, A)$ is finite, we have $proj.dim_{A\Gamma}\Omega^t(N)\otimes_A B(\Gamma, A)$ is finite, from which it follows, courtesy the fact that $B(\Gamma, A)$ is A-free, that $proj.dim_{A\Gamma}\Omega^t(N\otimes_A B(\Gamma, A)) < \infty$, and so we have $proj.dim_{A\Gamma}N \otimes_A B(\Gamma, A) < \infty$, for all $A\Gamma$-modules N.

As $Gcd_A(\Gamma) < \infty$, by Lemma 4.1.5 and Theorem C of [22], $\operatorname{fin.dim}(A\Gamma) < \infty$.

$M \otimes_A B(\Gamma, A)$ occurs as a kernel in a doubly infinite exact sequence of $A\Gamma$-projectives $(F_i \otimes_A B(\Gamma, A), d_i \otimes id)_{i\in\mathbb{Z}}$ where all the kernels are of the form (some $A\Gamma$-module) $\otimes_A B(\Gamma, A)$, and are therefore of finite projective dimension as $A\Gamma$-modules as shown earlier. Now since $\operatorname{fin.dim}(A\Gamma) < \infty$, we can directly use Lemma 4.2.8 and say that $M \otimes_A B(\Gamma, A)$ is projective as an $A\Gamma$-module, and we are done. □

We believe for any group Γ and any commutative ring A of finite global dimension, settling Conjecture 4.2.1 should be sufficient for settling Conjecture 4.2.7 even when the invariants in Conjecture 4.2.1 are not finite.

Conjecture 4.2.10. *For any group Γ and any commutative ring A of finite global dimension, Conjecture 4.2.1 and Conjecture 4.2.7 are equivalent.*

4.3 Relating Conjecture 4.2.1 with other conjectures

Proposition 4.3.1. *For any group Γ and any commutative ring A of finite global dimension, Conjecture 4.1.12 implies Conjecture 4.2.1.*

Proof. Again, courtesy Theorem 4.2.6, we can assume that $proj.dim_{A\Gamma}B(\Gamma, A)$ is not finite. If $Gcd_A(\Gamma)$ is also not finite, Conjecture 4.2.1 holds, and if it is finite, Conjecture 4.1.12 tells us that Γ is of type Φ over A, and since $B(\Gamma, A)$ is AG-free for every finite subgroup G of Γ by Lemma 1.3.7., we have from Definition 1.2.6 that $proj.dim_{A\Gamma}B(\Gamma, A)$ is finite, which is not possible. □

Also, when $A = \mathbb{Z}$, if Conjecture 4.1.12 is true, then $\mathscr{F}_\phi = H_1\mathscr{F}$, and therefore $LH\mathscr{F}_\phi = LH\mathscr{F}$ (see Definition 1.2.1 and Definition 1.2.6). The following remark is straightforward to note now.

Remark 4.3.2. *It is clear from Theorem 4.2.6 that if Conjecture 4.2.1 fails for some group Γ and some commutative ring A of finite global dimension, then $proj.dim_{A\Gamma}B(\Gamma, A)$ will have to be infinite and $Gcd_A(\Gamma)$ will have to be finite. For $A = \mathbb{Z}$, all known examples of groups with finite Gorenstein cohomological dimension, i.e. groups that admit complete resolutions over $\mathbb{Z}$ are in $H_1\mathscr{F}$. The examples of groups, constructed in [36], that are in $H\mathscr{F}$ but not in $H_1\mathscr{F}$ do not admit complete resolutions. In light of this discussion, looking at Conjecture 4.1.12, it can be deemed conceivable therefore that, over $\mathbb{Z}$ at least, if Conjecture 4.2.1 fails for some group, that group will have to be in $H_1\mathscr{F}$, which will in turn be impossible since Conjecture 4.2.7 has already been proved for a large class of groups containing $H_1\mathscr{F}$ (see Remark 6.4.7 in Chapter 6) and Conjecture 4.2.7 implies Conjecture 4.2.1 according to Theorem 4.2.9.a.*

We end this section with the following comment on what we can say about groups satisfying Conjecture 4.2.1 without knowing or checking whether they also satisfy Conjecture 4.2.7.

Remark 4.3.3. *Fix a commutative ring A of finite global dimension. Define the group cohomology functors in the usual way - $H^*(\Gamma, ?) := \mathrm{Ext}^*_{A\Gamma}(A, ?)$. For any $A\Gamma$-module M, define the complete cohomology in the following way using colimits - $\widehat{H}^j(\Gamma, M) :=$*

$\varinjlim_{n\in\mathbb{N}} H^{j+n}(\Gamma,\Omega^n(M))$. *When* Γ *admits complete resolutions, one can compute the complete cohomology using those complete resolutions, and that would give the same answer (see [47]). Using the language of finitary functors, one can define a functor* F *between two abelian categories to be* 0*-finitary if for any colimit system* $(M_\lambda)_{\lambda\in\Lambda}$ *satisfying* $\varinjlim_{\lambda\in\Lambda} M_\lambda = 0$*, one has* $\varinjlim_{\lambda\in\Lambda} F(M_\lambda) = 0$ *(see Section* 6.1 *of [42]).*

In Theorem 2.7 *of [42], it is shown that if* $\Gamma \in LH\mathscr{F}$*, and if the functor* $\widehat{H}^j(\Gamma, ?)$ *is* 0*-finitary for all* j*, then* $proj.dim_{A\Gamma}B(\Gamma,A) < \infty$*. The treatment in [42] is over the ring of integers, but one can make all the same arguments over any commutative ring of finite global dimension. Noting how big the class* $LH\mathscr{F}$ *is and the fact that all known examples of groups* Γ *satisfying* $proj.dim_{A\Gamma}B(\Gamma,A) < \infty$ *are in* $H_1\mathscr{F}$*, it is worth observing that the condition that* $\widehat{H}^*(G,?)$ *be* 0*-finitary seems like a very strong requirement. And using Theorem* 2.7 *of [42] in conjunction with Theorem 4.2.6 therefore, we get that all* $LH\mathscr{F}$*-groups* Γ *for which* $\widehat{H}^*(G,?)$ *is* 0*-finitary, Conjecture 4.2.1 is satisfied. Here, we are not making use of the fact that Conjecture 4.2.7 is known to hold true for* $LH\mathscr{F}$*-groups and Theorem 4.2.9.*

It has been mentioned in [42] that it would be interesting to see whether there are any groups lying outside $LH\mathscr{F}$ that satisfy Theorem 2.7 of [42]. We can streamline that query and ask the following.

Question 4.3.4. *For any arbitrary group* Γ *and any fixed commutative ring* A *of finite global dimension, is it true that* $\widehat{H}^*(\Gamma,?)$ *is* 0*-finitary iff* $proj.dim_{A\Gamma}B(\Gamma,A) < \infty$?

4.4 Main results, II

Our main result in this section is the following.

Theorem 4.4.1. *Let* $\Gamma \in LH\mathscr{F}_{\phi,A}$ *with* A *being a commutative ring of global dimension* t*. Then,*

$$proj.dim_{A\Gamma}B(\Gamma,A) = Gcd_A(\Gamma)$$

and

$$proj.dim_{A\Gamma}B(\Gamma,A) \leq \mathrm{fin.\,dim}(A\Gamma) = \mathrm{silp}(A\Gamma) = \mathrm{spli}(A\Gamma) = k(A\Gamma) \leq proj.dim_{A\Gamma}B(\Gamma,A)+t$$

The first equality in Theorem 4.4.1 is proved by noting that Conjecture 4.2.7 holds for groups in $LH\mathscr{F}_{\phi,A}$ (this is proved later in Chapter 6). In Chapter 6, we also prove an interesting connection between weak complete resolutions and complete resolutions for groups in $LH\mathscr{F}_{\phi,A}$. We collect these two results below.

Theorem 4.4.2. *Let $\Gamma \in LH\mathscr{F}_{\phi,A}$ where A is a commutative ring of finite global dimension. Then,*

a) The class of Benson's cofibrant $A\Gamma$-modules and Gorenstein projective $A\Gamma$-modules coincide. Therefore, from Theorem 4.2.9, it follows that $proj.dim_{A\Gamma}B(\Gamma, A) = Gcd_A(\Gamma)$.

b) Any $A\Gamma$-module M admits a weak complete resolution iff it admits a complete resolution.

For the sake of logical clarity, we provide a short proof of Theorem 4.4.2 here. We need the following lemma, which again is close to the concepts dealt with in Chapter 6.

Lemma 4.4.3. *(follows from Lemma 3.14 of [46]) Let $\Gamma \in \mathscr{F}_{\phi,A}$, where A is a commutative ring of finite global dimension. Then, every weak Gorenstein projective $A\Gamma$-module is a Gorenstein projective $A\Gamma$-module.*

Proof. As noted in the proof of Lemma 3.14 and Remark 3.15 of [46], all we need to check here is to make sure $\mathrm{silp}(A\Gamma) < \infty$.

As $\Gamma \in \mathscr{F}_{\phi,A}$, by Lemma 1.3.7, we have $proj.dim_{A\Gamma}B(\Gamma, A) < \infty$, and therefore by Theorem 4.2.6, $Gcd_A(\Gamma) < \infty$. Now, Lemma 4.1.5 gives us that $\mathrm{silp}(A\Gamma) < \infty$. ◻

Proof of Theorem 4.4.2. a) In [25], Dembegioti and Talelli prove (over $A = \mathbb{Z}$) that if M is a weak Gorenstein projective $A\Gamma$-module such that $M \otimes_A B(\Gamma, A)$ is projective over all finite subgroups, then $M \otimes_A B(\Gamma, A)$ is projective over all $LH\mathscr{F}$-subgroups of Γ. They first check that $M \otimes_A B(\Gamma, A)$ is projective over all $\mathscr{F}$-subgroups (follows from the hypothesis). Then,

I. They first prove that $M \otimes_A B(\Gamma, A)$ is projective over $H\mathscr{F}$-subgroups, and

II. Then they show that $M \otimes_A B(\Gamma, A)$ is projective over $LH\mathscr{F}$-subgroups.

The Dembegioti-Talelli method, i.e. Steps I and II from above, can be replicated with $\mathscr{F}_{\phi,A}$ replacing $\mathscr{F}$, as long as we check that $M \otimes_A B(\Gamma, A)$ is projective over all $\mathscr{F}_{\phi,A}$-subgroups of Γ. To do this checking, first note that since M is A-projective (by Lemma 1.1.3) and $B(\Gamma, A)$ is of finite projective dimension over $\mathscr{F}_{\phi,A}$-subgroups, $M \otimes_A B(\Gamma, A)$ is of finite projective dimension over all $\mathscr{F}_{\phi,A}$-subgroups. Now, let us fix a $\mathscr{F}_{\phi,A}$-subgroup of Γ, Γ'. As M is a weak Gorenstein projective $A\Gamma$-module, it is also a weak Gorenstein projective $A\Gamma'$-module. As $B(\Gamma, A)$ is A-free, $M \otimes_A B(\Gamma, A)$ is a weak Gorenstein projective $A\Gamma'$-module. By Lemma 4.4.3, $M \otimes_A B(\Gamma, A)$ is a Gorenstein projective $A\Gamma'$-module, and by Proposition 1.1.13, we have that $M \otimes_A B(\Gamma, A)$ is $A\Gamma'$-projective.

Thus, we have that for $\Gamma \in LH\mathscr{F}_{\phi,A}$, all weak Gorenstein projective $A\Gamma$-modules are Benson's cofibrants. Now using Theorem 4.2.3 therefore, we have that the class of weak Gorenstein proejctive $A\Gamma$-modules coincides with the class of Benson's cofibrant $A\Gamma$-modules, which in turn coincides with the class of Gorenstein projective $A\Gamma$-modules.

b) Part (b) follows directly from the coincidence between weak Gorenstein projectives and Benson's cofibrants, as noted in the proof of Corollary D in [25] over the integers and exactly the same proof works over A in our case. □

Now, we are in a position to prove the following result. Note that a proof of the same was claimed for $H\mathscr{F}$-groups in the proof of Theorem C of [22] but the authors of that paper overlooked a condition on the base ring that was present in the hypothesis of a key theorem that they were citing. We expand on this more towards the end of this section in Remark 4.4.7.

Lemma 4.4.4. *Let $\Gamma \in LH\mathscr{F}_{\phi,A}$ where A is a commutative ring of finite global dimension. Then,* $\mathrm{silp}(A\Gamma) \leq \mathrm{spli}(A\Gamma)$.

Proof. Assume $\mathrm{spli}(A\Gamma) < \infty$ because otherwise we have nothing to prove. $\mathrm{spli}(A\Gamma) < \infty$ implies that all $A\Gamma$-modules admit weak complete resolutions by Theorem 4.1.1. By Theorem 4.4.2.b., since $\Gamma \in LH\mathscr{F}_{\phi,A}$, it now follows that every $A\Gamma$-module admits complete resolutions, and therefore $Gcd_A(\Gamma) < \infty$ by Theorem 4.1.3. Since A is of finite global dimension, we get using Lemma 4.1.5 that $\mathrm{silp}(A\Gamma) < \infty$. As it is known that the silp and spli invariants, over any ring, coincide when it is known that they are both finite (see Theorem 1.3.4), we are done. □

We now prove the following three inequalities involving five different invariants that are known in the literature.

Lemma 4.4.5. *([49], [22]) Let A be a commutative ring and let Γ be a group. Then,*

a) $\underline{cd}_A(\Gamma) \leq \operatorname{silp}(A\Gamma)$.

b) $\operatorname{fin.dim}(A\Gamma) \leq \operatorname{silp}(A\Gamma)$.

If, in addition, A is of finite global dimension, then

c) $\operatorname{spli}(A\Gamma) \leq k(A\Gamma)$.

Proof. a) This has been noted in [49]. It is obvious from the definitions - it follows from the definition of $\operatorname{silp}(A\Gamma)$ that it is $sup\{n \in \mathbb{N} : \operatorname{Ext}^n_{A\Gamma}(X, P) \neq 0$ for some $A\Gamma$-module X and some $A\Gamma$-projective $P\}$, and $\underline{cd}_A(\Gamma) := sup\{n \in \mathbb{Z}_{\geq 0} : \operatorname{Ext}^n_{A\Gamma}(M, F) \neq 0$ for some A-free M and some $A\Gamma$-free $F\}$. Since, free modules are projective, the inequality follows.

b) This again follows from definitions and has been noted in the proof of Theorem C of [22]. We can assume that $\operatorname{silp}(A\Gamma) = r < \infty$ because otherwise we have nothing to prove. From the definition of injective dimension, it follows that $r = sup\{n \in \mathbb{Z} : \operatorname{Ext}^n_{A\Gamma}(M, P) \neq 0$ for some $A\Gamma$-module M and some $A\Gamma$-projective $P\}$.

Take an $A\Gamma$-module T of finite projective dimension, say k. There exists an $A\Gamma$-module X such that $\operatorname{Ext}^k_{A\Gamma}(T, X) \neq 0$ because otherwise $proj.dim_{A\Gamma}T \leq k - 1$. Take F to be the $A\Gamma$-free module on X and we get a short exact sequence $0 \to \Omega(X) \to F \to X \to 0$, that gives us a long exact Ext-sequence $.. \to \operatorname{Ext}^k_{A\Gamma}(T, \Omega(X)) \to \operatorname{Ext}^k_{A\Gamma}(T, F) \to \operatorname{Ext}^k_{A\Gamma}(T, X) \to \operatorname{Ext}^{k+1}_{A\Gamma}(T, \Omega(X)) \to ...$ Here, $\operatorname{Ext}^{k+1}_{A\Gamma}(T, \Omega(X)) = 0$ as $proj.dim_{A\Gamma}T = k$, so if $\operatorname{Ext}^k_{A\Gamma}(T, F) = 0$, we get an embedding $\operatorname{Ext}^k_{A\Gamma}(T, X) \hookrightarrow \operatorname{Ext}^{k+1}_{A\Gamma}(T, \Omega(X)) = 0$ implying $\operatorname{Ext}^k_{A\Gamma}(T, X) = 0$, which is not possible.

So, $\operatorname{Ext}^k_{A\Gamma}(T, F) \neq 0$, and since F is $A\Gamma$-projective as it is free, we get from the definition of $\operatorname{silp}(A\Gamma)$ that $k \leq r = \operatorname{silp}(A\Gamma)$. Thus, $\operatorname{fin.dim}(A\Gamma) \leq \operatorname{silp}(A\Gamma)$.

c) This, again, has been covered in the proof of Theorem C of [22]. We can assume that $k(A\Gamma) = n < \infty$. If I is an injective $A\Gamma$-module, then for any finite $G \leq \Gamma$, I is an injective AG-module with finite projective dimension as an AG-module since A has finite global dimension. So, by definition of $k(A\Gamma)$, $proj.dim_{A\Gamma}I \leq n$, and therefore, $\operatorname{spli}(A\Gamma) \leq n$. $\square$

We now prove our last major inequality involving the invariants.

Lemma 4.4.6. *Let $\Gamma \in LH\mathscr{F}_{\phi,A}$ where A is of finite global dimension. Then, $k(A\Gamma) \leq \operatorname{fin.dim}(A\Gamma)$.*

Proof. Assume that $\operatorname{fin.dim}(A\Gamma) = r < \infty$.

It was shown in the proof of Theorem C of [22] that if $\Gamma \in H\mathscr{F}$, then for every $A\Gamma$-module M such that $proj.dim_{AG} M < \infty$ for every finite $G \leq \Gamma$, $proj.dim_{A\Gamma} \operatorname{Ind}_{\Gamma'}^{\Gamma}(M) \leq r$ for every subgroup $\Gamma' \leq \Gamma$. We start with proving the same claim with $\mathscr{F}_{\phi,A}$ replacing $\mathscr{F}$ in quite an identical way. Since $H\mathscr{F}_{\phi,A}$ is subgroup-closed, given a subgroup $\Gamma' \leq \Gamma$, there is a smallest ordinal α such that $\Gamma' \in H_\alpha \mathscr{F}_{\phi,A}$. If $\Gamma' \in H_0\mathscr{F}_{\phi,A} = \mathscr{F}_{\phi,A}$, then as M has finite projective dimension over finite subgroups, it also has finite projective dimension over Γ' as Γ' is of type Φ; so clearly in this case, $proj.dim_{A\Gamma} \operatorname{Ind}_{\Gamma'}^{\Gamma}(M) < \infty$, and therefore by definition of the finitistic dimension, $proj.dim_{A\Gamma} \operatorname{Ind}_{\Gamma'}^{\Gamma}(M) \leq r$.

Now we take as our induction hypothesis that for all ordinals $\beta < \alpha$,

$proj.dim_{A\Gamma} \operatorname{Ind}_{\Gamma'}^{\Gamma}(M) \leq r$, for every $H_\beta\mathscr{F}_{\phi,A}$-subgroup of Γ, and for every $A\Gamma$-module M with finite projective dimension over $\mathscr{F}_{\phi,A}$-subgroups of Γ. If $\Gamma' \in H_\alpha\mathscr{F}_{\phi,A}$, then there is a contractible CW-complex X of finite dimension, say n, on which Γ' acts by permuting its cells with all cell-stabilisers in $H_{<\alpha}\mathscr{F}_{\phi,A}$. If we take the augmented cellular complex of this action and tensor it by M as an $A\Gamma'$-module, and induce up to Γ, we get an exact sequence of $A\Gamma$-modules: $0 \to \operatorname{Ind}_{\Gamma'}^{\Gamma}(C_n \otimes_{\mathbb{Z}} M) \to .. \to \operatorname{Ind}_{\Gamma'}^{\Gamma}(C_0 \otimes_{\mathbb{Z}} M) \to \operatorname{Ind}_{\Gamma'}^{\Gamma}(M) \to 0$ where each C_i is a permutation module that we get from the action of Γ' as a group of permutations of the i-dimensional cells of X. So, C_i can be written as a direct sum of the trivial module induced up to Γ' from subgroups of Γ' that are of the form Γ'_σ, where Γ'_σ denotes the stabilizer of the cell σ (note that $\Gamma'_\sigma \in H_{<\alpha}\mathscr{F}_{\phi,A}$) with σ running over the set of Γ'-representatives for the i-dimensional cells (we can denote this set by Δ). So, $\operatorname{Ind}_{\Gamma'}^{\Gamma}(C_i \otimes_{\mathbb{Z}} M) = \bigoplus_{\sigma\in\Delta} \operatorname{Ind}_{\Gamma'_\sigma}^{\Gamma}(M)$. By our induction hypothesis, M has finite projective dimension as an $A\Gamma'_\sigma$-module, for each $\sigma \in \Delta$, so $\operatorname{Ind}_{\Gamma'_\sigma}^{\Gamma}(M)$ has projective dimension at most r as an $A\Gamma$-module as $\operatorname{fin.dim}(A\Gamma) = r < \infty$, for each $\sigma \in \Delta$. So, as $A\Gamma$-modules, each term in the exact sequence except $\operatorname{Ind}_{\Gamma'}^{\Gamma}(M)$ has finite projective dimension, and therefore so does $\operatorname{Ind}_{\Gamma'}^{\Gamma}(M)$. Now, if we take $\Gamma' = \Gamma$, we are done.

Now let Γ be an $LH\mathscr{F}_{\phi,A}$-group not necessarily in $H\mathscr{F}_{\phi,A}$. Let $\{\Gamma_\lambda : \lambda \in \Lambda\}$ be the family of all finitely generated subgroups of Γ. Then, Γ can be written as the direct limit of the Γ_λ's, and M (as before, M is an $A\Gamma$-module with finite projective dimension over all finite subgroups) can be written as $\varinjlim_{\lambda\in\Lambda} \mathrm{Ind}_{\Gamma_\lambda}^{\Gamma}(M)$ (see the proof of Result 3.2 in [41]). Here, each $\Gamma_\lambda \in H\mathscr{F}_{\phi,A}$, and so as shown above, M has finite projective dimension over each Γ_λ (note that we have $\mathrm{fin.dim}(A\Gamma_\lambda) < \infty$ as $\mathrm{fin.dim}(A\Gamma) < \infty$), so $proj.dim_{A\Gamma} \mathrm{Ind}_{\Gamma_\lambda}^{\Gamma}(M) < \infty$. That means we have $proj.dim_{A\Gamma} \mathrm{Ind}_{\Gamma_\lambda}^{\Gamma}(M) \leq r$ as $\mathrm{fin.dim}(A\Gamma) = r$. So, $proj.dim_{A\Gamma} M \leq r$, and we are done.

□

We can finish the proof of Theorem 4.4.1 now.

Proof of Theorem 4.4.1. The first equality in the statement of Theorem 4.4.1 follows from Theorem 4.4.2.a. Putting together the results of Lemma 4.4.4, Lemma 4.4.5.b., Lemma 4.4.5.c. and Lemma 4.4.6, we get that $\mathrm{fin.dim}(A\Gamma) = \mathrm{silp}(A\Gamma) = \mathrm{spli}(A\Gamma) = k(A\Gamma)$. Note that if $\mathrm{silp}(A\Gamma)$ is not finite, then $Gcd_A(\Gamma)$ cannot be finite by Lemma 4.1.5, and so by the first part of this theorem, $proj.dim_{A\Gamma} B(\Gamma, A)$ cannot be finite. Now, assume that $\mathrm{silp}(A\Gamma)$ is finite. Then, $k(A\Gamma) = \mathrm{silp}(A\Gamma)$ is finite, and since $B(\Gamma, A)$ restricts to a free AG-module for every finite $G \leq \Gamma$ (see Lemma 1.3.7), we have $proj.dim_{A\Gamma} B(\Gamma, A) < \infty$, and so by Theorem 4.2.6, $Gcd_A(\Gamma) < \infty$, and now applying Theorem 4.1.9.b., we get that $\underline{cd}_A(\Gamma) = Gcd_A(\Gamma) = proj.dim_{A\Gamma} B(\Gamma, A) < \infty$. The second part of the theorem now follows from Lemma 4.1.5 and Lemma 4.4.5.a.

□

Remark 4.4.7. *In [22], Theorem C states that for $\Gamma \in H\mathscr{F}$ and for any commutative ring A of finite global dimension,* $\mathrm{fin.dim}(A\Gamma) = \mathrm{silp}(A\Gamma) = \mathrm{spli}(\Gamma) = k(A\Gamma)$. *The authors proved, without using the assumption $G \in H\mathscr{F}$, that* $\mathrm{fin.dim}(A\Gamma) \leq \mathrm{silp}(A\Gamma)$, $\mathrm{silp}(A\Gamma) \leq \mathrm{spli}(A\Gamma)$ *and* $\mathrm{spli}(A\Gamma) \leq k(A\Gamma)$. *The proofs of these results except* $\mathrm{silp}(A\Gamma) \leq \mathrm{spli}(A\Gamma)$ *that we provided while proving Lemma 4.4.5 were achieved using their tactics, as we have noted before. However, their proof of* $\mathrm{silp}(A\Gamma) \leq \mathrm{spli}(A\Gamma)$ *had a logical fallacy - they used Result* 2.4 *of [31] to say that* $\mathrm{silp}(A\Gamma)$ *must be finite if* $\mathrm{spli}(A\Gamma)$ *is finite, but that result of [31] requires A to be Noetherian, as noted in Remark 4.1.6. We resolved this problem with Lemma 4.4.4 and we, theoretically, broadened the class of groups for which those invariants would concur, going from groups in the hierarchy to groups locally in the hierarchy and changing the base class of groups from the class of finite groups to groups of type Φ. We use the phrase "theoretically broaden the class" because we changed the base class from the class of finite groups to the larger* *class of groups of type Φ over A. Whether or not there are concrete examples of groups that are not in $LH\mathscr{F}$ but are in $LH\mathscr{F}_{\phi,A}$, for some commutative ring A of finite global dimension, is not known - this is why we are saying we only "theoretically" broaden the class of groups.*

In [22], Corollary C of Theorem C, which we talk about in Remark 4.4.7 above, states that $H\mathscr{F}$-groups of type FP_∞ are of type Φ. Does the same corollary follow from our Theorem 4.4.1 with $LH\mathscr{F}_{\phi,A}$ replacing $H\mathscr{F}$ as expected, i.e. can we say that $LH\mathscr{F}_{\phi,A}$-groups of type FP_∞ over A are of type Φ over A, with A of finite global dimension? The answer is yes, but we need to state the following two results from [22] first.

Theorem 4.4.8. *Let A be a commutative ring and let $\mathscr{X}$ be a class of groups. Take $\Gamma \in LH\mathscr{X}$, M to be an $A\Gamma$-module of type FP_∞ over A, and N to be an $A\Gamma$-module that has finite projective dimension over all $\mathscr{X}$-subgroups of Γ. Then, $\widehat{Ext}^*_{A\Gamma}(M,N) = 0$. Here, for any i, $\widehat{Ext}^i_{A\Gamma}(M,?)$ is defined in terms of left satellite functors as $\varinjlim_{i\geq 0} S^{-i}\operatorname{Ext}^{n+i}_{A\Gamma}(M,?)$ (see [47] for more on satellite functors).*

Proof. This has been proved in Theorem A of [22] with $H\mathscr{X}$ in place of $LH\mathscr{X}$ and $\mathscr{X} = \mathscr{F}$. The proof works perfectly fine with the version we have stated here because in the proof of Theorem C of [22], the only part where the hierarchy is used is to invoke Result 3.2 of [41] and that result is stated in [41] in a general version for $LH\mathscr{X}$-groups. $\square$

Remark 4.4.9. *Note that in the statement of Theorem 4.4.8, taking $\mathscr{X} = \mathscr{F}_{\phi,A}$, we see that for any i, the functor $\widehat{Ext}^i_{A\Gamma}(M,?)$ vanishing on all $A\Gamma$-modules that have finite projective dimension over all $\mathscr{X}$-subgroups of Γ is equivalent to $\widehat{Ext}^i_{A\Gamma}(M,?)$ vanishing on all $A\Gamma$-modules that have finite projective dimension over all finite subgroups of Γ from the definition of type Φ groups.*

Theorem 4.4.10. *(part of Theorem B of [22]) Let A be some commutative ring, Γ some group and M some $A\Gamma$-module such that $\widehat{Ext}^0_{A\Gamma}(M,?)$ vanishes on all $A\Gamma$-modules that have finite projective dimension over all finite subgroups of Γ. Then, the following are equivalent:*

a) $proj.dim_{AG} M \otimes_A B(\Gamma, A) < \infty$, for all finite $G \leq \Gamma$.

b) $proj.dim_{A\Gamma} M \otimes_A B(\Gamma, A) < \infty$.

Now, we are in a position to prove our promised analog of Corollary C of [22]:

Proposition 4.4.11. *Let Γ be an $LH\mathscr{F}_{\phi,A}$-group of type FP_∞ over A, where A is a commutative ring of finite global dimension. Then, Γ is of type Φ over A.*

Proof. By Theorem 4.4.8 and Remark 4.4.9, $\widehat{Ext}^0_{A\Gamma}(A, ?)$ vanishes on all $A\Gamma$-modules that have finite projective dimension over all finite subgroups of Γ. So by taking $M = A$ in Theorem 4.4.10, we get that $proj.dim_{A\Gamma}B(\Gamma, A) < \infty$ as $B(\Gamma, A)$ restricts to a free module over finite subgroups. Now, we know by Theorem 4.4.1 that $k(A\Gamma) < \infty$, and therefore Γ is of type Φ over A.

□

4.5 Results on Conjecture 4.1.12

We first note the following complete characterisation of groups of type Φ in terms of the finiteness of one cohomological invariant.

Lemma 4.5.1. *Let A be a commutative ring of finite global dimension. Then, Γ is of type Φ over A iff $k(A\Gamma) < \infty$.*

Proof. It is obvious from the definition of $k(A\Gamma)$ and type Φ groups that if $k(A\Gamma) = n < \infty$, then for any $A\Gamma$-module M that has finite projective dimension over finite subgroups, $proj.dim_{A\Gamma}M \leq n$, so Γ is of type Φ over A.

Now, assume that Γ is of type Φ over A. Then, by definition of type Φ groups, $k(A\Gamma) = \operatorname{fin.dim}(A\Gamma)$ as the class of $A\Gamma$-modules with finite projective dimension is precisely the class of $A\Gamma$-modules with finite projective dimension over finite subgroups. If we assume that $\operatorname{fin.dim}(A\Gamma)$ is not finite, then for any integer n, we have an $A\Gamma$-module M_n such that $n \leq proj.dim_{A\Gamma}M_n < \infty$. Over finite subgroups, M_n has finite projective dimension bounded by the global dimension of A. Therefore, $\bigoplus_{n\in\mathbb{N}} M_n$ has does not have finite projective dimension as an $A\Gamma$-module but has finite projective dimension over finite subgroups which cannot be possible as Γ is of type Φ over A. □

Proposition 4.5.2. *Let $\Gamma \in LH\mathscr{F}_{\phi,A}$ where A is a commutative ring of finite global dimension. Then, statements (a) to (g) are equivalent in Conjecture 4.1.12.*

Proof. Using Lemma 4.5.1, we see that in Conjecture 4.1.12, (a) and (g) are always equivalent. Now, it follows from Theorem 4.4.1 that as $\Gamma \in LH\mathscr{F}_{\phi,A}$, statements (b) to (g) are equivalent. $\square$

Corollary 4.5.3. *For any commutative ring A of finite global dimension, $LH\mathscr{F} \cap \mathscr{F}_{\phi,A}$ is closed under extensions and taking Weyl groups with respect to finite subgroups.*

Proof. Let $0 \to \Gamma_1 \to \Gamma \to \Gamma_2 \to 0$ be a short exact sequence of groups where each Γ_i is of type Φ over A and in $LH\mathscr{F}$. Noting that $LH\mathscr{F} \subseteq LH\mathscr{F}_{\phi,A}$, using Proposition 4.5.2 we get that $Gcd_A(\Gamma_i) < \infty$, which implies that $Gcd_A(\Gamma) < \infty$ by Proposition 2.9 of [29]. $LH\mathscr{F}$ is extension-closed (Result 2.4 of [41]), so $\Gamma \in LH\mathscr{F}$ and since $Gcd_A(\Gamma) < \infty$, we can use Proposition 4.5.2 to say that Γ is of type Φ over A.

For any finite subgroup $G \leq \Gamma$, the Weyl group with respect to G is defined as $W_\Gamma(G) := N_\Gamma(G)/G$. Proposition 2.5 of [29] gives us that $Gcd_A(W_\Gamma(G)) \leq Gcd_A(\Gamma)$. And $LH\mathscr{F}$ is Weyl group closed (this follows from the fact that $H\mathscr{F}$ is Weyl group closed- see Proposition 7.1 of [43]). So, if Γ is an $LH\mathscr{F}$-group that is of type Φ over A, then,

a) From Proposition 4.5.2, $Gcd_A(\Gamma) < \infty$.

b) For any finite subgroup G of Γ, $W_\Gamma(G)$ satisfies $Gcd_A(W_\Gamma(G)) < \infty$. As noted above, $W_\Gamma(G) \in LH\mathscr{F}$. So, by Proposition 4.5.2, $k(AW_\Gamma(G)) < \infty$. Therefore, $W_\Gamma(G)$ is of type Φ over A.

$\square$

Remark 4.5.4. *We are not in a position to replace $LH\mathscr{F}$ by $LH\mathscr{F}_{\phi,A}$ in the statement of Corollary 4.5.3 because we do not know whether $LH\mathscr{F}_{\phi,A}$ is closed under extensions or under taking Weyl subgroups, which we do know for $LH\mathscr{F}$.*

Over the ring of integers, Talelli proved in [62], that $(a) \Rightarrow (c) \Rightarrow (b) \Rightarrow (f)$ in Conjecture 4.1.12. Now, when $\Gamma \in H_1\mathscr{F}$, which is Statement (h) in Conjecture 4.1.12, it is easy to show that Γ is of type Φ over A for any A of finite global dimension - see Lemma 5.1.7 for example, and therefore (h) implies (a) to (g) in Conjecture 4.1.12 since $H_1\mathscr{F} \subseteq H\mathscr{F} \subseteq H_{\mathscr{F}_{\phi,A}} \subseteq LH\mathscr{F}_{\phi,A}$. Whether any of the statements (a) to (g) in Conjecture 4.1.12 implies $\Gamma \in H_1\mathscr{F}$ when $A = \mathbb{Z}$ is an open question

As we noted in Chapter 1, in [36], it was shown that for any integer n, there are groups in $H_{n+1}\mathscr{F}$ that are not in $H_n\mathscr{F}$. Can we make a similar claim with $\mathscr{F}_{\phi,A}$

replacing $\mathscr{F}$? The answer is yes and the following result from [36] is the reason why.

Theorem 4.5.5. *(Theorem 4.1 of [36]) Let $\mathscr{X}$ be a class of groups containing the class of all finite groups such that there is a countable group in $H_1\mathscr{X} \setminus \mathscr{X}$. Then, $H_\alpha\mathscr{X} \neq H_\beta\mathscr{X}$, for any two distinct countable ordinals α and β.*

To see how we can use Theorem 4.5.5 to obtain that $H_\alpha\mathscr{F}_{\phi,A} \neq H_\beta\mathscr{F}_{\phi,A}$, for any two distinct countable ordinals, where $\mathscr{F}_{\phi,A}$, and A is a commutative ring of finite global dimension, we use the following lemma.

The following result has actually been proved in [21], but we provide a shorter proof of it.

Theorem 4.5.6. *(different proof in [21]) Let A be a commutative ring of finite global dimension. Then, for any $\Gamma \in H_1\mathscr{F}$, all $A\Gamma$-modules admit complete resolutions over A.*

Proof. Let $\Gamma \in H_1\mathscr{F}$. Then, there is an n-dimensional CW-complex, for some integer n, on which Γ acts with finite stabilisers. The augmented cellular complex looks like an exact sequence $0 \to C_n \to .. \to C_0 \to \mathbb{Z} \to 0$ where each C_i is a permutation module with finite stabilisers, which are all Gorenstein projective by Lemma 1.1.11, and therefore $Gcd_{\mathbb{Z}}(\Gamma) = Gpd_{\mathbb{Z}\Gamma}(\mathbb{Z}) < \infty$. By Proposition 1.1.7, we have $Gcd_A(\Gamma) < \infty$, for all commutative rings A, and now our result follows using Theorem 4.1.3 and Remark 4.1.4. $\square$

Remark 4.5.7. *Note that from the proof of Theorem 4.5.6 above, it is clear that we do not need the assumption that A is of finite global dimension to prove that the trivial $A\Gamma$-module A admits complete resolutions. It is only in showing that all $A\Gamma$-modules admit complete resolutions we need the assumption that A is of finite global dimension.*

Lemma 4.5.8. *For any commutative ring A of finite global dimension, a free abelian group is of type Φ over A iff it is of finite rank.*

Proof. Let Γ be of type Φ over A. Then, since $B(\Gamma, A)$ restricts to a free module over finite subgroups of Γ by Lemma 1.3.7, $proj.dim_{A\Gamma}B(\Gamma, A) < \infty$, and so by Theorem 4.2.6, $Gcd_A(\Gamma) < \infty$, i.e. Γ admits complete resolutions (see Theorem 4.1.3 and Remark 4.1.4). We know that free abelian groups of infinite rank cannot admit complete resolutions over $\mathbb{Z}$ (see Example 1.1.6), and the exact same proof works works for rings of finite global dimension.

Now, let Γ be a free abelian group of finite rank n. Then, it acts on an n-dimensional CW-complex with $\mathbb{R}^n$ as the underlying space, and therefore $\Gamma \in H_1\mathscr{F}$. So, Γ is of type Φ by Remark 1.2.7. $\square$

Using the above two results, we can prove the following result regarding how the classes in Kropholler's hierarchy differ when the base class is the class of type Φ groups.

Proposition 4.5.9. *Let A be a commutative ring of finite global dimension. Then, $H_\alpha\mathscr{F}_{\phi,A} \neq H_\beta\mathscr{F}_{\phi,A}$, for any two distinct countable ordinals α and β.*

Proof. Theorem 1.2.3 tells us that $\mathbb{A}_{\aleph_0}$, the free abelian group of rank $\aleph_0$ is in $H_2\mathscr{F}$ and as $H_1\mathscr{F}$-groups are of type Φ over A by Remark 1.2.7, we have that $\mathbb{A}_{\aleph_0}$ is in $H_1\mathscr{F}_{\phi,A}$ but it is not in $\mathscr{F}_{\phi,A}$ by Lemma 4.5.8. Thus, $\mathscr{F}_{\phi,A}$ satisfies the hypothesis of Theorem 4.5.5, and we are done. $\square$

Remark 4.5.10. *Since a theme running through our treatment here is replacing $\mathscr{F}$ with $\mathscr{F}_{\phi,A}$, it is worth noting that Theorem 4.5.6 is not true with $H_1\mathscr{F}_{\phi,A}$-groups because the free abelian group of rank $\aleph_0$ is in $H_1\mathscr{F}_{\phi,A}$ as noted in the proof of Proposition 4.5.9 above and by Lemma 4.5.8, it cannot admit complete resolutions over A. Note that this also tells us that the statement of Lemma 1.1.11 need not be true if we replaced "finite stabilisers" by "type Φ stabilisers".*

One can actually show that $H_1\mathscr{F}_{\phi,A} \neq H_2\mathscr{F}_{\phi,A}$ without making any use of Theorem 4.5.5. We get from Theorem 1.2.3 that the free abelian group of rank $\aleph_{\omega_0}$, where ω_0 is the first infinite ordinal, is in $H_3\mathscr{F}$ but not in $H_2\mathscr{F}$ - this straightaway implies that it is in $H_2\mathscr{F}_{\phi,A}$ as $H_1\mathscr{F} \subseteq \mathscr{F}_{\phi,A}$ and it is also easy to see that it cannot be in $H_1\mathscr{F}_{\phi,A}$ because if it were in $H_1\mathscr{F}_{\phi,A}$, then, since all of its $\mathscr{F}_{\phi,A}$-subgroups are free abelian groups of finite rank by Lemma 4.5.8 and since all such subgroups are in $H_1\mathscr{F}$ (by Theorem 1.2.3 again), it would be in $H_2\mathscr{F}$.

We now make a small detour in this section and show in Proposition 4.5.13 that without using Lemma 4.4.6 and by making a few changes to a result of Benson, we can prove that $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (g)$ in Conjecture 4.1.12 with the same extra assumption as before that $\Gamma \in LH\mathscr{F}_{\phi,A}$.

We first state the following result by Benson.

Theorem 4.5.11. *(Theorem 5.7 of [11]) Let $\Gamma \in LH\mathscr{F}$ and let A be a commutative ring. Take M to be an $A\Gamma$-module such that over finite subgroups M has projective dimension at most r and $proj.dim_{A\Gamma} M \otimes_A B(\Gamma, A) \leq r$. Then, $proj.dim_{A\Gamma} M \leq r$.*

Our variation on Theorem 4.5.11 is the following:

Theorem 4.5.12. *Let $\Gamma \in LH\mathscr{F}_{\phi,A}$ where A is a commutative ring of finite global dimension. Then, Γ is of type Φ over A if $proj.dim_{A\Gamma} B(\Gamma, A) < \infty$.*

Proof. First note that if $\Gamma \in LH\mathscr{F}$, Theorem 4.5.12 follows directly from Theorem 4.5.11. We explain why. Let $proj.dim_{A\Gamma} B(\Gamma, A) = m < \infty$ and let t be the global dimension of A and let M be an $A\Gamma$-module with finite projective dimension over finite subgroups of Γ. Then, $proj.dim_{AG} M \leq t$, for all finite $G \leq \Gamma$. Since, $\Omega^t(M)$ is A-projective, we have $proj.dim_{A\Gamma} \Omega^t(M) \otimes_A B(\Gamma, A) \leq m$, and since $B(\Gamma, A)$ is A-free by Lemma 1.3.7, this gives us $proj.dim_{A\Gamma} \Omega^t(M \otimes_A B(\Gamma, A)) \leq m$, and therefore $proj.dim_{A\Gamma} M \otimes_A B(\Gamma, A) \leq m + t$. So, if we take $r = m + t$ in the hypothesis of Theorem 4.5.11, we are done.

In [11], Theorem 4.5.11 is proved by first proving it for $H\mathscr{F}$-groups, and then proving it for $LH\mathscr{F}$-groups that are not necessarily in $H\mathscr{F}$, this second part can be replicated with $\mathscr{F}_{\phi,A}$ replacing $\mathscr{F}$. Proving Theorem 4.5.11 for $H\mathscr{F}$-groups is done by induction on α where $\Gamma \in H_\alpha\mathscr{F}$. Here again, the inductive step can be replicated with $\mathscr{F}_{\phi,A}$ replacing $\mathscr{F}$ (both the steps - the inductive step and the going into $LH\mathscr{F}_{\phi,A}$ from $H\mathscr{F}_{\phi,A}$ is similar to the technique shown in the proof of Lemma 4.4.6; it is the standard technique for such situations). For the base case $\alpha = 0$, note that since M has finite projective dimension over finite subgroups, it has finite projective dimension over $\mathscr{F}_{\phi,A}$-subgroups as well. □

We can now prove the following promised result on Conjecture 4.1.12.

Proposition 4.5.13. *Let $\Gamma \in LH\mathscr{F}_{\phi,A}$ where A is a commutative ring with finite global dimension. Then, without using Lemma 4.4.6, one can show that $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (g)$ in Conjecture 4.1.12.*

Proof. We have $(g) \Rightarrow (c)$ by Lemma 4.4.5.c., $(c) \Rightarrow (b)$ by Lemma 4.4.4, $(b) \Rightarrow (e)$ by Lemma 4.4.5.a., $(e) \Rightarrow (d)$ by Theorem 4.4.2.a., $(d) \Rightarrow (a)$ by Theorem 4.5.12, and $(a) \Leftrightarrow (g)$ by Lemma 4.5.1. □

We show below that the information obtained in Theorem 4.5.12 lets us almost completely derive Lemma 4.1.5 using methods independent of the ones used in [29]. Here again, we do not have to use Lemma 4.4.6.

Remark 4.5.14. *The proof of Theorem 4.5.12 gives us that if $\Gamma \in LH\mathscr{F}_{\phi,A}$ with A of finite global dimension t, then $k(A\Gamma) \leq proj.dim_{A\Gamma} B(\Gamma, A) + t$. Using Lemma 4.4.4, Lemma 4.4.5.c., Theorem 4.4.2.a, this gives us that for $\Gamma \in LH\mathscr{F}_{\phi,A}$, $\mathrm{silp}(A\Gamma), \mathrm{spli}(A\Gamma) \leq Gcd_A(\Gamma) + t$. We say this "almost" completely gives a new proof of Lemma 4.1.5 because we believe $Gcd_A(\Gamma) < \infty$ iff $\Gamma \in \mathscr{F}_{\phi,A}$ (see (a) and (e) in Conjecture 4.1.12) and all known examples of groups with finite Gorenstein cohomological dimension are in $H_1\mathscr{F}$.*

We end this section with a comment on the following conjecture of Dembegioti and Talelli:

Conjecture 4.5.15. *For any group Γ, $\mathrm{spli}(\mathbb{Z}\Gamma) = \underline{cd}_{\mathbb{Z}}(\Gamma) + 1$.*

Remark 4.5.16. *First, note that by Theorem 4.1.9, we have that for any group Γ, $\underline{cd}_{\mathbb{Z}}(\Gamma) = Gcd_{\mathbb{Z}}(\Gamma)$. Now let $\Gamma \in LH\mathscr{F}_{\phi}$. Taking $A = \mathbb{Z}$ in Theorem 4.4.1, it follows that $\mathrm{spli}(\mathbb{Z}\Gamma)$ and $\underline{cd}_{\mathbb{Z}}(\Gamma)$ are finite only when $proj.dim_{\mathbb{Z}\Gamma} B(\Gamma, \mathbb{Z})$ is finite, and when that is the case, Theorem 4.4.1 tells us that the conjecture looks like $\mathrm{fin.dim}(\mathbb{Z}\Gamma) = proj.dim_{\mathbb{Z}\Gamma} B(\Gamma, \mathbb{Z}) + 1$. Again, courtesy Theorem 4.4.1, noting the fact the global dimension of $\mathbb{Z}$ is 1, we see that Conjecture 4.5.15 will be settled for all $\Gamma \in LH\mathscr{F}_{\phi,\mathbb{Z}}$ if we can prove that $\mathrm{fin.dim}(\mathbb{Z}\Gamma) \neq proj.dim_{\mathbb{Z}\Gamma} B(\Gamma, \mathbb{Z})$, i.e. we need to find a $\mathbb{Z}\Gamma$-module whose projective dimension is strictly bigger than that of $B(\Gamma, \mathbb{Z})$ but finite. First, note that we can assume that $proj.dim_{\mathbb{Z}\Gamma} B(\Gamma, \mathbb{Z}) > 1$ - this is because in Corollary 4.7 of [28], Emmanouil settled the conjecture for the cases where the generalized cohomological dimension is bounded by 1, and as we have seen, $proj.dim_{\mathbb{Z}\Gamma} B(\Gamma, \mathbb{Z}) < \infty$ implies $proj.dim_{\mathbb{Z}\Gamma} B(\Gamma, \mathbb{Z}) = Gcd_{\mathbb{Z}}(\Gamma) = \underline{cd}_{\mathbb{Z}}(\Gamma)$ by Theorem 4.1.9 and Theorem 4.2.6. A candidate for a $\mathbb{Z}\Gamma$-module with finite but bigger projective dimension than that of $B(\Gamma, \mathbb{Z})$ can be $B(\Gamma, \mathbb{Z}/p\mathbb{Z})$ for any prime p because we have a short exact sequence $0 \to B(\Gamma, \mathbb{Z}) \xrightarrow{p} B(\Gamma, \mathbb{Z}) \to B(\Gamma, \mathbb{Z}/p\mathbb{Z}) \to 0$ where the first map is multiplication by p.*

4.6 On a few homological invariants

In the previous few sections of this chapter, all the invariants associated to groups that we dealt with were cohomological invariants. In this section, we work with some homological invariants that are similar in theme with some of the cohomological invariants that we discussed earlier. Some of the results in this section are achieved by juggling various results from the literature.

We start with some definitions.

Definition 4.6.1. *(made over $\mathbb{Z}$ in Definition 4.4 of [39]) For any commutative ring A and any group Γ, $\mathrm{silf}(A\Gamma)$ and $\mathrm{sfli}(A\Gamma)$ denote, respectively, the supremum over the injective dimensions of flat $A\Gamma$-modules and the supremum over the flat dimensions of injective $A\Gamma$-modules.*

Definition 4.6.2. *(made over $\mathbb{Z}$ in Section 3 of [35]) For any group Γ and any commutative ring A, the generalized homological dimension of Γ over A, denoted $\underline{hd}_A(\Gamma)$, is defined as $\sup\{n \in \mathbb{Z}_{\geq 0} : \mathrm{Tor}_n^{A\Gamma}(M, C) \neq 0$, for some A-torsion free M and some cofree $A\Gamma$-module $C\}$.*

The following is an easy observation.

Lemma 4.6.3. *([39], [49]) Let A be a commutative ring and Γ be a group. Then,*

a) $\mathrm{silp}(A\Gamma) \leq \mathrm{silf}(A\Gamma)$

b) $\mathrm{sfli}(A\Gamma) \leq \mathrm{spli}(A\Gamma)$.

Proof. It follows from the fact that projectives are flat. □

We collect the following deep results from the literature.

Theorem 4.6.4. *Let Γ be a group. Then,*

a) $\underline{hd}_{\mathbb{Z}}(\Gamma) \leq \mathrm{sfli}(\mathbb{Z}\Gamma) \leq \underline{hd}_{\mathbb{Z}}(\Gamma) + 1$ *(Proposition 4.5.(1) of [39]).*

b) If $\mathrm{sfli}(\mathbb{Z}\Gamma) < \infty$, *then* $\mathrm{silf}(\mathbb{Z}\Gamma) \leq \mathrm{sfli}(\mathbb{Z}\Gamma)$ *(Proposition 4.5.(2) of [39]).*

The following corollary is easy to observe.

Corollary 4.6.5. *For any group Γ, $\underline{hd}_{\mathbb{Z}}(\Gamma) < \infty$ iff $\underline{cd}_{\mathbb{Z}}(\Gamma) < \infty$.*

Proof. If $\underline{hd}_{\mathbb{Z}}(\Gamma) < \infty$ then Theorem 4.6.4.a. gives us that $\mathrm{sfli}(\mathbb{Z}\Gamma) < \infty$, and therefore by Theorem 4.6.4.b., $\mathrm{silf}(\mathbb{Z}\Gamma) < \infty$ and by Lemma 4.6.3, $\mathrm{silp}(\mathbb{Z}\Gamma) < \infty$, therefore $\underline{cd}_{\mathbb{Z}}(\Gamma) < \infty$ by Lemma 4.4.5.

And if $\underline{cd}_{\mathbb{Z}}(\Gamma) < \infty$, then by Theorem 4.1.9 and Lemma 4.1.5, $\mathrm{spli}(\mathbb{Z}\Gamma) < \infty$, so by Lemma 4.6.3.a., $\mathrm{sfli}(\mathbb{Z}\Gamma) < \infty$, and therefore by Theorem 4.6.4.a., $\underline{hd}_{\mathbb{Z}}(\Gamma) < \infty$. □

We are now in a position to prove the following interesting result.

Theorem 4.6.6. *For any group* Γ*,* $\underline{hd}_{\mathbb{Z}}(\Gamma) = 0$ *iff* Γ *is finite.*

Proof. Let $\underline{hd}_{\mathbb{Z}}(\Gamma) = 0$. Then, by Theorem 4.6.4.a., $\mathrm{sfli}(\mathbb{Z}\Gamma)$ is either 0 or 1, and it then follows from Theorem 4.6.4.b. that $\mathrm{silf}(\mathbb{Z}\Gamma)$ is either 0 or 1. Therefore, by Lemma 4.6.3.a., $\mathrm{silp}(\mathbb{Z}\Gamma)$ is either 0 or 1. If $\mathrm{silp}(\mathbb{Z}\Gamma) = 0$, then by Lemma 4.4.5, $\underline{cd}_{\mathbb{Z}}(\Gamma) = 0$, and so by Theorem 4.1.11, $\mathrm{spli}(\mathbb{Z}\Gamma) = 1$, and as $\mathrm{silp}(\mathbb{Z}\Gamma) = \mathrm{spli}(\mathbb{Z}\Gamma)$ (see Theorem 4.1.7), we have a contradiction. And, if $\mathrm{silp}(\mathbb{Z}\Gamma) = 1$, then by Theorem 4.1.7, $\mathrm{spli}(\mathbb{Z}\Gamma) = 1$, and so by Theorem 4.1.11, Γ is finite.

The converse direction, i.e. that Γ being finite implies $\underline{hd}_{\mathbb{Z}}(\Gamma) = 0$ follows from Proposition 7 of Section 3 of [35]. □

In Remark 4.5.16, we mentioned a conjecture by Dembegioti and Talelli (= Conjecture 4.5.15) that, for any group Γ, $\mathrm{spli}(\mathbb{Z}\Gamma) = \underline{cd}_{\mathbb{Z}}(\Gamma) + 1$. Before stating a homological version of this conjecture, we make the following remark which is now considered standard knowledge in the literature.

Remark 4.6.7. *Note that courtesy of Theorem 4.1.9.a., Lemma 4.4.5, Lemma 4.1.5, and Theorem 4.1.7, we always have that* $\underline{cd}_{\mathbb{Z}}(\Gamma) \leq \mathrm{spli}(\mathbb{Z}\Gamma) \leq \underline{cd}_{\mathbb{Z}}(\Gamma) + 1$*, for any group* Γ*.*

In [24], Dembegioti and Talelli proved Conjecture 4.5.15 for duality groups, fundamental groups of graphs of finite groups and fundamental groups of certain finite graphs of groups of type FP_{∞}.

The following is a homological version of Conjecture 4.5.15 in light of Theorem 4.6.4.e.:

Conjecture 4.6.8. *For any group* Γ, $\mathrm{sfli}(\mathbb{Z}\Gamma) = \underline{hd}_{\mathbb{Z}}(\Gamma) + 1$.

From Theorem 4.6.4 and Theorem 4.6.6, the following result, which mirrors Emmanouil's proof of the fact that Γ satisfies Conjecture 4.5.15 if $\underline{cd}_{\mathbb{Z}}(\Gamma)$ is 0 or 1 (Corollary 4.7 of [28]), is clear:

Theorem 4.6.9. *Conjecture 4.6.8 holds true when Γ is such that $\underline{hd}_{\mathbb{Z}}(\Gamma)$ is 0 or 1.*

Proof. When $\underline{hd}_{\mathbb{Z}}(\Gamma) = 0$, then by Theorem 4.6.4.a., $\mathrm{sfli}(\mathbb{Z}\Gamma)$ is either 0 or 1. $\mathrm{sfli}(\mathbb{Z}\Gamma) = 0$ cannot occur because then $\mathrm{silf}(\mathbb{Z}\Gamma) = 0$ (Theorem 4.6.4.b.), which implies $\underline{cd}_{\mathbb{Z}}(\Gamma) = \mathrm{silp}(\mathbb{Z}\Gamma) = 0$ by Lemma 4.6.3 and Lemma 4.4.5, which then implies the finiteness of Γ and forces $\mathrm{silp}(\mathbb{Z}\Gamma) = \mathrm{spli}(\mathbb{Z}\Gamma) = 1$ (Theorem 4.1.11 and Theorem 4.1.7), a contradiction.

When $\underline{hd}_{\mathbb{Z}}(\Gamma) = 1$, then Γ cannot be finite by Theorem 4.6.6. Theorem 4.6.4.a. tells us that $\mathrm{sfli}(\mathbb{Z}\Gamma)$ is 1 or 2. Here again, note that if $\mathrm{sfli}(\mathbb{Z}\Gamma) = 1$, then $\mathrm{silf}(\mathbb{Z}\Gamma)$ is 0 or 1, and therefore $\mathrm{silp}(\mathbb{Z}\Gamma) = 0$ or 1 by Lemma 4.6.3, and as noted above $\mathrm{silp}(\mathbb{Z}\Gamma) = 0$ cannot happen, so $\mathrm{spli}(\mathbb{Z}\Gamma) = \mathrm{silp}(\mathbb{Z}\Gamma) = 1$, i.e. Γ is finite by Theorem 4.1.11, which is a contradiction. Therefore, $\mathrm{sfli}(\mathbb{Z}\Gamma) = 2$. $\square$

It is noted in Remark 3.16 of [5] that Conjecture 4.6.8 also holds true for locally finite groups and free abelian groups of finite rank.

We end this section with the following result on connecting the "silp / spli" invariants with "silf / sfli" invariants for group rings where the base ring is Noetherian and of finite global dimension.

Proposition 4.6.10. *Let A be a commutative Noetherian ring of finite global dimension and let Γ be a group. Then,*

$$\mathrm{silp}(A\Gamma) = \mathrm{silf}(A\Gamma) = \mathrm{sfli}(A\Gamma) = \mathrm{spli}(A\Gamma)$$

.

Proof. From Lemma 4.6.3.b., we have $\mathrm{sfli}(A\Gamma) \leq \mathrm{spli}(A\Gamma)$.

To show $\mathrm{silf}(A\Gamma) \leq \mathrm{sfli}(A\Gamma)$, we can start with the assumption that $\mathrm{sfli}(A\Gamma)$ is finite. For $A = \mathbb{Z}$, this has been handled in Proposition 4.5.(2) of [39] (see Theorem 4.6.4.b.), and the same proof works in this situation. The proof uses the theory of character modules which works irrespective of the base ring (see Chapter 3 of [57]).

$\mathrm{silp}(A\Gamma) \leq \mathrm{silf}(A\Gamma)$, as noted in Lemma 4.6.3.a.

It follows from Theorem 4.1.7 that $\mathrm{silp}(A\Gamma) = \mathrm{spli}(A\Gamma)$.

□

It is reasonable to expect the following conjecture to be true in light of Corollary 4.6.5, Conjecture 4.5.15, Conjecture 4.6.8 and Proposition 4.6.10.

Conjecture 4.6.11. *For any group Γ, $\underline{cd}_{\mathbb{Z}}(\Gamma) = \underline{hd}_{\mathbb{Z}}(\Gamma)$.*

Remark 4.6.12. *It is easy to observe that, if for any group Γ, any two of Conjecture 4.5.15, Conjecture 4.6.8 and Conjecture 4.6.11 hold, the third one holds as well.*

Using Lemma 4.6.3 and since the proof of Theorem 4.6.4.b. given in [39] works with $\mathbb{Z}$ replaced by any commutative ring of finite global dimension, we can slightly extend Theorem 4.4.1 in the following way:

Theorem 4.6.13. *Let A be a commutative ring of finite global dimension t and $\Gamma \in LH\mathscr{F}_{\phi,A}$. Then,*

$$\mathrm{fin.\,dim}(A\Gamma) = \mathrm{silp}(A\Gamma) = \mathrm{silf}(A\Gamma) = \mathrm{sfli}(A\Gamma) = \mathrm{spli}(A\Gamma) = k(A\Gamma)$$

and the common value of these invariants lies between $proj.dim_{A\Gamma}B(\Gamma, A)$ and $t+$ $proj.dim_{A\Gamma}B(\Gamma, A)$. Note that, here, if any of these invariants is not finite, then none of the other invariants can be finite.

In light of Theorem 4.6.13, we can make a small extension to Conjecture 4.1.12:

Conjecture 4.6.14. *For any group Γ and any commutative ring A of finite global dimension, the following are equivalent:*

a) Γ *is of type* Φ *over* A.

b) $\mathrm{silp}(A\Gamma) < \infty$.

c) $\mathrm{spli}(A\Gamma) < \infty$.

d) $\mathrm{silf}(A\Gamma) < \infty$.

e) $\mathrm{sfli}(A\Gamma) < \infty$.

f) $proj.dim_{A\Gamma}B(\Gamma, A) < \infty$.

g) $Gcd_A(\Gamma) < \infty$.

h) $\mathrm{fin.\,dim}(A\Gamma) < \infty$.

i) $k(A\Gamma) < \infty$.

Remark 4.6.15. *Clearly, Theorem 4.6.13 proves Conjecture 4.6.14 for* $\Gamma \in LH\mathscr{F}_{\phi,A}$ *once we note that* $\Gamma \in \mathscr{F}_{\phi,A} \Leftrightarrow k(A\Gamma) < \infty$ *by Lemma 4.5.1.*

Chapter 5

Comparing Two Hierarchies and Related Questions

In the eighties, Bruce Ikenaga [35] devised a hierarchical system of groups and extended the study of complete cohomology to them. Complete cohomology had previously been studied for finite groups and groups of finite virtual cohomological dimension [17]; Ikenaga's classes contained these groups and it was clear that his classes had some more groups as well. It was not clear how big those classes were exactly. We have previously looked at Kropholler's hierarchy. A key property of groups in Ikenaga's hierarchy is the admission of complete resolutions, and this property is shared by groups in the first level of Kropholler's hierarchy with the class of finite groups as the base class. A direct connection between the two hierarchies was established by Mislin and Talelli in [49], and it became clear that Kropholler's hierarchy is strictly bigger than Ikenaga's.

In Section 5.1 of this chapter, we shall show how this question and a few other similar questions can be affected by some other related conjectures in the field. Section 5.1 can be treated as a section on some preliminaries. In Section 5.2, we use some results from Section 5.1 along with some others from the literature, including some that we proved in Chapter 4, to derive a result on classifying spaces that is connected to a question raised in Section 5.1. In Section 5.3, we prove some interesting properties of derived categories of chain complexes of modules over these groups and observe how questions raised in Section 5.1 relate to concepts and properties in the context of derived categories.

5.1 Ikenaga's hierarchy

We now introduce Ikenaga's classes of groups.

Definition 5.1.1. *(based on Section* 5*, [35]) Let $\mathscr{X}$ be a class of groups. Define $C_0(\mathscr{X}) := \mathscr{X}$, and a group $\Gamma \in C_n(\mathscr{X})$ iff there exists an acyclic simplicial complex X on which Γ acts by permuting the simplices such that $\Gamma_\sigma \in C_{n-1}(\mathscr{X})$, for each simplex $\sigma \in X$, where Γ_σ denotes the stabiliser of σ, and $sup_{\sigma\in\Sigma}\{dim(\sigma) + \underline{cd}_{\mathbb{Z}}(\Gamma) + dim(\sigma)\} < \infty$, where Σ is the set of representatives of X modulo the Γ-action.*

$C_\infty(\mathscr{X}) := \bigcup_{n\geq 0} C_n(\mathscr{X})$.

For groups in $C_\infty(\mathscr{F})$, the following was proved in [35].

Theorem 5.1.2. *Groups in $C_\infty(\mathscr{F})$ have finite generalized cohomological dimension over $\mathbb{Z}$ and they admit weak complete resolutions over $\mathbb{Z}$.*

Although it was not noted in [35], groups in $C_\infty(\mathscr{F})$ actually admit complete resolutions, which we can show using the following result.

Lemma 5.1.3. *(done over $\mathbb{Z}$ in Lemma* 2.2 *of [49], same proof works here) If Γ admits weak complete resolutions over a commutative ring A of finite global dimension and* $\mathrm{silp}(A\Gamma) < \infty$*, then Γ admits complete resolutions over A.*

Corollary 5.1.4. *$C_\infty(\mathscr{F})$-groups admit complete resolutions over any commutative ring A of finite global dimension.*

Proof. Let $\Gamma \in C_\infty(\mathscr{F})$. Then by Theorem 5.1.2, $\underline{cd}_{\mathbb{Z}}(\Gamma) < \infty$. Note that $\underline{cd}_{\mathbb{Z}}(\Gamma) = Gcd_{\mathbb{Z}}(\Gamma)$ by Theorem 4.1.9. So by Lemma 4.1.5, $\mathrm{silp}(\mathbb{Z}\Gamma) < \infty$. So, by Lemma 5.1.3 and Theorem 5.1.2, Γ admits complete resolutions over $\mathbb{Z}$. The result translates to all commutative rings of finite global dimension due to Proposition 1.1.7, Theorem 4.1.3 and Remark 4.1.4.

□

One can form Ikenaga's classes of groups starting with the class of all groups of type Φ as the base class. Whether or not we get any groups that we do not get when we start with the class of all finite groups as the base class is part of a conjecture (See Conjecture 5.1.10) that we make later.

Remark 5.1.5. *Since both the definitions of Ikenaga's classes and Kropholler's hierarchy involves a kind of iteration on the definition of a level to get to the next level, it is natural to wonder whether one can do something similar with type Φ groups by iterating Definition 1.2.6. It turns out we can't as we explain below.*

Let's fix an A of finite global dimension, and call type Φ groups type Φ^1. For all $n \geq 1$, define a group Γ to be of type Φ^n if, for any $A\Gamma$-module M, M is of finite projective dimension as an $A\Gamma$-module iff it is of finite projective dimension over all type Φ^{n-1} subgroups.

If Γ is type Φ^n, and M is of finite projective dimension over finite subgroups, then M is of finite projective dimension over type Φ subgroups, and by the iterative definition above, it is of finite projective dimension over type Φ^2 groups, and going on like this, it is of finite projective dimension over type Φ^{n-1} groups, from which it follows from the iterative definition above again, that $proj.dim_{A\Gamma} M < \infty$. Thus, Γ is of type Φ.

Remark 5.1.6. *As noted before, it is now known that for every integer n, $H_{n+1}\mathscr{F}$ is a strictly bigger class than $H_n\mathscr{F}$. No such result is known for Ikenaga's classes and that is why we feature this as a conjecture in Conjecture 5.1.13.*

The following are some handy connections between the classes of groups we have introduced.

Lemma 5.1.7. *a) $C_n(\mathscr{F}) := \{\Gamma \in H_n\mathscr{F} : \mathrm{spli}(\mathbb{Z}\Gamma) < \infty\}$; $C_1(\mathscr{F}) = H_1\mathscr{F}$. (Corollary* 2.6 *of [49])*

b) $C_\infty(\mathscr{F}) \subseteq \mathscr{F}_{\phi,A}$, for any A of finite global dimension.

Proof. The only bit that's new here is $C_\infty(\mathscr{F}) \subseteq \mathscr{F}_{\phi,A}$. Since $C_\infty(\mathscr{F}) \subseteq H\mathscr{F}$ by Lemma 5.1.7.a., it follows from Theorem 4.4.1 and (a) that $Gcd_{\mathbb{Z}}(\Gamma) < \infty$ and thus by Proposition 1.1.7, for any commutative ring A of finite global dimension, $Gcd_A(\Gamma) < \infty$. So, Γ is of type Φ over A by Proposition 4.5.2 or Proposition 4.5.13. □

It is noteworthy that the operator L is quite powerful in that when applied to classes of groups like $C_\infty(\mathscr{F}), \mathscr{F}_{\phi,A}$ (for any A of finite global dimension) and $H_1\mathscr{F}$, it gives a strictly larger class of groups:

Proposition 5.1.8. *For any commutative ring A of finite global dimension, $LH_1\mathscr{F} \neq H_1\mathscr{F}$; $LC_\infty(\mathscr{F}) \neq C_\infty(\mathscr{F})$; $L\mathscr{F}_{\phi,A} \neq \mathscr{F}_{\phi,A}$.*

Proof. Take Γ to be a free abelian group of infinite rank, then any finitely generated subgroup of it, say a free abelian groups of finite rank n, acts on an n-dimensional CW-complex with $\mathbb{R}^n$ as the underlying space, and therefore $\Gamma \in H_1\mathscr{F}$ and by Lemma 5.1.7.a., is in $C_\infty(\mathscr{F})$ and $\mathscr{F}_{\phi,A}$. Γ does not admit complete resolutions over A by Example 1.1.6, so it is not in $H_1\mathscr{F}, C_\infty(\mathscr{F})$ or $\mathscr{F}_{\phi,A}$. □

It follows from Theorem 4.4.1 and Lemma 4.5.1 that for type Φ groups all of our invariants are finite and well-behaved (see Remark 5.1.14 below). Below, we state a close restatement of Conjecture 4.1.12, with the difference being that we include a statement on the classifying space of proper actions. We need to define classifying space of proper actions of a group first.

Definition 5.1.9. *For any group Γ, $\underline{E}\Gamma$ denotes a CW-complex on which Γ acts cellularly with finite stabilisers such that for any finite subgroup G of Γ, the fixed point subcomplex $\underline{E}\Gamma^G$ is contractible. (it is known for any group, such a complex exists)*

Conjecture 5.1.10. *Let A be a commutative ring of finite global dimension. For any group Γ, the following are equivalent.*

a) Γ *is of type* Φ *over* A

b) $Gcd_A(\Gamma) < \infty$.

c) $\mathrm{spli}(A\Gamma) < \infty$.

d) $\mathrm{silp}(A\Gamma) < \infty$.

e) $\mathrm{fin.dim}(A\Gamma) < \infty$.

f) $k(A\Gamma) < \infty$.

When $A = \mathbb{Z}$, we can add the following statement:

g) Γ *admits a finite dimensional model for* $\underline{E}\Gamma$.

We deal with classifying spaces later in Proposition 5.1.15 and then in Section 5.2.

Remark 5.1.11. *Conjecture 5.1.10 looks very similar to Conjecture 4.1.12, except (g) of Conjecture 4.1.12 is the statement that $\Gamma \in H_1\mathscr{F}$, but here (g) of Conjecture 5.1.10 is the statement that Γ admits a finite dimensional model for $\underline{E}\Gamma$. Although it is clear that if Γ satisfies the latter it is definitely in $H_1\mathscr{F}$ (see Section 4 of [48]), whether the converse holds is still open to conjecture (see Conjecture 43.1 of [19]).*

It seems a sensible question to ask whether one could place all groups with complete resolutions within a known hierarchical class. The following result sheds some light in that direction.

Proposition 5.1.12. *(follows from Proposition 4.5.2/4.5.13) Let A be a commutative ring of finite global dimension. Then, any $LH\mathscr{F}_{\phi,A}$-group that admits complete resolutions over A is in $\mathscr{F}_{\phi,A}$.*

Proof. Let Γ be a group in $LH\mathscr{F}_{\phi,A}$ that admits complete resolutions over A. Then, $Gcd_A(\Gamma) < \infty$ by Theorem 4.1.3. Now, Proposition 4.5.2/4.5.13 tells us that $k(A\Gamma)$ is finite. So, by Lemma 4.5.1, $\Gamma \in \mathscr{F}_{\phi,A}$. □

Whether or not Proposition 5.1.12 can in any way be stated with the base class $\mathscr{F}$ instead of $\mathscr{F}_{\phi,A}$, i.e. whether we can say that any $H\mathscr{F}$-group with complete resolutions has to be in a particular level of Kropholler's hierarchy, is an interesting question and it forms one of our conjectured statements below. In Conjecture 5.1.13 below, most of the statements are expectations based on evidence of a lack of examples to indicate otherwise. 5.1.13.b., for example, is a standard question to ask once all the different hierarchical classes have been defined in any hierarchy in general. The same logic applies to asking 5.1.13.c/d/f. For the following conjecture, we denote by $\mathrm{CR}(\mathbb{Z})$ the class of all groups that admit complete resolutions over the integers.

Conjecture 5.1.13. *The following statements are true.*

a) $H\mathscr{F} \cap \mathrm{CR}(\mathbb{Z}) = H_1\mathscr{F}$.

b) $C_1(\mathscr{F}) = C_2(\mathscr{F}) = \ldots$

c) $C_\infty(\mathscr{F}) = H_1\mathscr{F}$.

d) $C_\infty(\mathscr{F}) = \mathscr{F}_\phi$.

e) $C_\infty(\mathscr{F}) = \{\Gamma : \underline{cd}_{\mathbb{Z}}(\Gamma) < \infty\}$

f) $\mathscr{F}_\phi = H_1\mathscr{F}$.

g) $C_\infty(\mathscr{F}_\phi) = C_\infty(\mathscr{F})$.

h) $H\mathscr{F}_\phi = H\mathscr{F}$.

Remark 5.1.14. *Note that it follows from Theorem 4.1.3, Remark 4.1.4, Theorem 4.1.9, Lemma 4.5.1 and Theorem 4.4.1 that if $\Gamma \in \mathscr{F}_{\phi,A}$ with A of finite global dimension, then Γ admits complete resolutions over A and all the invariants - $\underline{cd}_A(\Gamma)$,*

$Gcd_A(\Gamma)$, $proj.dim_{A\Gamma}B(\Gamma, A)$, $\mathrm{silp}(A\Gamma)$, $\mathrm{spli}(A\Gamma)$, $\mathrm{fin.\,dim}(A\Gamma)$, $k(A\Gamma)$- *are finite. We are recording this here because we will be making repeated use of this in the proof of Proposition 5.1.15.*

In the following result, whenever we say $p_1 \overset{p_2}{\Longrightarrow} p_3$, or $p_1 \overset{p_2}{\Longleftrightarrow} p_3$, for some statements p_1, p_2, p_3, we mean $p_1 \Longrightarrow p_3$, or resp. $p_1 \Longleftrightarrow p_3$, if p_2 is assumed to be true.

Proposition 5.1.15. *The following implications are true involving the statements of Conjecture 5.1.13.*

a) 5.1.13.*a.* $\Longleftrightarrow$ 5.1.13.*b.* $\Longleftrightarrow$ 5.1.13.*c.*

b) 5.1.13.*c.* $\overset{5.1.13.f}{\Longleftrightarrow}$ 5.1.13.*d.*

c) 5.1.13.*e.* $\Longrightarrow$ 5.1.13.*d.*

d) 5.1.13.*c.* $\overset{5.1.13.e.}{\Longrightarrow}$ 5.1.13.*f. And,* 5.1.13.*f.* $\Longrightarrow$ 5.1.13.*c.*

e) 5.1.13.*d.* $\overset{5.1.13.a.}{\Longrightarrow}$ 5.1.13.*f. And,* 5.1.13.*f.* $\Longrightarrow$ 5.1.13.*d.*

f) 5.1.13.*f.* $\Longrightarrow$ 5.1.13.*h.* $\Longrightarrow$ 5.1.13.*g.*

g) If, in Conjecture 5.1.10, when $A = \mathbb{Z}$, 5.1.10.b./c./d./e. $\Longrightarrow$ 5.1.10.g., then 5.1.13.a.-5.1.13.h. is true.

Proof. a) Note that by Theorem 4.1.3, Theorem 4.1.7 and Lemma 5.1.3, $\mathrm{spli}(\mathbb{Z}\Gamma) < \infty \Longleftrightarrow \Gamma \in \mathrm{CR}(\mathbb{Z})$. So, 5.1.13.*a.* $\Longleftrightarrow$ 5.1.13.*c.* by Lemma 5.1.7.a. 5.1.13.*b.* $\Longleftrightarrow$ 5.1.13.*c.* also follows from the fact that $C_1(\mathscr{F}) = H_1\mathscr{F}$ by Lemma 5.1.7.a.

b) This is obvious.

c) We know from Lemma 5.1.7.b. that $C_\infty(\mathscr{F}) \subseteq \mathscr{F}_\phi$. Now, if $\Gamma \in \mathscr{F}_\phi$, $\underline{cd}_{\mathbb{Z}}(\Gamma) < \infty$ by Remark 5.1.14, and by 5.1.13.e., $\Gamma \in C_\infty(\mathscr{F})$.

d) To show that 5.1.13.*c.* $\Longrightarrow$ 5.1.13.*f.* if we assume 5.1.13.e., note that $H_1\mathscr{F} \subseteq \mathscr{F}_\phi$ by Lemma 5.1.7 and if $\Gamma \in \mathscr{F}_\phi$, then $\underline{cd}_{\mathbb{Z}}(\Gamma) < \infty$ again by Remark 5.1.14, and therefore by 5.1.13.e., $\Gamma \in C_\infty(\mathscr{F}) = H_1\mathscr{F}$ (the last equality is from the hypothesis 5.1.13.c.).

If 5.1.13.f. is true, then $C_\infty(\mathscr{F}) \subseteq \mathscr{F}_\phi$ (by Lemma 5.1.7) $= H_1\mathscr{F}$ (by 5.1.13.f.). We already know courtesy Lemma 5.1.7 that $H_1\mathscr{F} = C_1(\mathscr{F}) \subseteq C_\infty(\mathscr{F})$.

e) To show that 5.1.13.*d.* $\Longrightarrow$ 5.1.13.*f.* if we assume 5.1.13.a., note that if $\Gamma \in \mathscr{F}_\phi$, then $\Gamma \in \mathrm{CR}(\mathbb{Z})$ by Remark 5.1.14, and since 5.1.13.d. gives us that $\Gamma \in C_\infty(\mathscr{F})$, we get from 5.1.13.a. and Lemma 5.1.7.a. that $\Gamma \in H_1\mathscr{F}$. Again, $H_1\mathscr{F} \subseteq \mathscr{F}_\phi$ follows from Lemma 5.1.7.b.

5.1.13.f. $\Longrightarrow$ 5.1.13.d. is easy to see as $H_1\mathscr{F} = C_1(\mathscr{F})$ by Lemma 5.1.7.a.

f) 5.1.13.f. $\Longrightarrow$ 5.1.13.h. is obvious as $\mathscr{F}_\phi = H_1\mathscr{F}$ implies $H\mathscr{F}_\phi = H(H_1\mathscr{F}) = H\mathscr{F}$. 5.1.13.$h$. $\Longrightarrow$ 5.1.13.g. is easy to see as well because it follows from the proof of Lemma 5.1.7.a. in [49] that $C_\infty(\mathscr{F}_\phi) = \{\Gamma \in H\mathscr{F}_\phi : \operatorname{spli}(\mathbb{Z}\Gamma) < \infty\}$.

g) It follows from Lemma 4.4.5 that we can streamline our hypothesis to 5.1.10.e. $\Longrightarrow$ 5.1.10.g. (we denote this statement by $(*)$). We assume $(*)$ is true.

$(*) \Longrightarrow$ 5.1.13.c.: Now, if $\Gamma \in C_\infty(\mathscr{F})$, then $\operatorname{fin.dim}(\mathbb{Z}\Gamma) < \infty$ by Lemma 5.1.7 and Remark 5.1.14, so there is a finite dimensional model for $\underline{E}\Gamma$, so clearly $\Gamma \in H_1\mathscr{F}$ (see Remark 5.1.11). Thus 5.1.13.c. holds, and so 5.1.13.a-c. hold as well by part (a) of this proposition.

$(*) \Longrightarrow$ 5.1.13.d.: If $\Gamma \in \mathscr{F}_\phi$, $\operatorname{fin.dim}(\mathbb{Z}\Gamma) < \infty$ by Remark 5.1.14, and since $(*)$ holds, there is a finite dimensional model for $\underline{E}\Gamma$, therefore $\Gamma \in H_1\mathscr{F} = C_1(\mathscr{F}) \subseteq C_\infty(\mathscr{F})$. So, 5.1.13.d. holds as we already know that $C_\infty(\mathscr{F}) \subseteq \mathscr{F}_\phi$ by Lemma 5.1.7.b.

$(*) \Longrightarrow$ 5.1.13.e.: If Γ be a group such that $\underline{cd}_{\mathbb{Z}}(\Gamma) < \infty$, then by Theorem 4.1.9.a., Lemma 4.1.5 and Lemma 4.4.5, $\operatorname{fin.dim}(\mathbb{Z}\Gamma) < \infty$, and therefore there is a finite dimensional model for $\underline{E}\Gamma$, so $\Gamma \in H_1\mathscr{F} = C_1(\mathscr{F})$. Again note that we already know that if groups in Ikenaga's classes have finite generalized cohomological dimension over the integers (Theorem 5.1.2).

$(*) \Longrightarrow$ 5.1.13.f.: We already know that $H_1\mathscr{F} \subseteq \mathscr{F}_\phi$ by Lemma 5.1.7.b. Now let $\Gamma \in \mathscr{F}_\phi$. Then, by Remark 5.1.14, $\operatorname{fin.dim}(\mathbb{Z}\Gamma) < \infty$ and by $(*)$, there exists a finite dimensional model for $\underline{E}\Gamma$ and therefore $\Gamma \in H_1\mathscr{F}$.

Thus, $(*)$ implies 5.1.13.g. and 5.1.13.h. as well by part (f) of this proposition. $\square$

5.2 A small result on classifying spaces

As we saw in Proposition 5.1.15.g., the finiteness of almost any cohomological invariant for Γ implying the existence of a finite dimensional model for $\underline{E}\Gamma$ is quite strong. In this section, we show using a key result from [44] that some of the classes of groups that we have dealt with admit finite dimensional models for their classifying space of proper actions if an additional condition is satisfied. To introduce this additional condition, we need the following definition.

Definition 5.2.1. *For a finite group G, define the length of G, denoted $l(G)$, as the* *supremum over n such that there is a nested sequence $H_0 \subsetneq H_1 \subsetneq .. \subsetneq H_n$ where each H_i is a subgroup of G.*

Definition 5.2.2. *For any integer d, a group Γ is said to be of type $b(d)$ if for every $\mathbb{Z}\Gamma$-module M that is projective over finite subgroups, $proj.dim_{\mathbb{Z}\Gamma} M \leq d$. Γ is said to be of type $B(d)$ if, for every finite $G \leq \Gamma$, $W_\Gamma(G) := N_\Gamma(G)/G$ is of type $b(d)$.*

It is easy to note that groups of type Φ over the integers are of type $b(d)$ for some $d \geq 0$:

Lemma 5.2.3. *Let $\Gamma \in \mathscr{F}_\phi$ ($:= \mathscr{F}_{\phi,\mathbb{Z}}$, see Definition 1.2.6). Then, Γ is of type $b(k(\mathbb{Z}\Gamma))$.*

Proof. It follows from Remark 5.1.14 that $k(\mathbb{Z}\Gamma) < \infty$. If M is projective over finite subgroups of Γ, then by the definition of $k(\mathbb{Z}\Gamma)$, $proj.dim_{\mathbb{Z}\Gamma} M \leq k(\mathbb{Z}\Gamma)$.

□

The following is the key result from [44] that we will be using in this section.

Theorem 5.2.4. *(Theorem 1.10 of [44]) Let Γ be a group of type $B(d)$ for some $d \geq 0$ and let l be the bound on the length of all finite subgroups of Γ. Then, Γ admits a finite dimensional model for $\underline{E}\Gamma$.*

Using Theorem 5.2.4, we can prove the following result.

Proposition 5.2.5. *Let $\Gamma \in LH\mathscr{F} \cap \mathscr{F}_\phi$ such that there is a bound on the length of all finite subgroups of Γ. Then, there exists a finite dimensional model for $\underline{E}\Gamma$.*

Proof. From Theorem 5.2.4, it follows that we will be done if we show that Γ is of type $B(d)$ for some $d \geq 0$. From Remark 5.1.14, it follows that $Gcd_{\mathbb{Z}}(\Gamma) < \infty$.

For any finite subgroup $G \leq \Gamma$, $W_\Gamma(G)$ is in $LH\mathscr{F}$ (this follows from the fact that $H\mathscr{F}$ is Weyl group closed - see Proposition 7.1 of [43]). So, it follows from Theorem 4.4.1 that $k(\mathbb{Z}W_\Gamma(G)) \leq Gcd_{\mathbb{Z}}(W_\Gamma(G)) + 1 \leq Gcd_{\mathbb{Z}}(\Gamma) + 1$ (the last inequality is by Proposition 2.5 of [29]). Thus, $W_\Gamma(G)$ is of type $b(Gcd_{\mathbb{Z}}(\Gamma) + 1)$. So we have shown that Γ is of type $B(Gcd_{\mathbb{Z}}(\Gamma) + 1)$, and we are done.

□

It is interesting to note that we can replace the hypothesis $\Gamma \in LH\mathscr{F} \cap \mathscr{F}_\phi$ in the statement of Proposition 5.2.5 by $\Gamma \in C_\infty(\mathscr{F})$.

Corollary 5.2.6. *Let Γ be in $C_\infty(\mathscr{F})$ with a bound on the length of its finite subgroups. Then, there is a finite dimensional model for $\underline{E}\Gamma$.*

Proof. This follows directly from Proposition 5.2.5 using Lemma 5.1.7. $\square$

5.3 Derived categories of modules over groups from the hierarchies

For the whole of this section, we shall use the following notations. For any ring R, we denote the derived category of unbounded chain complexes of R-modules by $\mathcal{D}(Mod\text{-}R)$, and for any class of objects $\mathscr{U} \subseteq \mathcal{D}(Mod\text{-}R)$, we denote the smallest localising subcategory of $\mathcal{D}(Mod\text{-}R)$ (localising subcategory= triangulated subcategory closed under coproducts) containing $\mathscr{U}$ by $\langle \mathscr{U} \rangle$ (note that we deal with these concepts again in Chapter 8 where we use a different notation for this notion of generation in triangulated categories with coproducts). In general, for any triangulated category $\mathscr{T}$, $\Delta_{\mathscr{T}}(\mathscr{U})$ denotes the smallest triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$.

Interestingly, the finiteness of the silp and spli invariants over a ring R can be shown to imply the coincidence of $\Delta_{\mathscr{T}}(\text{Projectives})$ and $\Delta_{\mathscr{T}}(\text{Injectives})$ where $\mathscr{T} = \mathcal{D}^b(Mod\text{-}R)$, the derived category of bounded chain complexes of R-modules. Before showing this, we need the following standard result which we have provided a full proof of in Chapter 8.

Lemma 5.3.1. *(= Lemma 8.2.2) Let R be a ring and let $\mathscr{T}$ be a triangulated subcategory of $\mathcal{D}(Mod\text{-}R)$ or $\mathcal{D}^b(Mod\text{-}R)$. Then, any chain complex, X_*, of the form $0 \to X_n \to X_{n-1} \to \ldots \to X_0 \to 0$ is in $\mathscr{T}$ if each X_i, when considered as a chain complex concentrated in degree zero, is in $\mathscr{T}$.*

Proposition 5.3.2. *Let R be a ring and let $\mathscr{T} = \mathcal{D}^b(Mod\text{-}R)$ or $\mathcal{D}(Mod\text{-}R)$. Denote by $Proj\text{-}R$ and $Inj\text{-}R$ the classes of all projective R-modules and all injective R-modules respectively. We treat these classes as classes of chain complexes concentrated in degree 0. Then,*

a) $\text{silp}(R) < \infty \Longrightarrow \Delta_{\mathscr{T}}(Proj\text{-}R) \subseteq \Delta_{\mathscr{T}}(Inj\text{-}R)$.

b) $\text{spli}(R) < \infty \Longrightarrow \Delta_{\mathscr{T}}(Inj\text{-}R) \subseteq \Delta_{\mathscr{T}}(Proj\text{-}R)$.

Therefore, $\text{silp}(R), \text{spli}(R) < \infty \Longrightarrow \Delta_{\mathscr{T}}(Inj\text{-}R) = \Delta_{\mathscr{T}}(Proj\text{-}R)$.

Proof. We prove (a). The proof for (b) is similar.

If $\operatorname{silp}(R) < \infty$, every projective module, as a chain complex concentrated in degree zero, is quasi-isomorphic to a bounded complex of injectives. So, by Lemma 5.3.1, $Proj\text{-}R \subseteq \Delta_{\mathscr{T}}(Inj\text{-}R)$, and therefore $\Delta_{\mathscr{T}}(Proj\text{-}R) \subseteq \Delta_{\mathscr{T}}(Inj\text{-}R)$.

□

Remark 5.3.3. *Note that if R does not have finite injective dimension over itself, i.e. if it is not of finite self-injective dimension, then $\Delta_{\mathscr{T}}(Inj\text{-}R) \neq \Delta_{\mathscr{T}}(Proj\text{-}R)$, where $\mathscr{T} = \mathcal{D}^b(Mod\text{-}R)$, because R cannot be quasi-isomorphic to a bounded complex of injectives.*

We now move to derived unbounded categories for convenience. It was shown in Theorem 4.3 of [55] that if R is a finite dimensional algebra over a field, then $\mathcal{D}(Mod\text{-}R) = \langle Inj\text{-}R\rangle \implies \operatorname{fin.dim}(R) < \infty$, and it was noted in Section 8 of [55] that there is no known R that is a finite dimensional algebra such that $\mathcal{D}(Mod\text{-}R) \neq \langle Inj\text{-}R\rangle$. Now, although the group algebras of the classes of groups that we have been dealing with are seldom finite dimensional and we seldom work over fields, it is an interesting question as to how much we can state about the relation between $\mathcal{D}(Mod\text{-}A\Gamma) = \langle Inj\text{-}A\Gamma\rangle$ and $\operatorname{fin.dim}(A\Gamma) < \infty$ for some group Γ and some commutative ring A.

Lemma 5.3.4. *Let R be a ring such that* $\operatorname{silp}(R) < \infty$*. Then,*

a) $\mathcal{D}(Mod\text{-}R) = \langle Inj\text{-}R\rangle$.

b) $\operatorname{fin.dim}(R) < \infty$.

Proof. a) It follows from Proposition 5.3.2.a., that $\langle Proj\text{-}R\rangle \subseteq \langle Inj\text{-}R\rangle$. It is standard fact that $\langle Proj\text{-}R\rangle = \mathcal{D}(Mod\text{-}R)$ (see Proposition 2.2 of [55]), so we are done.

b) This follows directly from Lemma 4.4.5. Although in Lemma 4.4.5 we work with group rings, the same proof works for general rings. □

Note that if $(e) \implies (d)$ in Conjecture 5.1.10, then by Lemma 5.3.4, $\operatorname{fin.dim}(A\Gamma) < \infty \implies \mathcal{D}(Mod\text{-}A\Gamma) = \langle Inj\text{-}A\Gamma\rangle$, for any group Γ and any commutative ring A of finite global dimension. However, we can get the same result for groups in $LH\mathscr{F}_{\phi,A}$:

Proposition 5.3.5. *Let $\Gamma \in LH\mathscr{F}_{\phi,A}$, with A of finite global dimension. Then,* $\operatorname{fin.dim}(A\Gamma) < \infty \implies \mathcal{D}(Mod\text{-}A\Gamma) = \langle Inj\text{-}A\Gamma\rangle$.

Proof. This follows directly from Theorem 4.4.1 which gives us that if $\mathrm{fin.\,dim}(A\Gamma) < \infty$, then $\mathrm{silp}(A\Gamma) = \mathrm{fin.\,dim}(A\Gamma) < \infty$, and Lemma 5.3.4.a. □

Since Proposition 5.3.5 forces generation results in the derived unbounded category with just the finiteness of an invariant as the hypothesis, it is relevant to state the following interesting generation property admitted by derived unbounded categories modules over groups in Kropholler's hierarchy.

Theorem 5.3.6. *(part of Theorem 8.2.5) Let $\Gamma \in H_n\mathscr{F}$, for some integer n and let A be a commutative ring. Then, $\mathcal{D}(Mod\text{-}A\Gamma) = \langle I(\Gamma, \mathscr{F})\rangle$, where $I(\Gamma, \mathscr{F})$ is the class of all modules induced up from finite subgroups of Γ.*

From Lemma 5.1.7, it follows that Theorem 5.3.6 is true with for groups in $C_\infty(\mathscr{F})$ as well.

We end this section with the following couple of questions that can be easily seen to be related to Conjecture 5.1.10, Proposition 5.3.5 and all the results discussed in this section including Theorem 5.3.6.

Question 5.3.7. *a) If Γ is an $H\mathscr{F}$-group satisfying $\mathcal{D}(Mod\text{-}A\Gamma) = \langle Inj\text{-}A\Gamma\rangle$, for some commutative ring A, then is $\Gamma \in H_1\mathscr{F}$?*

b) It follows from Lemma 5.1.7, Remark 5.1.14, Lemma 5.3.4 and Theorem 5.3.6 that if Γ is an $H_1\mathscr{F}$-group, then $\langle I(\Gamma, \mathscr{F})\rangle = \langle Inj\text{-}A\Gamma\rangle = \mathcal{D}(Mod\text{-}A\Gamma)$, where A is of finite global dimension. Now, does $\langle I(\Gamma, \mathscr{F})\rangle = \langle Inj\text{-}A\Gamma\rangle$, for all A of finite global dimension, imply that $\Gamma \in H_1\mathscr{F}$?

Also, can we find a group Γ such that for some A, $\langle I(\Gamma, \mathscr{F})\rangle = \langle Inj\text{-}A\Gamma\rangle$ but $\langle I(\Gamma, \mathscr{F})\rangle, \langle Inj\text{-}A\Gamma\rangle \neq \mathcal{D}(Mod\text{-}A\Gamma)$? It follows from Theorem 5.3.6 that such a Γ cannot be in $H_n\mathscr{F}$ for any integer n; whether it can still be in $H_\alpha\mathscr{F}$ for some higher ordinal α is unclear.

Chapter 6

Gorenstein Projectives and Benson's Cofibrants

In this chapter, we will be principally investigating two classes of modules over any given group ring - the class of Gorenstein projectives and the class of Benson's cofibrants. From Definition 1.3.7, we recall that for an $A\Gamma$-module M, M is called "Benson's cofibrant" iff $M \otimes_A B(\Gamma, A)$ is projective as an $A\Gamma$-module. Using the word "cofibrant" without a the context of a model category might seem strange, so the following is a natural question to ask.

Question 6.0.1. *What is the motivation behind using the word "cofibrant" in Definition 1.3.7?*

Answer 6.0.2. *In [12], Benson put a closed model category structure on the module category of $A\Gamma$-modules, for any group Γ and any commutative ring A, and defined cofibrations in the closed model category with the class of modules that we are calling "Benson's cofibrants" as the cofibrant objects. He additionally showed (Theorem 10.10 of [12]) that the homotopy category of the closed model category, denoted $Ho.Mod(A\Gamma)$, is equivalent to the "stable module category" where the objects are the $A\Gamma$-modules and the arrows between modules M and N are given by $\widehat{Ext}^0_{A\Gamma}(M, N)$. One can repeat the same construction of the closed model category structure for type Φ groups with Benson's "stable module category" replaced by the stable module category for groups of type Φ constructed by Mazza and Symonds in [46] (we discuss this construction in Section 8.4.2) and check very easily that even with the Mazza-Symonds stable module* *category, we get it equivalent to the homotopy category of the closed model category on the module category.*

There is a conjecture connecting the two classes of Gorenstein projectives and Benson's cofibrants that we briefly mentioned in Chapter 4, which we restate here for convenience.

Conjecture 6.0.3. *(= Conjecture 4.2.7) For any commutative ring A of finite global dimension and any group Γ, the class of Benson's cofibrants coincides with the class of Gorenstein projectives.*

In this chapter, we first show that for any group Γ and any commutative ring A, both the class of Gorenstein projectives and Benson's cofibrants are good classes (see Definition 2.2.5). We then investigate the relations between the classes of modules generated by those two classes and show how we can derive information about the coincidence of these classes from the coincidence of the classes generated by them. And finally we prove some general results on the coincidence of Gorenstein projectives and Benson's cofibrants for some classes of groups.

6.1 Both classes are good classes

For the rest of this chapter, for a group Γ and a commutative ring A, we shall denote by $GProj(A\Gamma)$ the class of Gorenstein projective $A\Gamma$-modules and by $CoF(A\Gamma)$ the class of $A\Gamma$-modules M such that $M \otimes_A B(\Gamma, A)$ is projective. Our first result here is that $GProj(A\Gamma)$, for any group Γ and any commutative ring A, is a good class.

Lemma 6.1.1. *Let Γ be a group and let A be a commutative ring. Then, $GProj(A\Gamma)$ is a good class.*

Proof. This was mentioned in Remark 2.2.8 and it is easy to see - Lemma 1.1.10 shows us that $GProj(A\Gamma)$ satisfies Statement (a) of Proposition 3.3.2, so we are done. □

Before we show that $CoF(A\Gamma)$ is a good class for all Γ and A, note that since $B(\Gamma, A)$ is A-free, if $P_* \twoheadrightarrow M$ is an $A\Gamma$-projective resolution of an $A\Gamma$-module M, then $P_* \otimes_A B(\Gamma, A) \twoheadrightarrow M \otimes_A B(\Gamma, A)$ is a projective resolution is an $A\Gamma$-projective resolution of $M \otimes_A B(\Gamma, A)$.

Lemma 6.1.2. *For any group Γ and any commutative ring A, $[CoF(A\Gamma)]$ and the class of all $A\Gamma$-modules M satisfying $proj.dim_{A\Gamma} M \otimes_A B(\Gamma, A) < \infty$ coincide.*

Proof. Let $M \in [CoF(A\Gamma)]$. Then, there exists an exact sequence

$$0 \to C_n \to C_{n-1} \to ... \to C_1 \to C_0 \to M \to 0$$

where each C_i is a cofibrant module, i.e. $C_i \otimes_A B(\Gamma, A)$ is projectives for all i. Since $B(\Gamma, A)$ is A-free, we can tensor the exact sequence by $B(\Gamma, A)$ to get

$$0 \to C_n \otimes_A B(\Gamma, A) \to ... \to C_1 \otimes_A B(\Gamma, A) \to C_0 \otimes_A B(\Gamma, A) \to M \otimes_A B(\Gamma, A) \to 0$$

Now, as each term in the exact sequence, other than $M \otimes_A B(\Gamma, A)$, is projective, we can say that $proj.dim_{A\Gamma} M \otimes_A B(\Gamma, A) < \infty$. Thus, $[CoF(A\Gamma)]$ is a subclass of the class of all $A\Gamma$-modules M satisfying $proj.dim_{A\Gamma} M \otimes_A B(\Gamma, A) < \infty$.

Now let M be an $A\Gamma$-module satisfying $proj.dim_{A\Gamma} M \otimes_A B(\Gamma, A) < \infty$. Let $(P_i, d_i)_{i \geq 0}$ be a projective resolution of M, and, for any positive integer r, let $K_r(M)$ denote the r-th kernel in this resolution. Let $proj.dim_{A\Gamma} M \otimes_A B(\Gamma, A) = t$. Then, $K_t(M) \otimes_A B(\Gamma, A)$ is projective. By definition of $K_t(M)$, we have the following exact sequence

$$0 \to K_t(M) \to P_{t-1} \to P_{t-2} \to ... \to P_1 \to P_0 \to M \to 0$$

$P_i \otimes_A B(\Gamma, A)$ is projective for all i (as $B(\Gamma, A)$ is A-free), so each P_i is cofibrant. And, $K_t(M) \otimes_A B(\Gamma, A)$ is projective as well, so $K_t(M)$ is cofibrant. Thus, M is in $[CoF(A\Gamma)]$. Therefore, the class of all $A\Gamma$-modules M satisfying $proj.dim_{A\Gamma} M \otimes_A B(\Gamma, A) < \infty$ is a subclass of $[CoF(A\Gamma)]$. □

Lemma 6.1.3. *For any group Γ and any commutative ring A, $CoF(A\Gamma)$ is a good class. In particular, we claim $[[CoF(A\Gamma)]] = [CoF(A\Gamma)]$.*

Proof. It is obvious that $[CoF(A\Gamma)]$ is a subclass of $[[CoF(A\Gamma)]]$.

Now let M be a module in $[[CoF(A\Gamma)]]$. There exists an exact sequence

$$0 \to C_n \to C_{n-1} \to ... \to C_1 \to C_0 \to M \to 0$$

where each C_i is in $[CoF(A\Gamma)]$. We can tensor the above exact sequence by $B(\Gamma, A)$, which is A-free, to get the following exact sequence.

$$0 \to C_n \otimes_A B(\Gamma, A) \to C_{n-1} \otimes_A B(\Gamma, A) \to ... \to C_0 \otimes_A B(\Gamma, A) \to M \otimes_A B(\Gamma, A) \to 0$$

By Lemma 6.1.2, $proj.dim_{A\Gamma}C_i \otimes_A B(\Gamma, A) < \infty$ for $i = 0, 1, .., n$.

Therefore, $proj.dim_{A\Gamma}M \otimes_A B(\Gamma, A) < \infty$, and by Lemma 6.1.2 again, M is in $[CoF(A\Gamma)]$. Thus, the classes $[[CoF(A\Gamma)]]$ and $[CoF(A\Gamma)]$ coincinde. Therefore, $CoF(A\Gamma)$ is a good class by Proposition 3.3.2. □

6.2 Relations between the classes generated by Gorenstein projectives and the cofibrants

We start this section with a technical result, a loose version of which is related to the previously recorded Lemma 4.2.8.

Proposition 6.2.1. *Let Γ be a group and A be a commutative ring. Let $\mathscr{T}$ be a class of $A\Gamma$-modules satisfying the following conditions:*

a) $\mathscr{T}$ is closed under summands and under tensoring with A-free modules.

b) For any $A\Gamma$-module M, there exists a surjective map of $A\Gamma$-modules $\phi : T_M \to M$ for some $T_M \in \mathscr{T}$.

c) For any $A\Gamma$-module M, if $\mathscr{T}$-$dim(M) \leq n$, then in any exact sequence $0 \to K_n \to T_{n-1} \to T_{n-2} \to ... \to T_1 \to T_0 \to M \to 0$ where each $T_i \in \mathscr{T}$, $K_n \in \mathscr{T}$.

For any A-free $A\Gamma$-module F, let $\mathscr{X}_{F,\mathscr{T}} := \{M \in Mod\text{-}A\Gamma : M \otimes_A F \in \mathscr{T}\}$, and let $\mathscr{S}$ denote the class of all $A\Gamma$-modules that occur as kernels of doubly infinite exact sequences of modules in $\mathscr{T}$. Then, $\mathscr{S} \cap [\mathscr{X}_{F,\mathscr{T}}] = \mathscr{S} \cap \mathscr{X}_{F,\mathscr{T}}$ if $F\mathscr{T}D(A\Gamma) < \infty$. Here, $F\mathscr{T}D(A\Gamma) := sup\{\mathscr{T}\text{-}dim(M) : M \in Mod\text{-}A\Gamma$ satisfying $\mathscr{T}\text{-}dim(M) < \infty\}$.

Proof. First we prove that $[\mathscr{X}_{F,\mathscr{T}}] = \{M \in Mod\text{-}A\Gamma : \mathscr{T}\text{-}dim(M \otimes_A F) < \infty\}$. To show this, we start with an $A\Gamma$-module M satisfying $\mathscr{T}\text{-}dim(M \otimes_A F) = r < \infty$. We know from condition (b) of our hypothesis, that there exists a surjective $A\Gamma$-linear map $\phi_0 : T_0 \to M$ for some $T_0 \in \mathscr{T}$. Similarly, there exists an $A\Gamma$-surjective map $\phi_1 : T_1 \to \text{Ker}(\phi_0)$. Going on like this, we get an exact sequence $0 \to \text{Ker}(\phi_{r-1}) \hookrightarrow T_{r-1} \to T_{r-2} \to ... \to T_1 \to T_0 \to M \to 0$, where each $T_i \in \mathscr{T}$. When we tensor this exact sequence by F which is A-free, we get an exact sequence $0 \to \text{Ker}(\phi_{r-1}) \otimes_A F \to T_{r-1} \otimes_A F \to ... \to T_0 \otimes_A F \to M \otimes_A F \to 0$, where each $T_i \otimes_A F \in \mathscr{T}$ as $\mathscr{T}$ is closed under tensoring with A-free modules by condition (a) of our hypothesis. Now, as $\mathscr{T}\text{-}dim(M \otimes_A F) = r$, by condition (c) of our hypothesis, $\text{Ker}(\phi_{r-1}) \otimes_A F \in \mathscr{T}$, i.e. $\operatorname{Ker}(\phi_{r-1}) \in \mathscr{X}_{F,\mathscr{T}}$. Thus, $M \in [\mathscr{X}_{F,\mathscr{T}}]$. Now if we take $M \in [\mathscr{X}_{F,\mathscr{T}}]$, then we have an exact sequence $0 \to X_n \to ... \to X_1 \to X_0 \to M \to 0$ for some n and some $X_0, ..., X_n$ where each $X_i \in \mathscr{X}_{F,\mathscr{T}}$. Tensoring this sequence by the A-free F, we get a finite length resolution of $M \otimes_A F$ of modules in $\mathscr{T}$ as each $X_i \otimes_A F \in \mathscr{T}$. Thus, $\mathscr{T}\text{-}dim(M \otimes_A F) < \infty$.

We shall now prove that if we have a short exact sequence $0 \to L \hookrightarrow T \twoheadrightarrow M \to 0$ where $T \in \mathscr{T}$ and $\mathscr{T}\text{-}dim(L) = r > 0$, then $\mathscr{T}\text{-}dim(M) = r + 1$. To see this, we start with a minimal length resolution of modules in $\mathscr{T}$ admitted by $L : 0 \to T_r \to ... \to T_0 \to L \to 0$, merging which with our short exact sequence we get $0 \to T_r \to ... \to T_1 \to T_0 \to T \to M \to 0$ which implies that $\mathscr{T}\text{-}dim(M) \leq r + 1$. Now if $\mathscr{T}\text{-}dim(M) < r + 1$, then one of the maps in the above exact sequence splits, i.e. L is either a summand of T in which case $L \in \mathscr{T}$ as $\mathscr{T}$ is closed under summands by condition (a) of our hypothesis but this cannot happen as $\mathscr{T}\text{-}dim(L) > 0$, or $0 < \mathscr{T}\text{-}dim(L) < r$ and this cannot happen as well. So, $\mathscr{T}\text{-}dim(M) = r + 1$.

We are now in a position to complete our proof of the proposition. Note that since $\mathscr{X}_{F,\mathscr{T}}$ is a subclass of $[\mathscr{X}_{F,\mathscr{T}}]$, we have $\mathscr{S} \cap \mathscr{X}_{F,\mathscr{T}}$ to be a subclass of $\mathscr{S} \cap [\mathscr{X}_{F,\mathscr{T}}]$. To prove the other direction, we start with an $A\Gamma$-module $M_0 \in \mathscr{S} \cap [\mathscr{X}_{F,\mathscr{T}}]$. From our first paragraph, it follows that $\mathscr{T}\text{-}dim(M_0 \otimes_A F) < \infty$. We need to show $M_0 \otimes_A F \in \mathscr{T}$, so we start with the assumption that it is not the case and that $\mathscr{T}\text{-}dim(M_0 \otimes_A F) = r > 0$. By definition of $\mathscr{S}$, there exists an exact sequence $... \to T_1 \to T_0 \to T_{-1} \to T_{-2} \to ...$ where each $T_i \in \mathscr{T}$ and $M_0 = \operatorname{Ker}(T_0 \to T_{-1})$. For $i \neq 0$, let $M_i := \operatorname{Ker}(T_i \to T_{i-1})$. We can tensor the short exact sequence $0 \to M_0 \hookrightarrow T_0 \twoheadrightarrow M_{-1} \to 0$ by F, which is A-free, to get the short exact sequence $0 \to M_0 \otimes_A F \hookrightarrow T_0 \otimes_A F \twoheadrightarrow M_{-1} \otimes_A F \to 0$ where $T_0 \otimes_A F \in \mathscr{T}$ since $\mathscr{T}$ is closed under tensoring with A-free modules. By the last paragraph, we can say that $\mathscr{T}\text{-}dim(M_{-1} \otimes_A F) = r + 1$. Similarly, we get that $\mathscr{T}\text{-}dim(M_{-F\mathscr{T}D(A\Gamma)} \otimes_A F) = r + F\mathscr{T}D(A\Gamma) > F\mathscr{T}D(A\Gamma)$ which is not possible. So, $M_0 \otimes_A F \in \mathscr{T}$, i.e. $M_0 \in \mathscr{X}_{F,\mathscr{T}}$. Thus, $\mathscr{S} \cap [\mathscr{X}_{F,\mathscr{T}}]$ is a subclass of $\mathscr{S} \cap \mathscr{X}_{F,\mathscr{T}}$, and we are done. $\square$

The proposition above is a general result. We can use it to show that when when $\operatorname{fin.dim}(A\Gamma) < \infty$, the class generated by Benson's cofibrants coincides with the class generated by Gorenstein projectives if and only if the class of Benson's cofibrants coincides with the class of Gorenstein projectives. To see why that is true, it is only appropriate to recall Theorem 4.2.3.

Theorem 6.2.2. *(= Theorem 4.2.3) For any group Γ and any commutative ring A, any $A\Gamma$-module M such that $proj.dim_{A\Gamma} M \otimes_A B(A,\Gamma) < \infty$ admits a complete resolution. So, $CoF(A\Gamma) \subseteq GProj(A\Gamma)$, and $\langle CoF(A\Gamma)\rangle \subseteq \langle GProj(A\Gamma)\rangle$.*

Corollary 6.2.3. *Let Γ be a group and let A be a commutative ring of finite global dimension such that $\operatorname{fin.dim}(A\Gamma) < \infty$. Then, the following are equivalent:*

a) $\langle GProj(A\Gamma)\rangle = \langle CoF(A\Gamma)\rangle$.

b) $GProj(A\Gamma) = CoF(A\Gamma)$.

Proof. This follows directly from Theorem 6.2.2 and from Proposition 6.2.1 taking $\mathscr{T}$ to be the class of projectives, F to be $B(\Gamma, A)$ and $\mathscr{S}$ to be the class of Gorenstein projectives. □

We can actually determine whether the class of modules generated by $CoF(A\Gamma)$ contains every module just by looking at the projective dimension of $B(\Gamma, A)$ as long as the ring A is of finite global dimension. Similarly, we can determine whether the class of modules generated by $GProj(A\Gamma)$ contains all modules just by checking whether the trivial module is generated by it. It follows from Theorem 6.2.2 that if the class of modules generated by the cofibrants includes everything, then the same is true of the class of modules generated by the Gorenstein projectives. The reverse is true if one of our earlier conjectures is assumed to be true:

Proposition 6.2.4. *For any group Γ and a commutative ring A which has finite global dimension, the following implications hold with the following statements: $(a) \iff (b) \implies (c)$, and Conjecture 4.2.1 $\implies ((c) \implies (b))$.*

a) $\langle CoF(A\Gamma)\rangle$ contains all $A\Gamma$-modules.

b) $proj.dim_{A\Gamma} B(\Gamma, A) < \infty$.

c) $\langle GProj(A\Gamma)\rangle$ contains all $A\Gamma$-modules.

Proof. $(a) \implies (b)$: If $\langle CoF(A\Gamma)\rangle$ contains all $A\Gamma$-modules, then it contains the trivial module, and since $CoF(A\Gamma)$ is a good class, we have $\langle CoF(A\Gamma)\rangle = [CoF(A\Gamma)] = \{M \in Mod\text{-}A\Gamma : proj.dim_{A\Gamma} M \otimes_A B(\Gamma, A) < \infty\}$ (by Lemma 6.1.2 and Lemma 6.1.3), and so $A \otimes_A B(\Gamma, A) = B(\Gamma, A)$ must have finite projective dimension.

$(b) \implies (a)$: Let t be the global dimension of A. Let M be an $A\Gamma$-module. Let $(P_i, d_i)_{i\geq 0}$ be a projective resolution admitted by M (denote by $K_r(M)$ the r-th kernel in this projective resolution, for any positive integer r). As $B(\Gamma, A)$ is A-free, $(P_i \otimes_A B(\Gamma, A), d_i \otimes id)_{i\geq 0}$ is a projective resolutiuon admitted by $M \otimes_A B(\Gamma, A)$ and $K_r(M) \otimes_A B(\Gamma, A)$ the r-th kernel in this projective resolution, for any positive integer r. $K_t(M)$ is A-projective for any $A\Gamma$-module M as t is the global dimension of A. As $proj.dim_{A\Gamma} B(\Gamma, A) < \infty$, we have $proj.dim_{A\Gamma} K_t(M) \otimes_A B(\Gamma, A) < \infty$ and since $K_t(M) \otimes_A B(\Gamma, A)$ is the t-th kernel in the projective resolution $P_* \otimes_A B(\Gamma, A) \twoheadrightarrow M \otimes_A B(\Gamma, A)$, we have $proj.dim_{A\Gamma} M \otimes_A B(\Gamma, A) < \infty$. So, M is in $[CoF(A\Gamma)]$, by Lemma 6.1.2. Thus, $[CoF(A\Gamma)]$ contains all $A\Gamma$-modules, and, as $CoF(A\Gamma)$ is a good class (by Lemma 6.1.3), $\langle CoF(A\Gamma)\rangle$ contains all $A\Gamma$-modules.

$(a) \implies (c)$: obvious from Theorem 6.2.2.

Conjecture 4.2.1 $\implies$ $((c) \implies (b))$: note that if $\langle GProj(A\Gamma)\rangle$ is the class of all $A\Gamma$-modules, then $Gpd_{A\Gamma}(A) = Gcd_A(\Gamma) < \infty$, and therefore by Conjecture 4.2.1, $proj.dim_{A\Gamma} B(\Gamma, A) < \infty$.

□

We showed in Theorem 4.2.9 that if for a group Γ and a commutative ring A of finite global dimension, Conjecture 6.0.3 holds true if Conjecture 4.2.1 holds true, and conjectured that we expect the converse to be true as well. Here, we show that Conjecture 4.2.1 is equivalent to a slightly weaker version of Conjecture 6.0.3 closely related to the theme of Proposition 6.2.4:

Lemma 6.2.5. *For any group Γ and any commutative ring A of finite global dimension, the following are equivalent:*

a) $\langle GProj(A\Gamma)\rangle$ contains all $A\Gamma$-modules iff $\langle CoF(A\Gamma)\rangle$ contains all $A\Gamma$-modules.

b) $Gcd_A(\Gamma) = proj.dim_{A\Gamma} B(\Gamma, A)$.

Proof. $(a) \implies (b)$: It follows from Remark 4.3.2 that for (b) to not hold true, $Gcd_A(\Gamma)$ has to be finite and $proj.dim_{A\Gamma}$ has to be infinite. But, $Gcd_A(\Gamma) < \infty \implies Gpd_{A\Gamma}(M) < \infty$ for all $A\Gamma$-modules M (see Theorem 4.1.3) $\implies [GProj(A\Gamma)] = \langle GProj(A\Gamma)\rangle$ (note that $GProj(A\Gamma)$ is a good class) contains all $A\Gamma$-modules, and therefore by (a), $\langle CoF(A\Gamma)\rangle$ contains all $A\Gamma$-modules as well, and this implies that $proj.dim_{A\Gamma} B(\Gamma, A) < \infty$ by Proposition 6.2.4.

$(b) \implies (a)$: This part is covered by Proposition 6.2.4.

□

It is clear that if Conjecture 6.0.3 is satisfied for Γ and A, then (a) of Lemma 6.2.5 will also be satisfied. The following corollary of Lemma 6.2.5 gives a concrete example of a module that would lie beyond the class generated by Benson's cofibrants should (a) of Lemma 6.2.5 fail to hold.

Corollary 6.2.6. *Let Γ be a group and A a commutative ring of finite global dimension such that only one of $\langle GProj(A\Gamma)\rangle$ and $\langle CoF(A\Gamma)\rangle$ contains all $A\Gamma$-modules, then the trivial module $A \in \langle GProj(A\Gamma)\rangle \setminus \langle CoF(A\Gamma)\rangle$.*

Proof. Obvious from Proposition 6.2.4 and Lemma 6.2.5.

□

The following too now follows directly from Proposition 6.2.4 and our earlier technical result Proposition 6.2.1.

Theorem 6.2.7. *Let Γ be a group and let A be a commutative ring of finite global dimension. Then, if $proj.dim_{A\Gamma} B(\Gamma, A) < \infty$, then $CoF(A\Gamma) = GProj(A\Gamma)$.*

Proof. We have covered a similar argument before. We start by noting that it follows from Theorem 4.2.6 that $Gcd_A(\Gamma) < \infty$. By Lemma 4.1.5 and Lemma 4.4.5, $\operatorname{fin.dim}(A\Gamma) < \infty$.

It follows from Proposition 6.2.4 that $\langle GProj(A\Gamma)\rangle = \langle CoF(A\Gamma)\rangle = Mod\text{-}A\Gamma$, and as $GProj(A\Gamma)$ and $CoF(A\Gamma)$ are both good classes, we therefore have $[GProj(A\Gamma)] = [CoF(A\Gamma)] = Mod\text{-}A\Gamma$. We are now done due to Corollary 6.2.3.

□

If Γ is of type Φ over A, $proj.dim_{A\Gamma} B(\Gamma, A) < \infty$ by the definition of type Φ (see Lemma 1.3.7), so Theorem 6.2.7 implies that groups of type Φ over any ring of finite global dimension satisfy Conjecture 6.0.3. We provide a simpler proof below (simpler in the sense that we do not need to work with cohomological invariants).

Theorem 6.2.8. *Let Γ be a group of type Φ over a ring A of finite global dimension. Then, $CoF(A\Gamma) = GProj(A\Gamma)$.*

Proof. We only need to show $GProj(A\Gamma) \subseteq CoF(A\Gamma)$ in light of Theorem 6.2.2. Now, if $M \in GProj(A\Gamma)$, then $M \otimes_A B(\Gamma, A)$, since $B(\Gamma, A)$ is A-free, occurs as a kernel in a doubly infinite acyclic complex of projectives. In [46], it was shown that acyclic complex of projectives for type Φ groups are totally acyclic, so $M \otimes_A B(\Gamma, A)$ is Gorenstein projective. Now, M is A-projective by Lemma 1.1.3 and over any finite subgroup G of Γ, $B(\Gamma, A)$ is AG-free by Lemma 1.3.7, so $M \otimes_A B(\Gamma, A)$ is AG-projective. As Γ is of type Φ, this means $M \otimes_A B(\Gamma, A)$ has finite projective dimension as an $A\Gamma$-module. Now, since $M \otimes_A B(\Gamma, A)$ is Gorenstien projective, it follows from Theorem 1.1.13 that $M \otimes_A B(\Gamma, A)$ is projective. Therefore, $M \in CoF(A\Gamma)$. $\square$

Remark 6.2.9. *It is actually quite easy to prove that groups of type Φ satisfy Conjecture 6.0.3. We provided two different proofs in this section, and a stronger result is proved later in Remark 6.4.5. Theorem 6.2.7 proves Conjecture 6.0.3 for a larger class of groups than the class of groups of type Φ if and only if* (d) *need not imply* (a) *in Conjecture 4.1.12.*

6.3 Two other questions on Gorenstein projectivity

We now look at some conjectured properties of Gorenstein projectivity and how those conjectures relate to Conjecture 6.0.3. We start by considering the class of modules called *weak Gorenstein projective* that was defined in Definition 1.1.1. For any group Γ and any commutative ring A, we denote the class of all weak Gorenstein projective $A\Gamma$-modules by $WGProj(A\Gamma)$.

Lemma 6.3.1. *Let A be a commutative ring of finite global dimension and let Γ be a group. Then, weak Gorenstein projectives over $A\Gamma$ are A-projective. Also, $WGProj(A\Gamma)$ is closed under direct sums.*

Proof. The first part is obvious from Lemma 1.1.3. The proof for weak Gorenstein projectives being closed under direct sums is obvious as direct sums of weak complete resolutions is still a weak complete resolution. $\square$

The following conjecture was made in by Dembegioti and Talelli over the ring of integers, we make it here over rings of finite global dimension.

Conjecture 6.3.2. *(made in [25] over $\mathbb{Z}$) For any group Γ and a commutative ring A of finite global dimension, an $A\Gamma$-module M admits a complete resolution iff it admits a weak complete resolution.*

The following conjecture regarding the class of weak Gorenstein projectives is equivalent to Conjecture 6.3.2.

Conjecture 6.3.3. *(made over $\mathbb{Z}$ in [25]) For any group Γ and a commutative ring A of finite global dimension, the class of weak Gorenstein projective $A\Gamma$-modules coincides with the class of Gorenstein projective $A\Gamma$-modules.*

Before going further in this section, we wish to state another question of Gorenstein projectivity which will be important in our study of groups that satisfy a key property related to Conjecture 6.0.3 locally in the next section.

Conjecture 6.3.4. *(see [25]) For any group Γ and any commutative ring A of finite global dimension, if M is Gorenstein projective as an $A\Gamma$-module then it is also Gorenstein projective as an $A\Gamma'$-module for any subgroup Γ' of Γ, i.e. Gorenstein projectivity is closed upon restriction to subgroups.*

The following is an immediate observation.

Lemma 6.3.5. *If Conjecture 6.3.3 is satisfied for a subgroup-closed class of groups $\mathscr{X}$ and a commutative ring A of finite global dimension, then $\mathscr{X}$-groups satisfy Conjecture 6.3.4 over A.*

Proof. Take $\Gamma \in \mathscr{X}$. Let M be a Gorenstein projective $A\Gamma$-module that occurs as a kernel in the complete resolution $(F_i, d_i)_{i\in\mathbb{Z}}$ of $A\Gamma$-projective modules. Upon restriction to a subgroup Γ' of Γ, M occurs as a kernel in the same weak complete resolution upon restriction to Γ' (it is still a weak complete resolution because projectivity is closed under restriction to subgroups). This means, upon restriction to Γ', M is weak Gorenstein projective, but since Conjecture 6.3.3 is satisfied for Γ' and A (this is because $\mathscr{X}$ is subgroup-closed), we have that M is Gorenstein projective upon restriction to Γ'. □

If it is known for some group Γ and some commutative ring A that the class of weak Gorenstein projective $A\Gamma$-modules coincides with the class of Benson's cofibrants, then Conjecture 6.3.2, in fact a slighty stronger version of it, follows.

Lemma 6.3.6. *If $WGProj(A\Gamma) = CoF(A\Gamma)$, then, over $A\Gamma$, every weak complete resolution is a complete resolution, i.e. Conjecture 6.3.2 holds for $A\Gamma$.*

Proof. This follows from Corollary D of [25] where they have dealt with the exactly similar situation in different language over $\mathbb{Z}$. The same proof works for any commutative ring A. $\square$

We now note the following useful result involving $B(\Gamma, A)$ that helps us to show why the statement of Conjecture 6.0.3 over a commutative ring A implies Conjecture 6.3.4.

Lemma 6.3.7. *For any group Γ and any commutative ring A, $B(\Gamma', A)$ is a summand of $\mathrm{Res}^{\Gamma}_{\Gamma'}(B(\Gamma, A))$, where Γ' is any subgroup of Γ.*

Proof. This result has been proved in [53] for $A = \mathbb{Z}$. It follows over any commutative ring A because $B(\Gamma, A) = B(\Gamma, \mathbb{Z}) \otimes_{\mathbb{Z}} A$ for any group Γ (see the proof of Lemma 3.4 of [11]). $\square$

The following lemma and the subsequent corollary are now trivial.

Lemma 6.3.8. *Let Γ be a group and A a commutative ring. Then, if M is cofibrant as an $A\Gamma$-module, M is cofibrant as an $A\Gamma'$-module for any subgroup $\Gamma' \leq \Gamma$ as well.*

Proof. Let M be cofibrant as an $A\Gamma$-module. So, $M \otimes_A B(\Gamma, A)$ is projective as an $A\Gamma$-module. This implies that $\mathrm{Res}^{\Gamma}_{\Gamma'}(M \otimes_A B(\Gamma, A))$ is projective as an $A\Gamma'$-module for any subgroup Γ' of Γ. By Lemma 6.3.7, $B(\Gamma', A)$ is a direct summand of $\mathrm{Res}^{\Gamma}_{\Gamma'}(B(\Gamma, A))$, and so $\mathrm{Res}^{\Gamma}_{\Gamma'}(M \otimes_A B(\Gamma', A))$ is a summand of $\mathrm{Res}^{\Gamma}_{\Gamma'}(M \otimes_A B(\Gamma, A))$ which is projective. So, $\mathrm{Res}^{\Gamma}_{\Gamma'} M$ is cofibrant as an $A\Gamma'$-module. $\square$

Corollary 6.3.9. *Let A be a commutative ring. If a group Γ satisfies Conjecture 6.0.3, then Conjecture 6.3.4 holds for Γ as well.*

Proof. When Conjecture 6.0.3 is satisfied for Γ, the class of Gorenstein projective $A\Gamma$-modules coincides with the class of $A\Gamma$-modules M such that $M \otimes_A B(\Gamma, A)$ is projective. Thus, by Lemma 6.3.8, our result follows. $\square$

6.4 More on groups satisfying Conjecture 6.0.3

Before going forward, we need to prove a technical result which is a close variant of Theorem B of [25]. Before we state and prove this result, we need to state two lemmas that will be crucial in its proof and in the proofs of some other results in this section.

Lemma 6.4.1. *(Lemma 5.6 of [11]) Let A be a commutative ring and let $\Gamma = \bigcup_{\alpha<\gamma} \Gamma_\alpha$ where $\{\Gamma_\alpha\}_{\alpha<\gamma}$ is an ascending chain of subgroups of Γ for some ordinal γ. Then for any $A\Gamma$-module M that is projective over each Γ_α, $proj.dim_{A\Gamma} M \leq 1$.*

Lemma 6.4.2. *(Lemma 2.5 of [36]) Every countable group admits an action on a tree with finitely generated vertex and edge stabilisers.*

Proposition 6.4.3. *(same proof done over $\mathbb{Z}$ and $\mathscr{F}$ and $B(\Gamma, A)$ in place of $B_{\Gamma,\mathscr{X}}$ in Theorem A of [25]) Let Γ be a group and let A be a commutative ring of finite global dimension. Let $\mathscr{X}$ be a class of groups and let $B_{\Gamma,\mathscr{X}}$ be an A-free $A\Gamma$-module that restricts to a projective module on every subgroup of Γ that is in $\mathscr{X}$. Now if M is a weak Gorenstein projective $A\Gamma$-module, then $M \otimes_A B_{\Gamma,\mathscr{X}}$ is projective over every subgroup of Γ that is in $LH\mathscr{X}$.*

Proof. We first show the result is true over all $H\mathscr{X}$-subgroups of Γ, i.e. over all $H_\alpha\mathscr{X}$-subgroups of Γ where α is some ordinal. We shall proceed by transfinite induction on α. When $\alpha = 0$, then $B_{\Gamma,\mathscr{X}}$ is projective over all $H_0\mathscr{X}$ subgroups of Γ by definition and since M is A-projective by Lemma 6.3.1, we have that $M \otimes_A B_{\Gamma,\mathscr{X}}$ is projective over all $H_0\mathscr{X}$ subgroups. Now assume that the result is true over $H_\beta\mathscr{X}$-subgroups of Γ for all $\beta < \alpha$ - this is our induction hypothesis, now if Γ' is a subgroup of Γ in $H_\alpha\mathscr{X}$, then the trivial module over Γ' admits a finite length resolution by direct sums of permutation modules with stabilisers in $H_\beta\mathscr{X}$ for some $\beta < \alpha$. We can tensor this resolution by $M \otimes_A B_{\Gamma,\mathscr{X}}$ and since by the induction hypothesis, $M \otimes_A B_{\Gamma,\mathscr{X}}$ is projective over all $H_\beta\mathscr{X}$-subgroups of Γ for any $\beta < \alpha$, what we get after tensoring is a finite length resolution of $M \otimes_A B_{\Gamma,\mathscr{X}}$, as an $A\Gamma'$-module, by modules that are projective over $A\Gamma'$. So, $proj.dim_{A\Gamma'} M \otimes_A B_{\Gamma,\mathscr{X}} < \infty$. Now note that this is true for all weak Gorenstein projectives M. So if $\sup\{proj.dimm_{A\Gamma'} M \otimes_A B_{\Gamma,\mathscr{X}} : M$ is weak Gorenstein projective over $A\Gamma\}$ is not finite, then for any n, there exists a a weak Gorenstein projective $A\Gamma$-module M_n such that $proj.dim_{A\Gamma'} M \otimes_A B_{\Gamma,\mathscr{X}} \geq n$. Then, since the class of weak

Gorenstein projectives is closed under direct sums, $proj.dim_{A\Gamma'} \bigoplus_{n\in\mathbb{N}} M_n \otimes_A B_{\Gamma,\mathscr{X}} \geq proj.dim_{A\Gamma'} M_n \otimes_A B_{\Gamma,\mathscr{X}} \geq n$, for all n, which is not possible. Now since $B_{\Gamma,\mathscr{X}}$ is A-free, if M is a kernel in a weak complete resolution $(F_i, d_i)_{i\in\mathbb{Z}}$ of $A\Gamma'$-modules (note that weak Gorenstein projectivity is closed under restriction to subgroups) then $M \otimes_A B_{\Gamma,\mathscr{X}}$ occurs as a kernel in the weak complete resolution $(F_i \otimes_A B_{\Gamma,\mathscr{X}}, d_i \otimes_A id)_{i\in\mathbb{Z}}$ and all the kernels in this weak complete resolution have finite projective dimension over $A\Gamma'$ as we have just showed, so by Proposition 6.2.1, $M \otimes_A B_{\Gamma,\mathscr{X}}$ is projective over Γ'.

Now we show the statement of the proposition is true for $LH\mathscr{X}$-subgroups of Γ. Let Γ' be an $LH\mathscr{X}$ subgroup of Γ. Then, if Γ' is countable, then by Lemma 6.4.2, Γ' is in $H\mathscr{F}$. Let Γ' be an uncountable $LH\mathscr{X}$-subgroup of Γ. Assume as the induction hypothesis that the statement of the proposition holds true for all $LH\mathscr{X}$-subgroups of Γ of cardinality strictly smaller than Γ'. Now as Γ' is uncountable, it can be written as $\bigcup_{\gamma<\alpha} \Gamma'_\gamma$ where $\{\Gamma'_\gamma\}_{\gamma<\alpha}$ is a strictly ascending chain of subgroups of Γ' and α is some ordinal, where each Γ'_γ is strictly smaller than Γ' in cardinality. By our induction hypothesis, $M \otimes_A B_{\Gamma,\mathscr{X}}$ is projective over $A\Gamma'_\gamma$ for each $\gamma < \alpha$, therefore by Lemma 6.4.1, $proj.dim_{A\Gamma'}(M \otimes_A B_{\Gamma,\mathscr{X}}) \leq 1$. Again, if M, as a weak Gorenstein projective $A\Gamma'$-module, occurs as the r-th kernel in a complete resolution $(F_i, d_i)_{i\in\mathbb{Z}}$ and if the $(r-1)$-th kernel, which is also a weak Gorenstein projective $A\Gamma'$-module, is denoted by $K_{-1}(M)$, then we have a short exact sequence $0 \to M \otimes_A B_{\Gamma,\mathscr{X}} \hookrightarrow F_{r-1} \otimes_A B_{\Gamma,\mathscr{X}} \twoheadrightarrow K_{-1}(M) \otimes_A B_{\Gamma,\mathscr{X}} \to 0$. So, if $proj.dim_{A\Gamma'} M \otimes_A B_{\Gamma,\mathscr{X}} = 1$, then $proj.dim_{A\Gamma'} K_{-1}(M) \otimes_A B_{\Gamma,\mathscr{X}} = 2$ which is not possible because since $K_{-1}(M)$ is weak Gorenstein projective, by Lemma 6.4.1, $proj.dim_{A\Gamma'} K_{-1}(M) \otimes_A B_{\Gamma,\mathscr{X}} \leq 1$. Thus, $proj.dim_{A\Gamma'} M \otimes_A B_{\Gamma,\mathscr{X}} = 0$.

□

Noting that, for any commutative ring A of finite global dimension t and any finite group G, if $proj.dim_{AG} M < \infty$ then $proj.dim_{AG} M \leq t$, we get the following corollary.

Corollary 6.4.4. *In the statement of Proposition 6.4.3, if $\mathscr{X} = \mathscr{F}$, we can replace the hypothesis that $B_{\Gamma,\mathscr{X}}$ is projective over $\mathscr{X}$-subgroups of Γ with the hypothesis that $B_{\Gamma,\mathscr{X}}$ has finite projective dimension over $\mathscr{X}$-subgroups of Γ, and change the conclusion "$M \otimes_A B_{\Gamma,\mathscr{X}}$ is projective over $LH\mathscr{X}$-subgroups of Γ" to "$M \otimes_A B_{\Gamma,\mathscr{X}}$ has finite projective dimension over $LH\mathscr{X}$-subgroups of Γ".*

Proposition 6.4.3 gives us a third proof of type Φ groups satisfying Conjecture 6.0.3.

The central theme of the proof of Proposition 6.4.3 can be simplified a little bit - see Corollary 6.4.6 below. Before stating this simplification, we do a small technical tweak on the definition of type Φ groups.

When we defined groups of type Φ, the motivation was to study the projectivity of modules by studying the projectivity with respect to finite subgroups. Although it is usually not much convenient, but we can replace the class of finite subgroups in the definition of type Φ and have in its place subgroups belonging to some other arbitrary class.

Definition 6.4.5. *A group Γ is said to be of type Φ-$\mathscr{X}$ over a commutative ring A if the following conditions are equivalent for all $A\Gamma$-modules M.*

a) $proj.dim_{A\Gamma} M < \infty$.

b) $proj.dim_{A\Gamma'} M < \infty$ for all subgroups $\Gamma' \leq \Gamma$ such that $\Gamma' \in \mathscr{X}$.

Corollary 6.4.6. *Let Γ be a group and let A be a commutative ring. Let $\mathscr{X}$ be a class of groups. Assume that for any weak Gorenstein projective $A\Gamma$-module M, $M \otimes_A B(\Gamma, A)$is projective over every $\mathscr{X}$-subgroup of Γ. Then, $M \otimes_A B(\Gamma, A)$ is projective every subgroup of Γ in Φ-$\mathscr{X}$ or $H\mathscr{X}$ or $L\mathscr{X}$.*

Proof. Let Γ' be an $L\mathscr{X}$-subgroup of Γ. Let $(\Gamma'_\gamma)_{\gamma \in \Gamma}$ be the family of all finitely generated subgroups of H. Now, Γ' can be viewed as the direct limit of the Γ'_γ's and any $A\Gamma'$-module N can be viewed as the direct limit of the induced modules $\mathrm{Ind}_{\Gamma'_\gamma}^{\Gamma'}(N)$. So $M \otimes_A B(\Gamma, A) = \varinjlim_\gamma \mathrm{Ind}_{\Gamma'_\gamma}^{\Gamma'}(M \otimes_A B(\Gamma, A))$ (see the proof of 3.2 in [41]). As Γ' is in $L\mathscr{X}$, each $\Gamma'_\gamma \in \mathscr{X}$, and so by the hypothesis of the theorem, $M \otimes_A B(\Gamma, A)$ is projective as an $A\Gamma'_\gamma$ module for all $\gamma \in \Gamma$, which implies that $M \otimes_A B(\Gamma, A)$ is projective over $A\Gamma'$.

The proof for $H\mathscr{X}$-subgroups has already been covered in the first part of the proof of Proposition 6.4.3. For Φ-$\mathscr{X}$-subgroups, note that if Γ' is a Φ-$\mathscr{X}$-subgroup of Γ, then as $M \otimes_A B(\Gamma, A)$ is projective over all $\mathscr{X}$-subgroups, it has finite projective dimension over Γ' by Definition 6.4.5, it now follows directly from the proof of Proposition 6.4.3 that this projective dimension cannot be non-zero.

□

Remark 6.4.7. *Taking $\mathscr{X} = \mathscr{F}$ in Corollary 6.4.6, we see that $WGProj(A\Gamma) = CoF(A\Gamma)$ (*) (this implies that $GProj(A\Gamma) = CoF(A\Gamma)$ as $CoF(A\Gamma) \subseteq GProj(A\Gamma) \subseteq WGProj(A\Gamma)$), where A is a commutative ring of finite global dimension, and Γ is in $\mathscr{F}_{\phi,A}$ or $H\mathscr{F}_{\phi,A}$, and reapplying Corollary 6.4.6 with $\mathscr{X} = H\mathscr{F}_{\phi,A}$, we get that $(*)$ holds for $\Gamma \in LH\mathscr{F}_{\phi,A}$ or Φ-$LH\mathscr{F}_{\phi,A}$. Also taking $\mathscr{X} = \mathscr{F}_{\phi,A}$, we get that $(*)$ holds for Φ-$\mathscr{F}_{\phi,A}$ groups (this class coincides with $\mathscr{F}_{\phi,A}$ by Remark 5.1.5), $L\mathscr{F}_{\phi,A}$-groups (already covered in $LH\mathscr{F}_{\phi,A}$, however we know courtesy Lemma 5.1.8 that $L\mathscr{F}_{\phi,A}$ is strictly bigger than $\mathscr{F}_{\phi,A}$). It is difficult to ascertain how or whether classes like Φ-$LH\mathscr{F}_{\phi,A}$ and Φ-$H\mathscr{F}_{\phi,A}$ differ from known classes like $LH\mathscr{F}$ or $H\mathscr{F}$.*

It follows from Lemma 6.3.6 that for groups in all these classes, the admission of a weak complete resolution by a module is equivalent to the admission of a complete resolution by it.

We end this chapter with the following easy observation related to Conjecture 6.3.4.

Corollary 6.4.8. *Fix a commutative ring A of finite global dimension. Then, Gorenstein projectivity is closed under restricting down to subgroups as long as the subgroups are from any of the classes mentioned in Remark 6.4.7.*

Proof. If M is a Gorenstein projective $A\Gamma$-module, then for any subgroup Γ' of Γ, $\mathrm{Res}^{\Gamma}_{\Gamma'}(M)$ is a weak Gorenstein projective $A\Gamma'$-module, and if Γ' is a subgroup of Γ from any of the mentioned classes in Remark 6.4.7, like Φ-$LH\mathscr{F}_{\phi,A}$ or $LH\mathscr{F}_{\phi,A}$, then by Remark 6.4.7, since over those subgroups weak Gorentein projectives are Gorenstein projective, we are done. $\square$

Chapter 7

Replacing Finite Groups by Type Φ Groups in Other Results on Kropholler's Hierarchy

In this chapter, we look at some results from the literature on groups in Kropholler's hierarchy and show how they can be modified with the base class of groups being changed from finite groups to groups of type Φ. We achieved similar modifications in Chapter 4 and Chapter 6. The results that we choose to modify here come from a wide collection of themes and topics and although we divide them into sections as per the papers they originally appeared in, they are mostly not connected to each other.

For the whole of this chapter, as usual, $\mathscr{F}_{\phi,A}$ denotes the class of groups of type Φ over A and $\mathscr{F}_{\phi} := \mathscr{F}_{\phi,\mathbb{Z}}$.

7.1 Groups not in $LH\mathscr{F}_{\phi}$ and $H\mathscr{F}_{\phi}$

It was shown in [41] that Thompson's group $F := \langle x_0, x_1, x_2, .. : x_k^{-1}x_nx_k = x_{n+1}\rangle$ is not in $LH\mathscr{F}$. Using basically the same argument, we can say that F is not in $LH\mathscr{F}_{\phi}$, as we show here.

First, we quote the following theorem from [41] which is one of the main results of that paper.

Theorem 7.1.1. *(Theorem A of [41]) Let $\mathscr{X}$ be a class of groups and let A be a commutative ring. Take an $LH\mathscr{X}$-group Γ and an $A\Gamma$-module M. Assume that*

$\text{Ext}^n_{A\Gamma}(M, ?)$ *commutes with direct limits for infinitely many non-negative* n*. Then, the following statements are equivalent.*

a) proj.dim$_{A\Gamma} M < \infty$.

b) proj.dim$_{A\Gamma'} M < \infty$*, for all* $\Gamma' \leq \Gamma$ *such that* $\Gamma' \in \mathscr{X}$.

Corollary 7.1.2. *Thompson's group* F *is not in* $LH\mathscr{F}_\phi$.

Proof. Note that for any group Γ and any commutative ring A, an $A\Gamma$-module M is of type FP_∞ iff the functors $\text{Ext}^*_{A\Gamma}(M, ?)$ commute with direct limits. Now take $\mathscr{X} = \mathscr{F}_\phi$ and $A = \mathbb{Z}$ in the statement of Theorem 7.1.1, and let M be of type FP_∞. Then, *proj.dim*$_{\mathbb{Z}\Gamma} M < \infty$ iff *proj.dim*$_{\mathbb{Z}\Gamma'} M < \infty$ for all type Φ subgroups $\Gamma' \leq \Gamma$, which in turn can happen iff *proj.dim*$_{\mathbb{Z}G} M < \infty$ for all finite subgroups $G \leq \Gamma$ (this last bit follows from the definition of type Φ groups).

It follows from Corollary 5.4 of [18] that F is of type FP_∞, i.e. the trivial module $\mathbb{Z}$ is of type FP_∞ as a $\mathbb{Z}F$-module, and it follows from Corollary 1.5 of [18] that F is torsion-free. So, the only finite subgroup of F is the trivial subgroup. It therefore follows from the preceding paragraph that if $F \in LH\mathscr{F}_\phi$, then *proj.dim*$_{\mathbb{Z}F}\mathbb{Z} < \infty$, i.e. F has finite cohomological dimension over $\mathbb{Z}$, which is not possible as F contains a free abelian group of infinite rank which has infinite cohomological dimension. $\square$

For a long time, since F was the most well-known group outside $LH\mathscr{F}$, the most common way to show a group was not in $H\mathscr{F}$ or $LH\mathscr{F}$ was to show that it had a subgroup isomorphic to F. In [3], the authors introduce a different set of methods that give examples of groups outside of Kropholler's hierarchy. We quote below one of the main theorems of [3].

Theorem 7.1.3. *(part of Theorem* 1.2 *of [3]) There exists an infinite finitely generated group which cannot act on any finite dimensional CW-complex without a global fixed point.*

Corollary 7.1.4. *The groups constructed in proving Theorem 7.1.3 in [3] are not in* $H\mathscr{F}_\phi$.

Proof. It has been noted in [3], and it is also easy to see, that if Q is a group satisfying the statement of Theorem 7.1.3, then $Q \in H\mathscr{X}$, for any class X, iff $Q \in \mathscr{X}$. Taking

$\mathscr{X} = \mathscr{F}_\phi$, we get that if $Q \in H\mathscr{F}_\phi$, then Q is of type Φ which is not possible because the groups Q constructed in [3] do not admit complete resolutions.

$\square$

Another known concrete example of a group outside $H\mathscr{F}$ is the first Grigorchuk group (Theorem 4.11 of [30]). A major ingredient in the proof of Theorem 4.11 of [30] is the following result of Petrosyan[54].

Theorem 7.1.5. *(Theorem 3.2 of [54]) Take A to be a commutative ring, and let Γ be a discrete group with no A-torsion such that it has jump cohomology of height k over A, which means that for any subgroup $\Gamma_1 \leq \Gamma$, $cd_A(\Gamma_1) < \infty$ implies $cd_A(\Gamma_1) \leq k$. If $\Gamma \in H\mathscr{F}$, then $cd_A(\Gamma) \leq k$. So, an $H\mathscr{F}$-group Γ can have jump cohomology of height k over $\mathbb{Q}$ if and only if $cd_{\mathbb{Q}}(\Gamma) \leq k$.*

Corollary 7.1.6. *The statement of Theorem 7.1.5 holds with $H\mathscr{F}$ replaced by $H\mathscr{F}_{\phi,A}$, for any commutative ring A of finite global dimension.*

Proof. Theorem 7.1.5 is proved in [54] by first proving it for the base case, i.e. when Γ is finite, and then proving it by transfinite induction on the level of Γ in $H\mathscr{F}$. We reproduce that proof for our case.

For our base case of type Φ groups, note that if Γ is of type Φ, then $cd_A(G) = 0$ for all finite $G \leq \Gamma$ since G needs to be torsion-free as per the hypothesis of Theorem 7.1.5. Thus, $cd_A(\Gamma) < \infty$ by definition of type Φ groups.

Now, assume as our induction hypothesis that, for some fixed ordinal α, $cd_A(\Gamma') < \infty$ for any $H_{<\alpha}\mathscr{F}_{\phi,A}$-subgroup Γ' of Γ. Let Γ'' be an $H_\alpha\mathscr{F}_{\phi,A}$-subgroup of Γ. Then, Γ'' acts on a finite dimensional contractible CW-complex X with stabilisers in $H_{<\alpha}\mathscr{F}_{\phi,A}$. By our induction hypothesis, all these stabilisers have cohomological dimension at most k over A. If the dimension of X is n, then using Lemma 3.3 of [54], we get that $cd_A(\Gamma'') \leq k + n$. From the hypothesis of Theorem 7.1.5, it now follows that $cd_A(\Gamma'') \leq k$.

We have thus proved that any $H\mathscr{F}_{\phi,A}$-subgroup of Γ has cohomological dimension at most k over A. So, if $\Gamma \in H\mathscr{F}_{\phi,A}$, then $cd_A(\Gamma) \leq k$.

$\square$

Corollary 7.1.7. *The first Grigorchuk group is not in $H\mathscr{F}_{\phi,\mathbb{Q}}$.*

Proof. It is shown in Theorem 4.11 of [30] that the first Grigorchuk group has jump rational cohomology of height 1 and has infinite cohomological dimension over the rationals, so by Corollary 7.1.6, it is not in $H\mathscr{F}_{\phi,\mathbb{Q}}$. □

7.2 Periodic cohomology and complete resolutions

We have seen before how for a group the property of admitting complete resolutions is quite helpful in dealing with many questions. A good indicator of whether a group admits (weak) complete resolutions or not, is checking whether it has periodic cohomology after a finite number of steps (Proposition 3.1 of [61]). We didn't work much with periodic cohomology elsewhere in this thesis, so we provide its definition here.

Definition 7.2.1. *(see [49], [61]) A group Γ is said to have periodic cohomology of period q after k steps iff the functors $H^i(\Gamma, ?)$ and $H^{i+q}(\Gamma, ?)$ are naturally equivalent for all $i > k$.*

The following important conjecture was made for groups with periodic cohomology in [61].

Conjecture 7.2.2. *(Conjecture A of [61]) A group Γ has periodic cohomology after some steps iff Γ admits a finite-dimensional free Γ-CW-complex, homotopy equivalent to a sphere.*

Talelli settled Conjecture 7.2.2 for $H\mathscr{F}$-groups in Corollary 3.5 of [61]. Almost the same proof works for $LH\mathscr{F}_{\phi}$-groups.

Theorem 7.2.3. *If $\Gamma \in LH\mathscr{F}_{\phi}$, Conjecture 7.2.2 holds true for Γ.*

Proof. We assume that Γ has periodic cohomology of period q after k steps. By Proposition 3.1 of [61], Γ admits a weak complete resolution, and since $\Gamma \in LH\mathscr{F}_{\phi}$, this implies that Γ admits complete resolutions by Remark 6.4.7, and therefore by Theorem 4.1.3 and Remark 4.1.4, $\mathrm{silp}(\mathbb{Z}\Gamma) < \infty$. So, by Theorem 3.2 and Corollary 3.3 of [61], the periodicity isomorphisms are induced by the cup product in $H^q(G, \mathbb{Z})$, and as noted in [61], Adem and Smith [1] proved that Conjecture 7.2.2 holds when this happens.

□

7.3 Results on stably flat modules

Stably flat modules arise in the study of complete cohomology of complete cohomology for infinite groups. This is again a concept that we haven't dealt with elsewhere in this thesis, so we provide a definition below:

Definition 7.3.1. *Let A be a commutative ring and let Γ be a group. An $A\Gamma$-module N is called stably flat iff $\widehat{Ext}^0_{A\Gamma}(M,N)=0$ for all $A\Gamma$-modules M of type FP_∞.*

Alcock [2] considered the question of whether, for some class of infinite groups and with certain conditions on the base ring, one can provide a complete characterisation of stably flat modules in terms of projective dimension and proved the following:

Theorem 7.3.2. *(Theorem A of [2]) Let A be a coherent commutative ring (i.e. every finitely generated ideal is finitely presented as a module) with finite global dimension and let $\Gamma \in H_1\mathscr{F}$. Then, for any $A\Gamma$-module N, the following are equivalent.*

a) N is stably flat as an $A\Gamma$-module.

b) $proj.dim_{A\Gamma}N < \infty$.

We have seen before that $H_1\mathscr{F} \subseteq \mathscr{F}_{\phi,A}$. We can now prove the statement of Theorem 7.3.2 replacing the condition $\Gamma \in H_1\mathscr{F}$ with $\Gamma \in \mathscr{F}_{\phi,A}$:

Theorem 7.3.3. *Let A be a coherent commutative ring of finite global dimension and let $\Gamma \in \mathscr{F}_{\phi,A}$. Then, for any $A\Gamma$-module N, the following are equivalent:*

a) N is stably flat as an $A\Gamma$-module.

b) $proj.dim_{A\Gamma}N < \infty$.

Proof. $(b) \Rightarrow (a)$. This is obvious as if $proj.dim_{A\Gamma}N < \infty$, then $\widehat{Ext}^0_{A\Gamma}(M,N)=0$ for all M of type FP_∞ because complete cohomology vanishes on modules with finite projective dimension.

$(a) \Rightarrow (b)$. If N is stably flat as an $A\Gamma$-module, then by Corollary 3.4 of [2], N is stably flat as an AG-module for all finite $G \leq \Gamma$, and by Theorem A' of [2] (or even just by Theorem 7.3.2), $proj.dim_{AG}N < \infty$ for all finite $G \leq \Gamma$, and therefore $proj.dim_{A\Gamma}N < \infty$ as Γ is of type Φ over A. □

7.4 Two general questions on projectivity

It is well-known that for a finite group G, a $\mathbb{Z}G$-module M is projective iff M is $\mathbb{Z}$-free and of finite projective dimension as a $\mathbb{Z}G$-module. In [38], the following question was asked:

Question 7.4.1. *(Question A of [38]) Fix $\mathbb{Z}$ to be the base ring. Is it only for finite groups G that a $\mathbb{Z}G$-module is projective iff it is $\mathbb{Z}$-free and of finite projective dimension as a $\mathbb{Z}G$-module?*

In making some progress on Question 7.4.1, the following theorem was proved in [38].

Theorem 7.4.2. *(Theorem 2.4 of [38]) Let Γ be a group such that every $\mathbb{Z}$-free $\mathbb{Z}\Gamma$-module of finite projective dimension is projective. If $\Gamma \in H\mathscr{F}$, then Γ is finite.*

Below we show how using some of the results that we have proved in this thesis, one can prove the statement of Theorem 7.4.2 replacing "$\Gamma \in H\mathscr{F}$" with "$\Gamma \in LH\mathscr{F}_{\phi}$":

Theorem 7.4.3. *Let Γ be a group such that every $\mathbb{Z}$-free $\mathbb{Z}\Gamma$-module of finite projective dimension is projective. If $\Gamma \in LH\mathscr{F}_{\phi}$, then Γ is finite.*

Proof. It is straightforward to see (also by Proposition 2.3 of [38]) that $\operatorname{fin.dim}(\mathbb{Z}\Gamma)$ is either 0 or 1. Since we are assuming that $\Gamma \in LH\mathscr{F}_{\phi}$, $\operatorname{fin.dim}(\mathbb{Z}\Gamma) = 0$ is an absurdity because if $\operatorname{fin.dim}(\mathbb{Z}\Gamma) = 0$, then by Theorem 4.4.1, $Gcd_{\mathbb{Z}}(\Gamma) = 0$ and by Theorem 4.1.11, Γ is finite, but for finite Γ, by Theorem 4.1.11, $\operatorname{spli}(\mathbb{Z}\Gamma) = 1$ and by Theorem 4.4.1, $\operatorname{fin.dim}(\mathbb{Z}\Gamma) = 1$, and so we have a contradiction.

Now, if $\operatorname{fin.dim}(\mathbb{Z}\Gamma) = 1$, then by Theorem 4.4.1, $\operatorname{spli}(\mathbb{Z}\Gamma) = 1$, and again by Theorem 4.1.11, Γ is finite. $\square$

The second question on projectivity that [38] tackles deals with stably flat modules as defined in Definition 7.3.1. The following was proved in [38]:

Theorem 7.4.4. *(Theorem 3.4 of [38]) Let $\Gamma \in LH\mathscr{F}$. Then, any stably flat $\mathbb{Z}\Gamma$-module M that is also a Benson's cofibrant (i.e. $M \otimes_{\mathbb{Z}} B(\Gamma, \mathbb{Z})$ is $\mathbb{Z}\Gamma$-projective), is projective.*

Using results proved in this thesis, we can prove the statement of Theorem 7.4.4 replacing $\mathbb{Z}$ with a coherent commutative ring A of finite global dimension and the condition "$\Gamma \in LH\mathscr{F}$" with "$\Gamma \in LH\mathscr{F}_{\phi,A}$":

Theorem 7.4.5. *Let A be a coherent commutative ring of finite global dimension and let $\Gamma \in LH\mathscr{F}_{\phi,A}$. For any $A\Gamma$-module M, if M is a stably flat as an $A\Gamma$-module and also a Benson's cofibrant, then it is projective.*

Proof. First, we deal with the case when $\Gamma \in H\mathscr{F}_{\phi,A}$. We proceed by transfinite induction on the smallest ordinal α such that $\Gamma \in H_\alpha\mathscr{F}_{\phi,A}$. If $\alpha = 0$, then by Theorem 7.3.3, $proj.dim_{A\Gamma}M < \infty$. Now, Benson's cofibrants coincide with Gorenstein projectives for $LH\mathscr{F}_{\phi,A}$-groups by Remark 6.4.7, so M is a Gorenstein projective with finite projective dimension, therefore it must be projective by Theorem 1.1.13. Now, as our induction hypothesis, assume that the statement of the theorem holds for all $\Gamma \in H_\beta\mathscr{F}_{\phi,A}$ for all ordinals $\beta < \alpha$. If, now, $\Gamma \in H_\alpha\mathscr{F}_{\phi,A}$, then Γ acts on a finite dimensional contractible CW-complex with stabilisers in $H_{<\alpha}\mathscr{F}_{\phi,A}$, and by tensoring the augmented cellular complex with M, we get a finite length resolution of M with modules that are direct sums of modules of the form $\mathrm{Ind}_{\Gamma'}^{\Gamma}(\mathrm{Res}_{\Gamma'}^{\Gamma}(M))$ for some $\Gamma' \in H_{<\alpha}\mathscr{F}_{\phi,A}$ (Here Ind and Res denote the induction and restriction functors respectively). As an $A\Gamma'$-module, $\mathrm{Res}_{\Gamma'}^{\Gamma} M$ is stably flat by Corollary 3.4 of [2] and also Benson's cofibrant by Lemma 6.3.8, and so by our induction hypothesis, it is projective as an $A\Gamma'$-module and $\mathrm{Ind}_{\Gamma'}^{\Gamma}(\mathrm{Res}_{\Gamma'}^{\Gamma}(M))$ is projective as an $A\Gamma$-module. Therefore, M has finite projective dimension as an $A\Gamma$-module, and as by Theorem 4.2.3, it is Gorenstein projective, M is projective by Theorem 1.1.13. This ends our proof for the case where $\Gamma \in H\mathscr{F}_{\phi,A}$.

Now, let $\Gamma \in LH\mathscr{F}_{\phi,A}$. We can assume that Γ is uncountable because if it is countable then since every countable group admits an action on a tree with finitely generated vertex and edge stabilizers (see Lemma 6.4.2), it follows that $\Gamma \in H\mathscr{F}_{\phi,A}$. Assume, as an induction hypothesis, that the theorem has been proved for all groups with cardinality strictly smaller than Γ. We can express Γ as an ascending union of subgroups $\{\Gamma_\lambda : \lambda \in \Lambda\}$ where each Γ_λ is of strictly smaller cardinality than Γ. By the induction hypothesis, M is projective over each Γ_λ (note that we are again using Corollary 3.4 of [2] and Lemma 6.3.8 to go down to subgroups here), and so by Lemma

6.4.1, $proj.dim_{A\Gamma} M \leq 1$. And again, this means M is Gorenstein projective with finite projective dimension, so it must be projective.

□

Our proof of Theorem 7.4.5 here is quite independent of the way Theorem 7.4.4 is proved in [38].

7.5 On the projective complete cohomological dimension

In [37], a cohomological invariant for groups, called the *projective complete cohomological dimension*, was introduced. We have dealt with many cohomological invariants in Chapter 4, Chapter 5 and Chapter 6 as well. But, we have not dealt with this invariant before, so we start this section by defining it as per [37].

Definition 7.5.1. *(done over $\mathbb{Z}$ in [37]) For any group Γ and any commutative ring A, the projective complete cohomological dimension of Γ over A, denoted $pccd_A(\Gamma)$, is defined to be the least integer n such that $H^i(\Gamma, M)$ is naturally isomorphic to $\widehat{H}^i(\Gamma, M)$ for any $A\Gamma$-module M, for all $i > n$.*

We know that when a group admits complete resolutions, since complete cohomology can be calculated using complete resolutions, the projective complete cohomlogical dimension is finite. The following general conjecture was made in [37] for all groups:

Conjecture 7.5.2. *(made over $\mathbb{Z}$ in Conjecture 3.1 of [37]) For any commutative ring A and any group Γ, $pccd_A(\Gamma) < \infty$.*

It is noted in [37] that Conjecture 7.5.2, with $A = \mathbb{Z}$, is known to hold true for $H_1\mathscr{F}$-groups. [37] introduces a hierarchy of groups similar to Kropholler's hierarchy with the additional condition that the groups act cocompactly on the complexes. For a base class of groups $\mathscr{X}$, this hierarchy is denoted $K\mathscr{X}$, with, expectedly, the levels denoted as $K_\alpha\mathscr{X}$ for any ordinal α. The following Theorem 7.5.4 was proved in [37] regarding groups satisfying Conjecture 7.5.2. A key lemma in proving that was the following result, an analogous version of which was proved for the generalized cohomological dimension by Ikenaga in [35]. It originally follows from an idea of Quillen, as noted in [35] by Ikenaga.

Lemma 7.5.3. *(Lemma 3.3 of [37]) Let Γ be a group and let X be an acyclic Γ-simplicial complex. Take Σ to be the set of representatives for the Γ-orbits of simplices of X. ([37] takes X to be a simplicial complex, so does [35], but one can state this result for CW-complexes as well)*

Then, for any commutative ring A, $pccd_A(\Gamma) \leq sup_{\sigma \in \Sigma}\{dim(\sigma) + pccd_A(\Gamma_\sigma)\}$, where Γ_σ denotes the isotropy subgroup corresponding to the simplex σ.

Proof. Although this has been proved in [37] with $A = \mathbb{Z}$, the proof follows a spectral sequence argument which works for any commutative ring A. □

Theorem 7.5.4. *(done with $\mathbb{Z}$ as the base ring in Theorem 3.4 of [37]) Let A be a commutative ring. A group Γ satisfied Conjecture 7.5.2 if it satisfies any of the following conditions:*

a) $\Gamma \in K\mathscr{F}$.

b) $proj.dim_{A\Gamma} B(\Gamma, A) < \infty$.

Proof. We omit this proof as the proof of Theorem 3.4 of [37] works fine if $\mathbb{Z}$ is replaced by any commutative ring. □

Theorem 7.5.5. *Theorem 7.5.4 works with $\mathscr{F}$ replaced by $\mathscr{F}_{\phi,A}$, for any commutative ring A.*

Proof. The proof of Theorem 7.5.4 in [37] proceeds by transfinite induction on the ordinal α such that $\Gamma \in K_\alpha \mathscr{F}_{\phi,A}$. The inductive step works fine, courtesy Lemma 3.3 of [37], irrespective of what the base class of groups is; so this part works fine when we replace $\mathscr{F}$ by $\mathscr{F}_{\phi,A}$. We need to just check the base case, i.e. when $\Gamma \in K_0\mathscr{F}_{\phi,A} = \mathscr{F}_{\phi,A}$. In this case, $proj.dim_{A\Gamma} B(\Gamma, A) < \infty$ by the definition of type Φ groups because $B(\Gamma, A)$ restricts to a free module over any finite subgroup of Γ, so by Theorem 7.5.4, we are done.

□

7.6 On an invariant related to complete homology

Just like the projective complete cohomological dimension that we briefly touched upon in Section 7.5, Jo also introduced an analogous *injective complete homological dimension* in [39], which we define below. In this context, it must be noted that the complete homology theory that we are referring to was formulated by Goichot[32] in developing on ideas of Vogel. See [32] or the first two sections of [39] for a survey of this area.

Definition 7.6.1. *(made over $\mathbb{Z}$ in Definition 3.1 of [39]) Let A be a commutative ring. Then, the injective complete homological dimension of a group Γ over A, denoted $ichd_A(\Gamma)$, is defined as the smallest integer n such that $H_i(\Gamma, M)$ is naturally isomorphic to $\widehat{H}_i(\Gamma, M)$, for all $A\Gamma$-modules M, and all $i > n\}$. Here, $\widehat{H}_i(\Gamma, ?)$ denotes the complete homology functors.*

Over the integers, the projective complete cohomological dimension and the injective complete homological dimension coincide when the group is of type FP_∞ (Proposition 3.5 of [39]). This gives an example of a group outside $LH\mathscr{F}$, namely Thompson's group F from Section 7.1, for which both of these dimensions over $\mathbb{Z}$ is -1.

A related question, somewhat analogous to Theorems 7.5.4 and 7.5.5, is if one can come up with a large class of infinite groups which has finite injective complete homological dimension. The following was proved in [39]:

Theorem 7.6.2. *(done over $\mathbb{Z}$ in Theorem 4.2 of [39]) Let A be a commutative ring of finite global dimension. Then, for any group Γ, $ichd_A(\Gamma) < \infty$ if Γ satisfies any of the following conditions:*

a) $\mathrm{spli}(A\Gamma) < \infty$.

b) $\Gamma \in K\mathscr{F}$.

Staying with the theme of this chapter, we prove the following result:

Theorem 7.6.3. *Theorem 7.6.2 holds true with the condition "$\Gamma \in K\mathscr{F}$" replaced by "$\Gamma \in K\mathscr{F}_{\phi,A}$"*

Proof. The proof is similar to the proof of Theorem 7.5.5, i.e. here too we proceed by transfinite induction on the ordinal α such that $\Gamma \in K_\alpha\mathscr{F}_{\phi,A}$. Unsurprisingly again, the inductive step works fine irrespective of what the base class of groups is because the analogous result of Lemma 7.5.3 with *ichd* in place of *pccd* is proved in Lemma 4.1 of [39] over the integers and the same argument works over any ring. For the base case, note that if $\Gamma \in \mathscr{F}_{\phi,A}$, then, by Remark 5.1.14, $\mathrm{spli}(A\Gamma) < \infty$ (this is where we are using the fact that A is of finite global dimension), and so by Theorem 7.6.2, we are fine.

□

7.7 Strongly Gorenstein flat modules

In this section, we use some results proved earlier in the thesis to simplify a result of Bahlekeh [6] regarding the coincidence of strongly Gorenstein flat modules and Gorenstein projective modules. First, we define strongly Gorenstein projective modules.

Definition 7.7.1. *(see Definition* 3.1 *of [6]) For any ring R, an R-module M is called strongly Gorenstein flat if M is a kernel of an acyclic complex of projectives $(P_i, d_i)_{i\in\mathbb{Z}}$ such that $Hom_R(P_*, F)$ is acyclic for all flat R-modules F.*

It is clear that Definition 7.7.1 closely resembles the definition of Gorenstein projectives (Definition 1.1.1). Bahlekeh chooses not to call the modules defined above Gorenstein flat because Gorenstein flats are usually defined in a different way (see Definition 3.1 of [34]) as kernels of acyclic complexes of flats which remain exact after applying $? \otimes_R (flat)$. If the base ring is coherent, there are examples where the class of strongly Gorenstein flats lies strictly between Gorenstein flats and projectives (see [27]).

Over group rings, it is a natural question to wonder whether strongly Gorenstein flats and Gorenstein projectives coincide. Bahlekeh proved the following in [6].

Theorem 7.7.2. *(done over $\mathbb{Z}$ in Theorem* 3.11 *of [6]) Let A be a commutative ring of finite global dimension and let Γ be a group. Then, the class of strongly Gorenstein flat $A\Gamma$-modules coincides with the class of Gorenstein projective $A\Gamma$-modules if Γ satisfies any of the following conditions.*

a) $\mathrm{fin.dim}(A\Gamma) < \infty$.

b) *Γ is an $H\mathscr{F}$-group of type FP_∞ over A.*

Question 7.7.3. *Can we replace $H\mathscr{F}$ by $LH\mathscr{F}_{\phi,A}$ or $H\mathscr{F}_{\phi,A}$ in condition* (*b*) *of the statement of Theorem 7.7.2?*

Answer 7.7.4. *We have seen in Proposition 4.4.11 that any $LH\mathscr{F}_{\phi,A}$-group Γ of type FP_∞ over A, where A is of finite global dimension, is of type Φ over A, and therefore satisfies* $\mathrm{fin.dim}(A\Gamma) < \infty$ *by Remark 5.1.14. So, in the statement of Theorem 7.7.2, we can replace $H\mathscr{F}$ by $LH\mathscr{F}_{\phi,A}$, but the previous sentence makes it clear why one can simplify Theorem 7.7.2 by dropping the condition* (*b*) *from the statement entirely.*

Chapter 8

Generation and Cogeneration of Derived Categories of Modules over Groups in Kropholler's Hierarchy

8.1 Generating modules in Kropholler's hierarchy

Notation: For any class of R-modules, $\mathscr{T}$, we denote by $\mathscr{T}^{\oplus}$ the smallest class containing $\mathscr{T}$ that is closed under arbitrary direct sums. For the whole of this section, our modules will be over group algebras, so we shall take the base ring to be a commutative ring denoted A.

Lemma 8.1.1. *Let Γ be a group that acts cellularly on a Γ-CW-complex X with stabilisers in a class $\mathscr{L}$. Let $I(\Gamma, \mathscr{L})$ be a class of $A\Gamma$-modules consisting of all modules of the form $\mathrm{Ind}_{\Gamma'}^{\Gamma}(M)$ where Γ' is some subgroup of Γ that is in $\mathscr{L}$ and M is some $A\Gamma'$-module. Then, the number of steps needed to generate trivial module from $I(\Gamma, \mathscr{L})^{\oplus}$ is bounded by the dimension of X.*

Proof. We can assume that the maximal dimension of cells in X is finite because if it is not finite we have nothing to prove. Let this number be n. The augmented cell complex is of the form $0 \to A_n \to ... \to A_1 \to A_0 \to A \to 0$ where each A_i is an $A\Gamma$-permutation module that we get from the action of Γ as a group of permutations of the i-dimensional cells of X. Each A_i is a direct sum of the trivial module induced up to Γ from subgroups of Γ that are in $\mathscr{L}$. By Lemma 2.1.9, the trivial module can be generated from $I(\Gamma, \mathscr{L})^{\oplus}$ in $(n+1)-1+0=n$ steps. $\square$

Definition 8.1.2. *For any group* Γ *and a class of groups* $\mathscr{X}$, $\Lambda_n(\Gamma, \mathscr{X}) := \{\mathrm{Ind}_{\Gamma'}^{\Gamma}(M) :$ M *is any* $A\Gamma'$*-module and* Γ' *is any* $H_n\mathscr{X}$*-subgroup of* $\Gamma\}$.

Before stating our next result, we recall that the successor of an ordinal number α is the smallest ordinal number greater than α. An ordinal number that is a successor is called a successor ordinal. If α is a successor ordinal, we define $\alpha - 1$ to be the ordinal number β whose successor is α (in ordinal addition notation, $\alpha = \beta + 1$).

Our next result, although never stated separately in the form we are stating it below, has been known for a while because the techniques in proving it are quite standard.

Lemma 8.1.3. *For any group* Γ*, a class of groups* $\mathscr{X}$ *and any successor ordinal* α*,* $\Lambda_\alpha(\Gamma, \mathscr{X}) \subseteq [\Lambda_{\alpha-1}(\Gamma, \mathscr{X})^{\oplus}]$.

Proof. Let Γ' be a $H_\alpha\mathscr{X}$-subgroup of Γ. Then, by definition, there exists a finite dimensional contractible complex T on which Γ' acts with stabilizers in $H_{\alpha-1}\mathscr{X}$. Its cellular chain complex is of the following form:

$$0 \to A_t \to ... \to A_1 \to A_0 \to A \to 0$$

where each A_i is a permutation module that we get from the action of Γ as a group of permutations of the i-dimensional cells of T.

Let X be an arbitrary $A\Gamma'$-module. If we tensor the above complex by X, we get the following complex:

$$0 \to A_t \otimes X \to ... \to A_1 \otimes X \to A_0 \otimes X \to X \to 0$$

Now, if we induce all these modules up to Γ, we get the following complex:

$$0 \to \mathrm{Ind}_{\Gamma'}^{\Gamma}(A_t \otimes X) \to ... \to \mathrm{Ind}_{\Gamma'}^{\Gamma}(A_1 \otimes X) \to \mathrm{Ind}_{\Gamma'}^{\Gamma}(A_0 \otimes X) \to \mathrm{Ind}_{\Gamma'}^{\Gamma}(X) \to 0$$

Each A_i can be written as a direct sum of the trivial module induced up to Γ' from subgroups of Γ' that are of the form Γ'_σ, where Γ'_σ denotes the stabilizer of the cell σ (note that $\Gamma'_\sigma \in H_{\alpha-1}\mathscr{X}$ for all σ), with σ running over the set of Γ'-orbit representatives for the i-dimensional cells (we can denote this set by Δ). Thus, $\mathrm{Ind}_{\Gamma'}^{\Gamma}(A_i \otimes X) = \bigoplus_{\sigma\in\Delta} \mathrm{Ind}_{\Gamma'_\sigma}^{\Gamma}(X) \in \Lambda_{\alpha-1}(\Gamma, \mathscr{X})^{\oplus}$. Thus, $\mathrm{Ind}_{\Gamma'}^{\Gamma}(X) \in [\Lambda_{\alpha-1}(\Gamma, \mathscr{X})^{\oplus}]$. $\square$

The following result follows straightforwardly from Lemma 8.1.3.

Corollary 8.1.4. *For any class of groups $\mathscr{X}$, any group Γ and any positive integer n,*

$$\Lambda_n(\Gamma,\mathscr{X}) \subseteq [\Lambda_{n-1}(\Gamma,\mathscr{X})^{\oplus}] \subseteq [[\Lambda_{n-2}(\Gamma,\mathscr{X})^{\oplus}]^{\oplus}] \subseteq \ldots \subseteq \underbrace{[[\ldots[}_{n\ times}\Lambda_0(\Gamma,\mathscr{X})\underbrace{^{\oplus}]^{\oplus}]\ldots^{\oplus}]}_{n\ times}$$

.

Theorem 8.1.5. *Let $\mathscr{X}$ be a class of groups and Γ a group. For any $n \geq 1$ and for any group J, let $d_{n,\mathscr{X}}(J) := inf\{dim(X) : X$ is a finite dimensional contractible CW-complex on which J acts with stabilisers in $H_{n-1}\mathscr{X}\}$, and let $t_n := sup\{d_{n,\mathscr{X}}(H) : H \leq G, H \in H_n\mathscr{X}\}$. Then, for any fixed n, $\Lambda_n(\Gamma,\mathscr{X})^{\oplus} \subseteq \langle\Lambda_{n-m}(\Gamma,\mathscr{X})^{\oplus}\rangle_{\Pi_{i=n-m+1}^{n}(1+t_i)-1}$, for any m such that $1 \leq m \leq n$.*

Proof. We shall proceed by induction on m.

Let $\mathrm{Ind}_H^{\Gamma}(M) \in \Lambda_n(\Gamma,\mathscr{X})$ where H is some $H_n\mathscr{X}$-subgroup of Γ and M is some RH-module. From the proof of Lemma 8.1.3, it follows that $\mathrm{Ind}_H^{\Gamma}(M) \in [\Lambda_{n-1}(\Gamma,\mathscr{X})^{\oplus}]_{t_n}$, and by Lemma 2.3.3, any arbitrary direct sum of modules in $\Lambda_n(\Gamma,\mathscr{X})$ has to be in $[\Lambda_{n-1}(\Gamma,\mathscr{X})^{\oplus}]_{t_n}$. Thus, $\Lambda_n(\Gamma,\mathscr{X})^{\oplus} \subseteq [\Lambda_{n-1}(\Gamma,\mathscr{X})^{\oplus}]_{t_n} \subseteq \langle\Lambda_{n-1}(\Gamma,\mathscr{X})^{\oplus}\rangle_{t_n}$. This proves the lemma for $m = 1$.

Similarly, we get $\Lambda_a(\Gamma,\mathscr{X})^{\oplus} \subseteq [\Lambda_{a-1}(\Gamma,\mathscr{X})^{\oplus}]_{t_a} \subseteq \langle\Lambda_{a-1}(\Gamma,\mathscr{X})^{\oplus}\rangle_{t_a}$ for any a between 0 and n(this follows from the definition of t_a and Lemma 2.2.2.b.).

We assume the statement of the theorem to be true for $m = d$. Now, let $m = d+1$. We have the following:

a) $\Lambda_{n-d}(\Gamma,\mathscr{X})^{\oplus} \subseteq \langle\Lambda_{n-d-1}(\Gamma,\mathscr{X})^{\oplus}\rangle_{t_{n-d}}$.

b) $\Lambda_n(\Gamma,\mathscr{X})^{\oplus} \subseteq \langle\Lambda_{n-d}(\Gamma,\mathscr{X})^{\oplus}\rangle_{\Pi_{i=n-d+1}^{n}(1+t_i)-1}$ (induction hypothesis)

By Lemma 2.1.7.b., therefore, every module in $\Lambda_n(\Gamma,\mathscr{X})^{\oplus}$ is generated from the class $\Lambda_{n-d-1}(\Gamma,\mathscr{X})^{\oplus}$ in $t_{n-d}\Pi_{i=n-d+1}^{n}(1+t_i) - t_{n-d} + t_{n-d} + (\Pi_{i=n-d+1}^{n}(1+t_i) - 1) = (1+t_{n-d})\Pi_{i=n-d+1}^{n}(1+t_i) - 1 = \Pi_{i=n-(d+1)+1}^{n}(1+t_i) - 1$ steps. This ends our induction. ∎

Corollary 8.1.6. *Let $\mathscr{X}$ be a class of groups and Γ be an $H_n\mathscr{X}$ group. Using the notations of Theorem 8.1.5, every $A\Gamma$-module is generated from $\Lambda_0(\Gamma,\mathscr{X})^{\oplus}$ in $\Pi_{i=1}^{n}(1+t_i) - 1$ steps.*

Proof. We can assume that all the t_i's are finite because if any of them are not we have nothing to prove. The corollary then follows by taking $m = n$ in the statement of Theorem 8.1.5 and by noting that as $\Gamma \in H_n\mathscr{X}$, $\Lambda_n(\Gamma, \mathscr{X})$ is the class of all $A\Gamma$-modules. $\square$

We end this section with the following remark.

Remark 8.1.7. *For any class of $A\Gamma$-modules, $\mathscr{U}$, let us define the finitistic $\mathscr{U}$-dimension of $A\Gamma$, denoted $\mathscr{F}\mathscr{U}$-dim$(A\Gamma)$, to be sup$\{\mathscr{U}$-dim$(M) : M \in Mod$-$A\Gamma$ such that $\mathscr{U}$-dim$(M) < \infty\}$. It is obvious that if $\mathscr{T} \subseteq \mathscr{U}$, then $\mathscr{F}\mathscr{T}$-dim$(A\Gamma) \leq \mathscr{F}\mathscr{U}$-dim$(A\Gamma)$.*

It is noteworthy that one can just replace $t_{n,\mathscr{X}}(G)$ by $t = \mathscr{F}\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$-dim$(A\Gamma)$ in the statement of Theorem 8.1.5. This way, it might be more algebraic for visualisation purposes. Also, from the inequality mentioned in the previous paragraph, it follows that we can replace all the t_i's by t. So, Corollary 8.1.6 can be restated as : for $\Gamma \in H_n\mathscr{X}$, every module can be generated from $\Lambda_m(\Gamma, \mathscr{X})^{\oplus}$, for any m satisfying $1 \leq m \leq n$, in $(1+t)^{n-m} - 1$ steps. This shows very clearly that the number of levels we go down in the hierarchy to generate our class of modules gets reflected in the degree of the polynomial in t that we get as the number of steps (note that this number of steps need not me optimal). Also, this is very much in line with the spirit of Kropholler's hierarchy because, taking $\mathscr{X}$ to be the class of all finite groups for example, we see that we are generating all modules from the class of modules induced up from finite subgroups closed under direct sums.

8.2 Generation in the derived categories

For the rest of this chapter, for any ring R, we will use the following notations.

$\mathcal{D}^+(Mod\text{-}R)$: the derived category of bounded above chain complexes of R-modules.

$\mathcal{D}^-(Mod\text{-}R)$: the derived category of bounded below chain complexes of R-modules.

$\mathcal{D}^b(Mod\text{-}R)$: the derived category of bounded chain complexes of R-modules.

$\mathcal{D}(Mod\text{-}R)$: the derived category of unbounded chain complexes of R-modules.

For any class of modules $\mathscr{C}$, when we write $\mathcal{D}^*(\mathscr{C})$, as we do in the statements of Theorem 8.2.3, we mean a subcategory of chain complexes in the relevant derived category where the modules in the chain complexes are from $\mathscr{C}$.

We begin straightaway with two very useful lemmas which are both standard knowledge (one can look at [51] for details).

Lemma 8.2.1. *Let R be a commutative ring. Let $\mathcal{D}(Mod\text{-}R)$ be the derived unbounded category of chain complexes of R-modules and let $\mathscr{U}$ be a triangulated subcategory of $\mathcal{D}(Mod\text{-}R)$. If $\mathscr{U}$ is closed under coproducts, then it is closed under direct limits of chain complexes, and if $\mathscr{U}$ is closed under products, then it is closed under inverse limits of chain complexes.*

Proof. Let $\mathscr{U}$ be closed under coproducts, and let $\{S_i\}_{i\geq 0}$ be a collection of chain complexes in $\mathscr{U}$ where we have maps $f_i : S_i \to S_{i+1}$. The direct limit $\varinjlim_{i\geq 0} S_i$, from the definition of homotopy colimit in the derived unbounded category, arises as a cokernel in the following short exact sequence $0 \to \bigoplus_{i\geq 0} S_i \xrightarrow{\bigoplus_{i\geq 0}(id_{S_i}-f_i)} \bigoplus_{i\geq 0} S_i \to \varinjlim_{i\geq 0} S_i \to 0$. Here the first two terms are in $\mathscr{U}$ as $\mathscr{U}$ is closed under coproducts, and therefore the third term is in $\mathscr{S}$ as well since $\mathscr{U}$ is a triangulated subcategory of $\mathcal{D}(Mod\text{-}R)$.

Similarly, now if we let $\mathscr{U}$ be closed under products where we have maps $f_i : S_i \to S_{i-1}$, then by the inverse limit $\varprojlim_{i\geq 0} S_i$ arises as a kernel in the following short exact sequence by the definition of homotopy limits: $0 \to \varprojlim_{i\geq 0} S_i \to \prod_{i\geq 0} S_i \xrightarrow{\prod_{i\geq 1}(id_{S_i}-f_i)} \prod_{i\geq 0} S_i \to 0$. Here the last two terms are in $\mathscr{U}$ as $\mathscr{U}$ is closed under products, and therefore the third term is in $\mathscr{U}$ since $\mathscr{U}$ is a triangulated subcategory of $\mathcal{D}(Mod\text{-}R)$. □

The following lemma is standard knowledge too. One can look up the proof of Proposition 2.1.f of [55] for an idea of the proof in the derived unbounded case, that same proof works for us.

Lemma 8.2.2. *Let R be a ring and let $\mathscr{T}$ be a triangulated subcategory of $\mathcal{D}(Mod\text{-}R)$ or $\mathcal{D}^+(Mod\text{-}R)$ or $\mathcal{D}^-(Mod\text{-}R)$ or $\mathcal{D}^b(Mod\text{-}R)$. Then, any chain complex, X_*, of the form $0 \to X_n \to X_{n-1} \to \ldots \to X_0 \to 0$ is in $\mathscr{T}$ if each X_i, when considered as a chain complex concentrated in degree zero, is in $\mathscr{T}$.*

Proof. Let us assume that we are working in $\mathcal{D}(Mod\text{-}R)$. We will prove this by induction on the length of X_*. Of course, if X_* is of length 1, then it is in $\mathscr{T}$ by the hypothesis and since triangulated subcategories are closed under shifts. We now assume that if X_* is of length $\leq n$ then it is in $\mathscr{T}$ - this is our induction hypothesis.

Let $X_* : 0 \to X_n \to X_{n-1} \to ... \to X_1 \to X_0 \to 0$ be a bounded complex of length $n+1$ where each X_i is in $\mathscr{T}$. We can fit this into a short exact sequence of bounded complexes as shown below.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r, "0"] \arrow[d] & X_n \arrow[r, "Id"] \arrow[d] & X_n \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r, "0"] \arrow[d] & X_2 \arrow[r, "Id"] \arrow[d] & X_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r, "0"] \arrow[d] & X_1 \arrow[r, "Id"] \arrow[d] & X_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_0 \arrow[r, "Id"] \arrow[d] & X_0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Here, the first chain complex is in $\mathscr{T}$ by the hypothesis of our lemma, and the third chain complex is in $\mathscr{T}$ by the induction hypothesis as each X_i is in $\mathscr{T}$ and it is a bounded complex of length $n-1$. So, since $\mathscr{T}$ is a triangulated subcategory of $\mathcal{D}(Mod\text{-}R)$, X_* is in $\mathscr{T}$, and we are done.

Note that the exact same proof works when $\mathscr{T}$ is a triangulated subcategory of $\mathcal{D}^+(Mod\text{-}R)$ or $\mathcal{D}^+(Mod\text{-}R)$ or $\mathcal{D}^b(Mod\text{-}R)$.

□

We are now in a position to prove the following theorem about generation of the derived bounded above derived bounded and derived unbounded categories of chain complexes of modules with respect to classes of modules induced up from subgroups in Kropholler's hierarchy. In many of the upcoming statements, we will come across classes of modules, in most cases considered as classes of chain complexes concentrated in degree zero, with the superscript "$^{\oplus}$", which means closed under direct sums as explained earlier, and in some cases with the superscript "$^{\oplus,\Pi}$", which means we are taking the direct-sum closed class and closing it under direct products of chain complexes.

Note that although the statements of our next two results, Theorem 8.2.3 and Theorem 8.2.5, might seem a little technical, the proofs are actually quite straightforward and easy to follow. The only reason why length-wise some of the proofs are quite long is we have chosen to elaborate in detail what limits we are using in what case and why.

Theorem 8.2.3. *Let Γ be a group and let A be a commutative ring. We fix a class of groups, $\mathscr{X}$. For any triangulated category $\mathscr{T}$ and a class of objects in it denoted $\mathscr{U}$, we shall denote the smallest triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$ by $\Delta_{\mathscr{T}}\mathscr{U}$.*

a) Let $\mathscr{T} = \mathcal{D}^-(Mod\text{-}A\Gamma)$. Then, for any $n \in \mathbb{N}$,

$$... = \Delta_{\mathscr{T}}\mathcal{D}^-(\Lambda_n(\Gamma, \mathscr{X})^{\oplus}) = \Delta_{\mathscr{T}}\mathcal{D}^-(\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}) = ... = \Delta_{\mathscr{T}}\mathcal{D}^-(\Lambda_0(\Gamma, \mathscr{X})^{\oplus})$$

b) Let $\mathscr{T} = \mathcal{D}^+(Mod\text{-}A\Gamma)$. If, for some $n \in \mathbb{N}$, $t_{n,\mathscr{X}}(G)$ or $\mathscr{F}\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$-dim$(A\Gamma)$ is finite, then

$$\Delta_{\mathscr{T}}\mathcal{D}^+(\Lambda_n(\Gamma, \mathscr{X})^{\oplus,\Pi}) = \Delta_{\mathscr{T}}\mathcal{D}^+(\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus,\Pi}) = ... = \Delta_{\mathscr{T}}\mathcal{D}^+(\Lambda_0(\Gamma, \mathscr{X})^{\oplus,\Pi})$$

c) Let $\mathscr{T} = \mathcal{D}(Mod\text{-}A\Gamma)$. In this case, for any class of objects $\mathscr{U}$ in $\mathscr{T}$, denote by $\mathscr{T}\text{-}\langle\mathscr{U}\rangle$ the smallest triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$ closed under products and coproducts (direct sums). Then,

$$\begin{array}{ccccccc}
.. = \mathscr{T}\text{-}\langle\mathcal{D}(\Lambda_n(\Gamma, \mathscr{X}))\rangle & = & \mathscr{T}\text{-}\langle\mathcal{D}(\Lambda_{n-1}(\Gamma, \mathscr{X}))\rangle & = & .. & = & \mathscr{T}\text{-}\langle\mathcal{D}(\Lambda_0(\Gamma, \mathscr{X}))\rangle \\
\| & & \| & & & & \| \\
.. = \mathscr{T}\text{-}\langle\Lambda_n(\Gamma, \mathscr{X})\rangle & = & \mathscr{T}\text{-}\langle\Lambda_{n-1}(\Gamma, \mathscr{X})\rangle & = & ... & = & \mathscr{T}\text{-}\langle\Lambda_0(\Gamma, \mathscr{X})\rangle
\end{array}$$

d) Let $\mathscr{T} = \mathcal{D}^b(Mod\text{-}A\Gamma)$. Then, for any n, $\Delta_{\mathscr{T}}\mathcal{D}^b(\Lambda_n(\Gamma, \mathscr{X})) \subseteq \Delta_{\mathscr{T}}\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$. If additionally $t_{n,\mathscr{X}}(G)$ or alternatively $\mathscr{F}\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$-dim$(A\Gamma)$ is finite, then

$$\begin{array}{ccccccc}
\Delta_{\mathscr{T}}\mathcal{D}^b(\Lambda_n(\Gamma, \mathscr{X})^{\oplus}) & = & \Delta_{\mathscr{T}}\mathcal{D}^b(\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}) & .. = & & & \Delta_{\mathscr{T}}\mathcal{D}^b(\Lambda_0(\Gamma, \mathscr{X})^{\oplus}) \\
\| & & \| & & & & \| \\
\Delta_{\mathscr{T}}\Lambda_n(\Gamma, \mathscr{X})^{\oplus} & = & \Delta_{\mathscr{T}}\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus} & = & ... & = & \Delta_{\mathscr{T}}\Lambda_0(\Gamma, \mathscr{X})^{\oplus}
\end{array}$$

Proof. The techniques used in this proof for each of the subparts have some similarities.

a) Note that for any n, $\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus} \subseteq \Lambda_n(\Gamma, \mathscr{X})^{\oplus}$. Thus,

$$\mathcal{D}^-(\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}) \subseteq \mathcal{D}^-(\Lambda_n(\Gamma, \mathscr{X})^{\oplus}) \subseteq \Delta_{\mathscr{T}}\mathcal{D}^-(\Lambda_n(\Gamma, \mathscr{X})^{\oplus})$$

and therefore $\Delta_{\mathscr{T}}\mathcal{D}^-(\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}) \subseteq \Delta_{\mathscr{T}}\mathcal{D}^-(\Lambda_n(\Gamma, \mathscr{X})^{\oplus})$.

Now take a bounded-below chain complex $X_* = ... \to X_{m+k} \to ... \to X_{m+1} \to X_m \to 0$ where each X_i in $\Lambda_n(\Gamma, \mathscr{X})^{\oplus}$. We now look at the following truncations of X_*.

$g_0(X_*) = ... \to 0 \to 0 \to X_m \to 0$

$g_1(X_*) = ... \to 0 \to X_{m+1} \to X_m \to 0$

$\vdots$

$g_k(X_*) = ... \to 0 \to X_{m+k} \to ... \to X_{m+1} \to X_m \to 0$

In $g_i(X_*)$ as defined above, we have X_{m+j} in degree $m+j$ for all $j \in \{0, 1, .., i\}$, and zero everywhere else. Note that each $X_i = \oplus_{\sigma\in\Sigma_i} X_{i,\sigma}$ for some indexing set Σ_i where each $X_{i,\sigma} \in \Lambda_n(\Gamma, \mathscr{X})$. Now, as $\Lambda_n(\Gamma, \mathscr{X}) \subseteq [\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}]$ by Lemma 8.1.3, each $X_{i,\sigma}$ admits a finite length resolution $I^*_{i,\sigma} \twoheadrightarrow X_{i,\sigma}$ with modules from $\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$. We thus get a resolution of $\oplus_{\sigma\in\Sigma_i} X_{i,\sigma}$ of possibly infinite length with modules from $\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$: $\oplus_{\sigma\in\Sigma_i} I^*_{i,\sigma} \twoheadrightarrow \oplus_{\sigma\in\Sigma_i} X_{i,\sigma} = X_i$. Thus, a chain complex with X_i in degree zero and zero everywhere else is quasi-isomorphic to a bounded below complex :.. $\to \oplus_{\sigma\in\Sigma_i} I^k_{i,\sigma} \to ... \to \oplus_{\sigma\in\Sigma_i} I^1_{i,\sigma} \to \oplus_{\sigma\in\Sigma_i} I^0_{i,\sigma} \to 0 \to 0 \to ..$ with $\oplus_{\sigma\in\Sigma_i} I^k_{i,\sigma} \in \Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$ in degree k for $k \geq 0$ and zero in every other degree. So, each X_i, considered as a complex concentrated in degree zero, is in $\Delta_{\mathscr{T}}\mathcal{D}^-(\Lambda_{n-1}(G, \mathscr{X})^{\oplus})$. Now note that each $g_i(X_*)$ is a bounded complex where each module, when considered as a complex concentrated in degree zero, is in $\Delta_{\mathscr{T}}\mathcal{D}^-(\Lambda_{n-1}(G, \mathscr{X})^{\oplus})$. By Lemma 8.2.2, it follows that each $g_i(X_*)$ is in $\Delta_{\mathscr{T}}\mathcal{D}^-(\Lambda_{n-1}(G, \mathscr{X})^{\oplus})$. Note that $\bigoplus_{i\in\mathbb{N}} g_i(X_*)$ is in $\Delta_{\mathscr{T}}\mathcal{D}^-(\Lambda_{n-1}(G, \mathscr{X})^{\oplus})$ as each $g_i(X_*)$ is bounded below at degree m. We now apply the homotopy colimit construction, that we invoked in the proof of Lemma 8.2.1, artifically. We have a sequence of chain maps $g_0(X_*) \overset{\phi_0}{\to} g_1(X_*) \overset{\phi_1}{\to} g_2(X_*) \overset{\phi_2}{\to} ...$ between complexes, where $\phi_i : g_i(X_*) \to g_{i+1}(X_*)$ is given by the identity map at every degree between m and $m+i$ and the zero map at every other degree. Over chain complexes, the direct limit of the $g_i(X_*)$'s is X_* and it follows from the definition of homotopy colimits that, in $\mathcal{D}(Mod\text{-}A\Gamma)$, we have a short exact sequence (see Lemma 8.2.1)

$$0 \to \bigoplus_{i\in\mathbb{N}} g_i(X_*) \overset{\bigoplus_{i\geq 0}(id_{g_i(X_*)} - \phi_i)}{\longrightarrow} \bigoplus_{i\in\mathbb{N}} g_i(X_*) \to X_* \to 0$$

Now note that $\bigoplus_{i\in\mathbb{N}} g_i(X_*), X_* \in \mathcal{D}^-(Mod\text{-}A\Gamma)$ which is a triangulated subcategory of $\mathcal{D}(Mod\text{-}A\Gamma)$ and so the above short exact sequence is a distinguished triangle in $\mathcal{D}^-(Mod\text{-}A\Gamma)$. We can see, in the short exact sequence above, the first two terms are in $\Delta_{\mathscr{T}}\mathcal{D}^-(\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus})$ which is a triangulated subcategory of $\mathcal{D}^-(Mod\text{-}A\Gamma)$, therefore it follows that $X_* \in \Delta_{\mathscr{T}}\mathcal{D}^-(\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus})$.

b) Again, just like we saw in (a), $\Delta_{\mathscr{T}}\mathcal{D}^+(\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus,\Pi}) \subseteq \Delta_{\mathscr{T}}\mathcal{D}^+(\Lambda_n(\Gamma, \mathscr{X})^{\oplus,\Pi})$.

We start with an arbitrary bounded above chain complex $X_* := .. \to 0 \to X_m \to X_{m-1} \to X_{m-2} \to ..$, where each X_i is in $\Lambda_n(\Gamma, \mathscr{X})^{\oplus,\Pi}$, with X_{m-i} in degree $m-i$ for all $i \geq 0$ and zero in every other degree. Now look at the following truncations of X_*.

$g_0(X_*) = .. \to 0 \to X_m \to 0 \to 0 \to ...$

$g_1(X_*) = .. \to 0 \to X_m \to X_{m-1} \to 0 \to ...$

$\vdots$

$g_k(X_*) = .. \to 0 \to X_m \to X_{m-1} \to .. \to X_{m-k} \to 0 \to 0 \to 0 \to ...$

In $g_i(X_*)$, we have the module X_{m-j} in degree $m-j$ for all $j \in \{0, 1, .., i\}$, and zero in every other degree. The chain map $\phi_{k+1} : g_{k+1}(X_*) \to g_k(X_*)$ is given by the identity map in every degree between m and $m-k$ and the zero map in every other degree. As $t_n := t_{n,\mathscr{X}}(G) < \infty$ or $\mathscr{F}\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$-$dim(A\Gamma) < \infty$, we denote either of these quantities by t and we have $\Lambda_n(\Gamma, \mathscr{X}) \subseteq [\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}]_t$, and since by Lemma 2.3.3, for any class of modules $\mathscr{T}$ that is closed under arbitrary direct sums, $[\mathscr{T}]_l$ is closed under arbitrary direct sums as well for any finite l, we have $\Lambda_n(\Gamma, \mathscr{X})^{\oplus} \subseteq [\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}]_t$. Now for any i, $X_i = \Pi_{j\in J} X_{i,j}$, for some indexing set J, where each $X_{i,j} \in \Lambda_n(\Gamma, \mathscr{X})^{\oplus}$, and we have a complex $.. \to 0 \to X_{i,j,t} \to X_{i,j,t-1} \to ... \to X_{i,j,0} \to 0 \to ..$, with $X_{i,j,k} \in \Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$ in degree k for all $k \in \{0, 1, .., t\}$ and zero in every other degree, quasi-isomorphic to the complex with $X_{i,j}$ in degree zero and zero in every other degree. Thus each X_i, when considered as a complex concentrated in degree zero, is in $\Delta_{\mathscr{T}}\mathcal{D}^+(\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus,\Pi})$ - note that here the direct product of exact sequences, each of length t, is still an exact sequence of length t because we are in the module category. So, by Lemma 8.2.2, each $g_i(X_*) \in \Delta_{\mathscr{T}}\mathcal{D}^+(\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus,\Pi})$. Every $g_i(X_*)$ has just the zero module in every degree higher than m, so the bounded above chain complex $\prod_{i\in\mathbb{N}} g_i(X_*)$ is in $\Delta_{\mathscr{T}}\mathcal{D}^+(\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus,\Pi})$. Over chain complexes, the inverse limit of the $g_i(X_*)$'s is X_* and from the definition of homotopy limit in the derived unbounded category, we get the following short exact sequence (see Lemma 8.2.1).

$$0 \to X_* \to \prod_{i\in\mathbb{N}} g_i(X_*) \xrightarrow{\Pi_{i\geq 1}(id_{g_i(X_*)} - \phi_i)} \prod_{i\in\mathbb{N}} g_i(X_*) \to 0$$

All the terms here are in $\mathcal{D}^+(Mod\text{-}A\Gamma)$ which is a triangulated subcategory of $\mathcal{D}(Mod\text{-}A\Gamma)$, so the above short exact sequence is a distinguished triangle in $\mathcal{D}^+(Mod\text{-}A\Gamma)$. Now, note that the last two terms are in $\Delta_{\mathscr{T}}\mathcal{D}^+(\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus,\Pi})$, and therefore X_* is in

$\Delta_{\mathscr{T}}\mathcal{D}^+(\Lambda_{n-1}(\Gamma,\mathscr{X})^{\oplus,\Pi})$ as well since $\Delta_{\mathscr{T}}\mathcal{D}^+(\Lambda_{n-1}(\Gamma,\mathscr{X})^{\oplus,\Pi})$ is a triangulated subcategory of $\mathcal{D}^+(Mod\text{-}A\Gamma)$ by definition.

Thus, $\Delta_{\mathscr{T}}\mathcal{D}^+(\Lambda_n(\Gamma,\mathscr{X})^{\oplus,\Pi}) = \Delta_{\mathscr{T}}\mathcal{D}^+(\Lambda_{n-1}(\Gamma,\mathscr{X})^{\oplus,\Pi})$. Note that $\mathscr{F}\Lambda_\alpha(\Gamma,\mathscr{X})^{\oplus}$-$dim(A\Gamma) \leq \mathscr{F}\Lambda_\beta(\Gamma,\mathscr{X})^{\oplus}$-$dim(A\Gamma)$ whenever $\alpha < \beta$ (similarly $t_{m,\mathscr{X}}(G) \leq t_{n,\mathscr{X}}(G)$ for all $m \leq n$), so $\mathscr{F}\Lambda_{n-1}(\Gamma,\mathscr{X})^{\oplus}$-$dim(A\Gamma) < \infty \implies \mathscr{F}\Lambda_{n-2}(\Gamma,\mathscr{X})^{\oplus}$-$dim(A\Gamma) < \infty$, and now we can show that $\Delta_{\mathscr{T}}\mathcal{D}^+(\Lambda_{n-1}(\Gamma,\mathscr{X})^{\oplus,\Pi}) = \Delta_{\mathscr{T}}\mathcal{D}^+(\Lambda_{n-2}(\Gamma,\mathscr{X})^{\oplus,\Pi})$. We can go all the way down to $\Delta_{\mathscr{T}}\mathcal{D}^+(\Lambda_0(\Gamma,\mathscr{X})^{\oplus,\Pi})$ like this.

c) We first show that for any $n \geq 1$, $\mathscr{T}$-$\langle\Lambda_n(\Gamma,\mathscr{X})\rangle = \mathscr{T}$-$\langle\Lambda_{n-1}(\Gamma,\mathscr{X})\rangle$ which follows quite straightforwardly from the fact that any complex in $\Lambda_n(\Gamma,\mathscr{X})$ is a module from the class of modules $\Lambda_n(\Gamma,\mathscr{X})$ concentrated in degree zero and by Lemma 8.1.3, such a complex in $\Lambda_n(\Gamma,\mathscr{X})$ is quasi-isomorphic to a bounded complex of modules from $\Lambda_{n-1}(\Gamma,\mathscr{X})^{\oplus}$ and is therefore in $\mathscr{T}$-$\langle\Lambda_{n-1}(\Gamma,\mathscr{X})^{\oplus}\rangle$ by Lemma 8.2.2 and since $\mathscr{T}$-$\langle\Lambda_{n-1}(\Gamma,\mathscr{X})^{\oplus}\rangle = \mathscr{T}$-$\langle\Lambda_{n-1}(\Gamma,\mathscr{X})\rangle$ because $\mathscr{T}$-$\langle\Lambda_{n-1}(\Gamma,\mathscr{X})\rangle$ is closed under arbitrary direct sums, we have $\mathscr{T}$-$\langle\Lambda_n(\Gamma,\mathscr{X})\rangle \subseteq \mathscr{T}$-$\langle\Lambda_{n-1}(\Gamma,\mathscr{X})\rangle$. The other direction $\mathscr{T}$-$\langle\Lambda_{n-1}(\Gamma,\mathscr{X})\rangle \subseteq \mathscr{T}$-$\langle\Lambda_n(\Gamma,\mathscr{X})\rangle$ is obvious as $\Lambda_{n-1}(\Gamma,\mathscr{X}) \subseteq \Lambda_n(\Gamma,\mathscr{X}) \subseteq \mathscr{T}$-$\langle\Lambda_n(\Gamma,\mathscr{X})\rangle$ implies $\mathscr{T}$-$\langle\Lambda_{n-1}(\Gamma,\mathscr{X})\rangle \subseteq \mathscr{T}$-$\langle\Lambda_n(\Gamma,\mathscr{X})\rangle$.

Now we show that for any $n \geq 0$, $\mathscr{T}$-$\langle\mathcal{D}(\Lambda_n(\Gamma,\mathscr{X}))\rangle = \mathscr{T}$-$\langle\Lambda_n(\Gamma,\mathscr{X})\rangle$. Again, it is clear that $\mathscr{T}$-$\langle\Lambda_n(\Gamma,\mathscr{X})\rangle \subseteq \mathscr{T}$-$\langle\mathcal{D}(\Lambda_n(\Gamma,\mathscr{X}))\rangle$. Now take an arbitrary unbounded chain complex X_* where each X_i is in $\Lambda_n(\Gamma,\mathscr{X})$. For any fixed m, it can be written as the inverse limit of its truncations given by

$g_{m,0}(X_*) : .. \to X_m \to 0 \to 0 \to ...$

$g_{m,1}(X_*) : .. \to X_m \to X_{m-1} \to 0 \to 0 \to ..$

$\vdots$

$g_{m,k}(X_*) : .. \to X_m \to X_{m-1} \to .. \to X_{m-k} \to 0 \to 0 \to ...$

In $g_{m,k}(X_*)$, we have X_{m-i} in degree $m-i$ for all $m \geq i \geq m-k$. The reason why we have an inverse limit here is because, like in the proof of (b) where we artificially used the short exact sequence used to define homotopy limits in the derived unbounded category, our chain maps are from $g_{m,k+1}(X_*)$ to $g_{m,k}(X_*)$ for all k - the map from $g_{m,k+1}(X_*)$ to $g_{m,k}(X_*)$ is given by the identity map in all degrees strictly higher than $m-k$ and the zero map in every other degree. Each $g_{m,k}(X_*)$ is a bounded below chain complex and, like in the proof of part (a), can be written as the direct limit of its truncations given by

$j_0(g_{m,k}(X_*)) : .. \to 0 \to X_{m-k} \to 0$

$j_1(g_{m,k}(X_*)) : .. \to X_{m-(k-1)} \to X_{m-k} \to 0$

$\vdots$

$j_t(g_{m,k}(X_*)) : .. \to X_{m-(k-t)} \to ... \to X_{m-k} \to 0$

Here we have a direct limit because our chain maps go from $j_t(g_{m,k}(X_*))$ to $j_{t+1}(g_{m,k}(X_*))$ for all t - the map from $j_t(g_{m,k}(X_*))$ to $j_{t+1}(g_{m,k}(X_*))$ is given by the identity map in all degrees between $m-k$ and $m-k+t$ and the zero map in all other degrees. Thus, $X_* = \varprojlim_k \varinjlim_t j_t(g_{m,k}(X_*))$. Now note that each $j_t(g_{m,k}(X_*))$ is a bounded complex of modules from $\Lambda_n(\Gamma, \mathscr{X})$, and so by Lemma 8.2.2, each $j_t(g_{m,k}(X_*))$ is in $\mathscr{T}$-$\langle\Lambda_n(\Gamma, \mathscr{X})\rangle$, and now since $\mathscr{T}$-$\langle\Lambda_n(\Gamma, \mathscr{X})\rangle$ is closed under both products and coproducts by definition, it is closed under both direct limits and inverse limits by Lemma 8.2.1, and therefore X_* is in $\mathscr{T}$-$\langle\Lambda_n(\Gamma, \mathscr{X})\rangle$.

d) Take a bounded chain complex $X_* = 0 \to X_m \to X_{m-1} \to ... \to X_0 \to 0$ where each X_i is in $\Lambda_n(\Gamma, \mathscr{X})$. By Lemma 8.1.3, it follows that each X_i, considered as a chain complex concentrated in degree zero, is quasi-isomorphic to a bounded chain complex in $\mathcal{D}^b(\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus})$, and by Lemma 8.2.2, that bounded chain complex is in $\Delta_{\mathscr{T}}\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$. This proves the first part.

Now note that for any k, we have
$\Lambda_{k-1}(\Gamma, \mathscr{X})^{\oplus} \subseteq \Lambda_k(\Gamma, \mathscr{X})^{\oplus} \subseteq \mathcal{D}^b(\Lambda_k(\Gamma, \mathscr{X})^{\oplus}) \subseteq \Delta_{\mathscr{T}}\mathcal{D}^b(\Lambda_k(\Gamma, \mathscr{X})^{\oplus})$ and that implies that $\Delta_{\mathscr{T}}\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus} \subseteq \Delta_{\mathscr{T}}\mathcal{D}^b(\Lambda_n(\Gamma, \mathscr{X})^{\oplus})$.

Denote $t_{n,\mathscr{X}}(G)$ (or $\mathscr{F}\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$-$dim(A\Gamma)$) by t. From Lemma 8.1.3, we have $\Lambda_n(\Gamma, \mathscr{X}) \subseteq [\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}]$, and when we have the additional assumption that $t < \infty$, we get $\Lambda_n(\Gamma, \mathscr{X}) \subseteq [\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}]_t$ which implies, courtesy Lemma 2.3.3, that $\Lambda_n(\Gamma, \mathscr{X})^{\oplus} \subseteq [\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}]_t$. Using Lemma 8.2.2, we can say that all chain complexes in $\mathcal{D}^b(\Lambda_n(\Gamma, \mathscr{X})^{\oplus})$ are in $\Delta_{\mathscr{T}}\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$, therefore $\Delta_{\mathscr{T}}\mathcal{D}^b(\Lambda_n(\Gamma, \mathscr{X})^{\oplus}) \subseteq \Delta_{\mathscr{T}}\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$. Thus, we have $\Delta_{\mathscr{T}}\mathcal{D}^b(\Lambda_n(\Gamma, \mathscr{X})^{\oplus}) = \Delta_{\mathscr{T}}\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$.

All the vertical equalities follow from Lemma 8.2.2.

Since $t_{n,\mathscr{X}}(G) \geq t_{m,\mathscr{X}}(G)$ (and similarly we have $\mathscr{F}\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$-$dim(A\Gamma) \geq \mathscr{F}\Lambda_{m-1}(\Gamma, \mathscr{X})^{\oplus}$-$dim(A\Gamma)$) for any $m \leq n$, we have $\Lambda_k(\Gamma, \mathscr{X})^{\oplus} \subseteq [\Lambda_{k-1}(\Gamma, \mathscr{X})^{\oplus}]_t$ for all $k \in \{1, 2, .., n\}$, and again using Lemma 8.2.2 the way we used it above, we get that $\Delta_{\mathscr{T}}\Lambda_k(\Gamma, \mathscr{X})^{\oplus} \subseteq \Delta_{\mathscr{T}}\Lambda_{k-1}(\Gamma, \mathscr{X})^{\oplus}$. Since the inclusion in the other direction is obvious, this gives us our horizontal chain of equalities, and we are done. $\square$

Definition 8.2.4. *Let Γ be a group and A a commutative ring. For any chain complex X_*, we denote by $deg_i(X_*)$ the module in degree i of X_*. We make the following definitions.*

a) Let $\mathscr{T} := \mathcal{D}(Mod\text{-}A\Gamma)$, and let $\mathscr{U}$ be a class of objects in $\mathscr{T}$. We denote by $\mathrm{Loc}_{\mathscr{T}}\text{-}\langle\mathscr{U}\rangle$, called the localising subcategory of $\mathscr{T}$ generated by $\mathscr{U}$, the smallest triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$ and closed under coproducts (direct sums), and if this is all of $\mathscr{T}$, we say $\mathscr{U}$ generates $\mathscr{T}$.

We denote by $\mathrm{Coloc}_{\mathscr{T}}\text{-}\langle\mathscr{U}\rangle$, called the colocalising subcategory of $\mathscr{T}$ generated by $\mathscr{U}$, the smallest triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$ and closed under products, and if this is all of $\mathscr{T}$, we say $\mathscr{U}$ cogenerates $\mathscr{T}$. We denote by $\mathscr{T}\text{-}\langle\mathscr{U}\rangle$, as in the statement of Theorem 8.2.3.c., the smallest triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$ closed under products and coproducts.

If $\mathrm{Loc}_{\mathscr{T}}\text{-}\langle\mathscr{U}\rangle = \mathscr{T}$ (resp. $\mathrm{Coloc}_{\mathscr{T}}\text{-}\langle\mathscr{U}\rangle = \mathscr{T}$), we say $\mathscr{U}$ generates (resp. cogenerates) $\mathscr{T}$. If $\mathscr{T}\text{-}\langle\mathscr{U}\rangle = \mathscr{T}$, we say $\mathscr{U}$ generates $\mathscr{T}$ with products and coproducts.

b) Let $\mathscr{T} := \mathcal{D}^+(Mod\text{-}A\Gamma)$, and let $\mathscr{U}$ be a class of objects in $\mathscr{T}$. We denote by $\underline{\mathrm{Loc}}_{\mathscr{T}}\text{-}\langle\mathscr{U}\rangle$ (resp. $\underline{\mathrm{Coloc}}_{\mathscr{T}}\text{-}\langle\mathscr{U}\rangle$) the smallest triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$ satisfying the following property:

If $\{X_*^\lambda\}_{\lambda\in\Lambda}$ is a class of chain complexes in $\mathscr{U}$ such that $\exists n \in \mathbb{Z}$ such that for every $\lambda \in \Lambda$, $deg_i(X_*^\lambda) = 0$, $\forall i > n$, then $\bigoplus_{\lambda\in\Lambda} X_*^\lambda$ (resp. $\Pi_{\lambda\in\Lambda} X_*^\lambda$) is in $\underline{\mathrm{Loc}}_{\mathscr{T}}\text{-}\langle\mathscr{U}\rangle$ (resp. $\underline{\mathrm{Coloc}}_{\mathscr{T}}\text{-}\langle\mathscr{U}\rangle$).

c) Let $\mathscr{T} := \mathcal{D}^-(Mod\text{-}A\Gamma)$, and let $\mathscr{U}$ be a class of objects in $\mathscr{T}$. We denote by $\underline{\mathrm{Loc}}_{\mathscr{T}}\text{-}\langle\mathscr{U}\rangle$ (resp. $\underline{\mathrm{Coloc}}_{\mathscr{T}}\text{-}\langle\mathscr{U}\rangle$) the smallest triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$ satisfying the following property:

If $\{X_*^\lambda\}_{\lambda\in\Lambda}$ is a class of chain complexes in $\mathscr{U}$ such that $\exists n \in \mathbb{Z}$ such that for every $\lambda \in \Lambda$, $deg_i(X_*^\lambda) = 0$, $\forall i < n$, then $\bigoplus_{\lambda\in\Lambda} X_*^\lambda$ (resp. $\Pi_{\lambda\in\Lambda} X_*^\lambda$) is in $\underline{\mathrm{Loc}}_{\mathscr{T}}\text{-}\langle\mathscr{U}\rangle$ (resp. $\underline{\mathrm{Coloc}}_{\mathscr{T}}\text{-}\langle\mathscr{U}\rangle$).

d) For any triangulated category $\mathscr{T}$, and any class of objects in it denoted $\mathscr{U}$, we denote by $\langle\mathscr{U}\rangle$ the smallest triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$. Note that we used a different notation for this in the statement of Theorem 8.2.3, but this $\langle\rangle$ notation is more convenient in the context of generation.

With the aid of the above definitions, we get the following generation results for groups that are themselves in Kropholler's hierarchy (note that in the statement of

Theorem 8.2.3, we did not require the big group Γ to be in Kropholler's hierarchy).

Theorem 8.2.5. *Let Γ be a group in $H_n\mathscr{X}$ for some class of groups $\mathscr{X}$. The following statements hold.*

a) If $\mathscr{T} = \mathcal{D}(Mod\text{-}A\Gamma)$, then

$$\begin{array}{ccc} \mathrm{Loc}_{\mathscr{T}}\text{-}\langle\Lambda_{n-1}(\Gamma,\mathscr{X})\rangle & = \mathrm{Loc}_{\mathscr{T}}\text{-}\langle\Lambda_{n-2}(\Gamma,\mathscr{X})\rangle = \dots = & \mathrm{Loc}_{\mathscr{T}}\text{-}\langle\Lambda_0(\Gamma,\mathscr{X})\rangle \\ \| & & \\ \mathcal{D}(Mod\text{-}A\Gamma) & = \mathscr{T}\text{-}\langle\Lambda_{n-1}(\Gamma,\mathscr{X})\rangle = \quad \dots \quad = & \mathscr{T}\text{-}\langle\Lambda_0(\Gamma,\mathscr{X})\rangle \\ \| & & \\ \mathrm{Coloc}_{\mathscr{T}}\text{-}\langle\Lambda_{n-1}(\Gamma,\mathscr{X})^{\oplus}\rangle & & \end{array}$$

b) If $\mathscr{T} = \mathcal{D}^+(Mod\text{-}A\Gamma)$ and in addition $t_{n,\mathscr{X}}(G)$ or $\mathscr{F}\Lambda_{n-1}(\Gamma,\mathscr{X})^{\oplus}$-$dim(A\Gamma)$ is finite, then

$$\begin{array}{ccc} \underline{\mathrm{Loc}}_{\mathscr{T}}\text{-}\langle\Lambda_{n-1}(\Gamma,\mathscr{X})\rangle & & \\ \| & & \\ \mathcal{D}^+(Mod\text{-}A\Gamma) & = \underline{\mathrm{Coloc}}_{\mathscr{T}}\text{-}\langle\Lambda_{n-1}(\Gamma,\mathscr{X})^{\oplus}\rangle = \dots = & \underline{\mathrm{Coloc}}_{\mathscr{T}}\text{-}\langle\Lambda_0(\Gamma,\mathscr{X})^{\oplus}\rangle \end{array}$$

c) If $\mathscr{T} = \mathcal{D}^-(Mod\text{-}A\Gamma)$, then

$$\begin{array}{ccc} \underline{\mathrm{Coloc}}_{\mathscr{T}}\text{-}\langle\Lambda_{n-1}(\Gamma,\mathscr{X})^{\oplus}\rangle & & \\ \| & & \\ \mathcal{D}^-(Mod\text{-}A\Gamma) & = \underline{\mathrm{Loc}}_{\mathscr{T}}\text{-}\langle\Lambda_{n-1}(\Gamma,\mathscr{X})\rangle = \dots = & \underline{\mathrm{Loc}}_{\mathscr{T}}\text{-}\langle\Lambda_0(\Gamma,\mathscr{X})\rangle \end{array}$$

d) $\mathcal{D}^b(Mod\text{-}A\Gamma) = \langle\Lambda_{n-1}(\Gamma,\mathscr{X})^{\oplus}\rangle$.

If, additionally, $t_{n,\mathscr{X}}(G)$ or $\mathscr{F}\Lambda_{n-1}(\Gamma,\mathscr{X})^{\oplus}$-$dim(A\Gamma)$ is finite (see Remark 8.1.7), then $\mathcal{D}^b(Mod\text{-}A\Gamma) = \langle\Lambda_k(\Gamma,\mathscr{X})^{\oplus}\rangle$, for all $k \geq 0$.

Proof. a) To prove the first horizontal line of equalities, first note that, for any k, $\Lambda_k(\Gamma,\mathscr{X}) \subseteq [\Lambda_{k-1}(\Gamma,\mathscr{X})^{\oplus}]$ by Lemma 8.1.3, so
$\mathrm{Loc}_{\mathscr{T}}\text{-}\langle\Lambda_k(\Gamma,\mathscr{X})\rangle \subseteq \mathrm{Loc}_{\mathscr{T}}\text{-}\langle\Lambda_{k-1}(\Gamma,\mathscr{X})^{\oplus}\rangle = \mathrm{Loc}_{\mathscr{T}}\text{-}\langle\Lambda_{k-1}(\Gamma,\mathscr{X})\rangle$ (the last equality follows from the fact that localising subcategories are closed under arbitrary direct sums (coproducts) by definition).

For the other direction, it is obvious that $\mathrm{Loc}_{\mathscr{T}}\text{-}\langle\Lambda_{k-1}(\Gamma,\mathscr{X})\rangle \subseteq \mathrm{Loc}_{\mathscr{T}}\text{-}\langle\Lambda_k(\Gamma,\mathscr{X})\rangle$ because $\Lambda_{k-1}(\Gamma,\mathscr{X}) \subseteq \Lambda_k(\Gamma,\mathscr{X})$. Note that up to here, we have not used the fact that $\Gamma \in H_n\mathscr{X}$.

Note that since $\Gamma \in H_n\mathscr{X}$, $\Lambda_n(\Gamma,\mathscr{X}) = Mod\text{-}A\Gamma$. We start by observing that the second horizontal line of equalities follow directly from the lower horizontal line of equalities in Theorem 8.2.3.c.

Now take an arbitrary unbounded chain complex of $A\Gamma$-modules $(X_*, d_*) = .. \to X_k \to X_{k-1} \to \dots$ We fix an m. We look at the following truncations.

$j_0(X_*)$: $\ldots \to X_{m+2} \to X_{m+1} \to \mathrm{Ker}(d_m) \to 0 \to 0 \to ..$

$j_1(X_*)$: $\ldots \to X_{m+2} \to X_{m+1} \to X_m \to \mathrm{Ker}(d_{m-1}) \to 0 \to 0 \to ..$

$\vdots$

$j_k(X_*)$: $\ldots \to X_{m+2} \to \ldots \to X_{m-(k-2)} \to X_{m-(k-1)} \to \mathrm{Ker}(d_{m-k}) \to 0 \to ..$

Here in $j_k(X_*)$, there is X_{m-i} in degree $m-i$ for all $i \leq k-1$ and $\mathrm{Ker}(d_{m-k})$ in degree $m-k$ and zero in every other degree. The map f_k between $j_k(X_*)$ and $j_{k+1}(X_*)$ is given by the identity map in every degree bigger than or equal to $m-(k-1)$, the inclusion map in degree $m-k$ and the zero map in every other degree. X_* can be written as the direct limit of these truncations, $\varinjlim_k j_k(X_*)$.

Each $j_k(X_*)$ is a bounded below chain complex, and as shown in the proof of part (b) of Theorem 8.2.3, can be written as the direct limit of their non-canonical truncations $g_i(j_k(X_*))$'s (using the g_i notation from the proof of Theorem 8.2.3.b.), each of which in turn are bounded chain complexes. So we have $X_* = \varinjlim_k \varinjlim_i g_i(j_k(X_*))$. Now note that each $g_i(j_k(X_*))$ is a bounded complex where each module, when considered as a chain complex concentrated in degree zero, is quasi-isomorphic to a bounded complex with modules from $\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$ by Lemma 8.1.3. So, each of the modules in the bounded chain complex $g_i(j_k(X_*))$, for any i and k, is in $\mathrm{Loc}_{\mathscr{T}}$-$\langle \Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus} \rangle$ by Lemma 8.2.2. Therefore, by Lemma 8.2.2 again, each $g_i(j_k(X_*))$ is in $\mathrm{Loc}_{\mathscr{T}}$-$\langle \Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus} \rangle = \mathrm{Loc}_{\mathscr{T}}$-$\langle \Lambda_{n-1}(\Gamma, \mathscr{X}) \rangle$ (the last equality follows from the fact that localising subcategories are closed under arbitrary direct sums by definition). Now localising subcategories are closed under coproducts by definition, so they are closed under direct limits by Lemma 8.2.1, therefore X_* is in $\mathrm{Loc}_{\mathscr{T}}$-$\langle \Lambda_{n-1}(\Gamma, \mathscr{X}) \rangle$.

To prove the result about colocalising subcategories, we start with the same chain complex X_* and the same fixed m and look at the following truncations.

$g_0(X_*)$: $.. \to X_m \to 0 \to 0 \to \ldots$

$g_1(X_*)$: $.. \to X_m \to X_{m-1} \to 0 \to 0 \to \ldots$

$\vdots$

$g_k(X_*)$: $.. \to X_m \to X_{m-1} \to \ldots \to X_{m-k} \to 0 \to 0 \to ..$

Here in $g_k(X_*)$, we have the module X_{m-i} in degree $m-i$ for all $i \leq k$ and zero in every other degree. The map between $g_{k+1}(X_*)$ and $g_k(X_*)$ is given by the identity map in every degree bigger than or equal to $m-k$ and the zero map in every other degree. X_* can be written as an inverse limit of these truncations, $\varprojlim_k g_k(X_*)$. Each

$g_k(X_*)$ is a bounded below chain complex, and they can be written as the inverse limit of their canonical truncations in the following way.

Let (Y_*, δ_*) be a bounded below chain complex where t is the smallest degree with a non-zero module. We define $j_k(Y_*) = .. \to 0 \to \mathrm{Ker}(\delta_{t+k}) \hookrightarrow Y_{t+k} \to Y_{t+k-1} \to ... \to Y_t \to 0$ where every degree i with $t \leq i \leq t+k$ has Y_i, degree $t+k+1$ has $\mathrm{Ker}(\delta_{t+k})$, and every other degree has zero. The map from $j_{k+1}(Y_*)$ and $j_k(Y_*)$ is given by δ_{t+k+1} in degree $t+k+1$, the identity map in every degree between t and $t+k$, and the zero map in every other degree. In this case, $Y_* = \varprojlim_k j_k(Y_*)$.

Using the above information, we can write each $g_k(X_*)$ as $\varprojlim_i j_i(g_k(X_*))$, and therefore we have $X_* = \varprojlim_k \varprojlim_i j_i(g_k(X_*))$. Again, each $j_i(g_k(X_*))$ is a bounded chain complex where each module, when considered as a chain complex concentrated in degree zero, is quasi-isomorphic to a bounded chain complex with modules from $\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$ by Lemma 8.1.3. So $j_i(g_k(X_*))$ is in $\mathrm{Coloc}_{\mathscr{T}}\text{-}\langle \Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus} \rangle$ by Lemma 8.2.2 as colocalising subcategories are triangulated subcategories. Now, colocalising subcategories are closed under products by definition and so they are closed under inverse limits by Lemma 8.2.1, therefore X_* is in $\mathrm{Coloc}_{\mathscr{T}}\text{-}\langle \Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus} \rangle$.

b) Here, the horizontal chain of equalities follows from Theorem 8.2.3.b.

To show that $\mathcal{D}^{+}(Mod\text{-}A\Gamma) = \underline{\mathrm{Loc}}_{\mathscr{T}}\text{-}\langle \Lambda_{n-1}(\Gamma, \mathscr{X}) \rangle$, we start with an arbitrary bounded above chain complex of $A\Gamma$-modules (X_*, d_*) with m being the biggest degree without the zero module, and note that, like we saw in the proof of (a), over chain complexes, X_* can be realised as the direct limit of its truncations $j_{m,k}(X_*) : .. \to 0 \to X_m \to X_{m-1} \to ... \to X_{m-(k-1)} \to \mathrm{Ker}(d_{m-k}) \to 0$. Denote by f_k the chain map between $j_{m,k}(X_*)$ and $j_{m,k+1}(X_*)$ which is given by the identity map in every degree between m and $m-k+1$, the inclusion map in degree $m-k$, and the zero map in every other degree. All $A\Gamma$-modules are in $[\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}]$ by Lemma 8.1.3. So, each of the modules in $j_{m,k}(X_*)$, when considered as a chain complex concentrated in degree zero, is quasi-isomorphic to a bounded chain complex of modules from $\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$ and, by Lemma 8.2.2, is therefore in $\underline{\mathrm{Loc}}_{\mathscr{T}}\text{-}\langle \Lambda_{n-1}(\Gamma, \mathscr{X}) \rangle$ (note that we do not need the $^{\oplus}$ sign here since any module in $\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$, as a chain complex concentrated in degree zero, can be written as a direct sum of chain complexes concentrated in degree zero with each of them having a module from $\Lambda_{n-1}(\Gamma, \mathscr{X})$ in degree zero, and is therefore in $\underline{\mathrm{Loc}}_{\mathscr{T}}\text{-}\langle \Lambda_{n-1}(\Gamma, \mathscr{X}) \rangle$). And since each $j_{m,k}(X_*)$ is a bounded chain complex, we can say using Lemma 8.2.2 again that each $j_{m,k}(X_*)$ is in $\underline{\mathrm{Loc}}_{\mathscr{T}}$-$\langle \Lambda_{n-1}(\Gamma, \mathscr{X})\rangle$. So, in $\mathcal{D}(Mod\text{-}A\Gamma)$, we have the short exact sequence (see Lemma 8.2.1)

$$0 \to \bigoplus_{k\in\mathbb{N}} j_{m,k}(X_*) \xrightarrow{\bigoplus_{i\geq 0}(id_{j_{m,i}(X_*)}-f_i)} \bigoplus_{k\in\mathbb{N}} j_{m,k}(X_*) \to X_* \to 0$$

and we note that the above short exact sequence is a distinguished triangle in $\mathcal{D}^+(Mod\text{-}A\Gamma)$. Now, the first two terms are in $\underline{\mathrm{Loc}}_{\mathscr{T}}$-$\langle \Lambda_{n-1}(\Gamma, \mathscr{X})\rangle$ by definition since each $j_{m,k}(X_*)$ is in $\underline{\mathrm{Loc}}_{\mathscr{T}}$-$\langle \Lambda_{n-1}(\Gamma, \mathscr{X})\rangle$ and is bounded above at degree m. Therefore, X_* is in $\underline{\mathrm{Loc}}_{\mathscr{T}}$-$\langle \Lambda_{n-1}(\Gamma, \mathscr{X})\rangle$ as $\underline{\mathrm{Loc}}_{\mathscr{T}}$-$\langle \Lambda_{n-1}(\Gamma, \mathscr{X})\rangle$ is a triangulated subcategory of $\mathcal{D}^+(Mod\text{-}A\Gamma)$.

c) Again, the horizontal chain of equalities follows from Theorem 8.2.3.a.

To show that $\mathcal{D}^-(Mod\text{-}A\Gamma) = \underline{\mathrm{Coloc}}_{\mathscr{T}}$-$\langle \Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}\rangle$, note that if $(X_i, d_i)_{i\geq m}$ is an arbitrary bounded below chain complex of $A\Gamma$-modules, over chain complexes, it is the inverse limit of its truncations $j_{m,k}(X_*) : .. \to 0 \to \mathrm{Ker}(d_{m+k}) \hookrightarrow X_{m+k} \to ... \to X_m \to 0$ which is bounded and again, like in the proof of part (b), it follows that these truncations are in $\underline{\mathrm{Coloc}}_{\mathscr{T}}$-$\langle \Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}\rangle$. Here, the chain map f_k between $j_{m,k}(X_*)$ and $j_{m,k-1}(X_*)$ is given by the identity map in every degree between m and $m+k-1$, d_{m+k} in degree $m+k$, and the zero map in every other degree. In $\mathcal{D}(Mod\text{-}A\Gamma)$, we have the short exact sequence (see Lemma 8.2.1)

$$0 \to X_* \to \prod_{k\in\mathbb{N}} j_{m,k}(X_*) \xrightarrow{\prod_{i\geq 1}(id_{j_{m,i}(X_*)}-f_i)} \prod_{k\in\mathbb{N}} j_{m,k}(X_*) \to 0$$

Again, we note that this is a distinguished triangle in $\mathcal{D}^-(Mod\text{-}A\Gamma)$, and since the last two terms are clearly in $\underline{\mathrm{Coloc}}_{\mathscr{T}}$-$\langle \Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}\rangle$ by definition, X_* is in $\underline{\mathrm{Coloc}}_{\mathscr{T}}$-$\langle \Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}\rangle$ as well because $\underline{\mathrm{Coloc}}_{\mathscr{T}}$-$\langle \Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}\rangle$ is a triangulated subcategory of $\mathscr{T} = \mathcal{D}^-(Mod\text{-}A\Gamma)$.

d) This follows from Theorem 8.2.3.d. □

The following corollary is obvious from the proof of Theorem 8.2.5.

Corollary 8.2.6. *Fix a commutative ring A. Let Γ be a group and let $\mathscr{X}$ be a class of groups. Assume that for some n, $\Lambda_n(\Gamma, \mathscr{X})^{\oplus}$ is closed under kernels, then* $\mathrm{Loc}_{\mathscr{T}}$-$\langle \Lambda_{n-1}(\Gamma, \mathscr{X})\rangle = \mathrm{Coloc}_{\mathscr{T}}$-$\langle \Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}\rangle$, *where* $\mathscr{T} = \mathcal{D}(Mod\text{-}A\Gamma)$.

Remark 8.2.7. *It follows from the proof of Theorem 8.2.5.a. that for an $H_n\mathscr{X}$-group Γ, we can generate the whole derived unbounded category in three distinct ways from the class $\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$- with coproducts, with products, and with using both products and coproducts - in the language of Definition 8.2.4.a., we can say that in this case* *the whole derived unbounded category is generated, cogenerated and "generated with products and coproducts" by $\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$. We can of course also state this result by replacing n with a limit ordinal α and having in place of $n-1$ some ordinal strictly smaller than α.*

Comparing the class of localising subcategories with the class of colocalising subcategories arising from a given triangulated category with arbitrary products and coproducts is a question of classical interest. The following was proved by Amnon Neeman in [50] and [52].

Theorem 8.2.8. *([50], [52]) Let A be a Noetherian ring. For $\mathscr{U}$ a triangulated subcategory of $\mathcal{D}(Mod\text{-}A)$, write $\phi(\mathscr{U}) := \{X \in \mathcal{D}(Mod\text{-}A) : Hom_{\mathcal{D}(Mod\text{-}A)}(U, X) = 0, \forall U \in \mathscr{U}\}$ and $\psi(\mathscr{U}) := \{X \in \mathcal{D}(Mod\text{-}A) : Hom_{\mathcal{D}(Mod\text{-}A)}(X, U) = 0, \forall U \in \mathscr{U}\}$. Then,*

a) If $\mathscr{U}$ is a localising subcategory of $\mathcal{D}(Mod\text{-}A)$, then $\phi(\mathscr{U})$ is colocalising, and if $\mathscr{U}$ is a colocalising subcategory of $\mathcal{D}(Mod\text{-}A)$, then $\psi(\mathscr{U})$ is localising. Also, if $\mathscr{U}$ is localising, then $\psi(\phi(\mathscr{U})) = \mathscr{U}$.

b) The assignment $\mathscr{U} \mapsto \phi(\mathscr{U})$ induces a bijection between the collection of localising subcategories of $\mathcal{D}(Mod\text{-}A)$ and the collection of colocalising subcategories of $\mathcal{D}(Mod\text{-}A)$.

Remark 8.2.9. *What Theorem 8.2.8 shows is that if the group ring is Noetherian, then we do not get any "new" colocalising subcategories other than the ones we get from the localising subcategories. For groups in Kropholler's hierarchy, group rings over arbitrary rings are seldom Noetherian. (To see an easy example, note that if we take a group not all of whose subgroups are finitely generated, then its group ring over any field is not Noetherian. Theorem 1.2.3 tells us, for example, that the free abelian group of rank $\aleph_\omega$, where ω is the first infinite ordinal, is in $H_3\mathscr{F}$ - so this group does not have a Noetherian group ring over fields.) So it is nice to see in Theorem 8.2.5.a that as far as the localising and colocalising subcategories generated by the smallest direct-sum closed class containing modules induced from subgroups in lower levels on the hierarchy are concerned, they coincide with each other. Can we find an example of a group Γ in $H_n\mathscr{X}$, for some n and $\mathscr{X}$, and a commutative ring A, such that $A\Gamma$ is not Noetherian, where we get some colocalising subcategories of $\mathcal{D}(Mod\text{-}A\Gamma)$ that do not come from localising subcategories the way shown in Theorem 8.2.8?*

We end this section with the following remark and a subsequent pair of questions.

Remark 8.2.10. *For simplicity, in this remark when there is no ambiguity over* Γ*, we shall denote the smallest direct sum closed class of modules induced up from* $H_\alpha\mathscr{X}$*-subgroups of* Γ *by* Λ_α*, where* α *is some ordinal.*

Can we replace n *in the statement of Theorem 8.2.5 by a limit ordinal* α*? If* $\Gamma \in H_\alpha\mathscr{X}$*, for some limit ordinal* α*, then it follows from Definition 1.2.1 that* $\Gamma \in H_\beta\mathscr{X}$ *for some successor ordinal* $\beta < \alpha$*. It follows from the arguments in the proof of Theorem 8.2.5 that for any* $H_\alpha\mathscr{X}$*-group* Γ *where* $\mathscr{X}$ *is a class of groups, we have the following equality of localising subcategories and filtration of colocalising subcategories of* $\mathcal{D}(Mod\text{-}A\Gamma)$ *for some fixed commutative ring* A*, where* δ *is the biggest limit ordinal strictly smaller than* α *:*

$$\mathrm{Loc}_{\mathscr{T}}\text{-}\langle\Lambda_\delta\rangle = \mathrm{Loc}_{\mathscr{T}}\text{-}\langle\Lambda_{\delta+1}\rangle = ... = \mathrm{Loc}_{\mathscr{T}}\text{-}\langle\Lambda_{\beta-1}\rangle = \mathcal{D}(Mod\text{-}A\Gamma)$$

and

$$Coloc_{\mathscr{T}}\text{-}\langle\Lambda_\delta\rangle \subseteq Coloc_{\mathscr{T}}\text{-}\langle\Lambda_{\delta+1}\rangle \subseteq ... \subseteq Coloc_{\mathscr{T}}\text{-}\langle\Lambda_{\beta-1}\rangle = \mathcal{D}(Mod\text{-}A\Gamma)$$

where $\mathscr{T} = \mathcal{D}(Mod\text{-}A\Gamma)$ *and each of the inclusion functors in the filtration is triangulated. The main reason, in short, as to why we do not get an equality for the generated colocalising subcategories is because they need not be closed under coproducts.*

Denoting ω *to be the first infinite ordinal, if we now take* $\alpha = \omega.n = \omega + \omega + ... + \omega$ *(*n *times), and we take a group* $\Gamma \in H_\alpha\mathscr{X}$ *for some* $\mathscr{X}$ *but not in* $H_\beta\mathscr{X}$ *for any* $\beta < \alpha$ *(note that we do not yet have examples of such groups - the best result that we have in the literature is that there are groups in* $H_\alpha\mathscr{F} \setminus H_{<\alpha}\mathscr{F}$ *for all ordinals* α *smaller than the first uncountable ordinal, this result is from [36]), then we do get a filtration of localising subcategories that need not be a chain of equalities (this is because Lemma 8.1.3 need not hold when* α *is not a successor ordinal): for any integer* k*, let* $Loc_{\mathscr{T}}\text{-}\langle\Lambda^{[k-1,k]}\rangle := Loc_{\mathscr{T}}\text{-}\langle\Lambda_\beta\rangle$*, for any successor ordinal* β *satisfying* $\omega.(k-1) < \beta < \omega.k$*. This is well defined because for any two successor ordinals* β_1 *and* β_2 *between* $\omega.(k-1)$ *and* $\omega.k$*,* $Loc_{\mathscr{T}}\text{-}\langle\Lambda_{\beta_1}\rangle = Loc_{\mathscr{T}}\text{-}\langle\Lambda_{\beta_2}\rangle$*, this follows from the above chain of equalities. Now, we have the following filtration of localising subcategories:*

$$Loc_{\mathscr{T}}\text{-}\langle\Lambda^{[0,1]}\rangle \subseteq Loc_{\mathscr{T}}\text{-}\langle\Lambda^{[1,2]}\rangle \subseteq ... \subseteq Loc_{\mathscr{T}}\text{-}\langle\Lambda^{[n-1,n]}\rangle = \mathcal{D}(Mod\text{-}A\Gamma)$$

We end this remark with a little comment on how the above filtration can be useful in studying the Krull dimension of the derived unbounded categories of groups in Kropholler's hierarchy which has not been studied at all before. Before providing the definition for the Krull dimension of triangulated categories, we recall that thick subcategories of a triangulated category are defined as triangulated subcategories closed under summands (see Definition 8.4.24). For any two subcategories $\mathscr{I}_1, \mathscr{I}_2$ of $\mathscr{T}$, we define $\mathscr{I}_1 * \mathscr{I}_2$ to be the full subcategory of $\mathscr{T}$ consisting of objects M such that there is a distinguished triangle $M_1 \to M \to M_2 \rightsquigarrow$ with $M_i \in \mathscr{I}_i$. Rouqier, in [58], defines a thick subcategory $\mathscr{I}$ of $\mathscr{T}$ to be irreducible if given two thick subcategories $\mathscr{I}_1$ and $\mathscr{I}_2$ of $\mathscr{I}$ such that $\mathscr{I}$ is the smallest thick subcategory of $\mathscr{T}$ containing $\mathscr{I}_1 * \mathscr{I}_2$, then at least one of the $\mathscr{I}_i$'s is $\mathscr{I}$. The Krull dimension of $\mathscr{T}$ is the length of the maximal chain of thick irreducible subcategories $0 \neq \mathscr{I}_0 \subset \mathscr{I}_1 \subset ... \subset \mathscr{I}_n = \mathscr{T}$. Now, can we use the above filtration of localising subcategories with the same Γ to comment on the Krull dimension of $\mathcal{D}(Mod\text{-}A\Gamma)$? Localising subcategories are thick (this is standard knowledge, see Lemma 8.4.14), and making the inclusions strict can be possible with the choice of our group or possibly with making sure that at every level of Kropholler's hierarchy below α, Γ has a subgroup which is not in any lower level. The most non-trivial part will be checking irreducibility of the localising subcategories, but that can possibly be handled with looking at the irreducible components.

We end this section with the following question on groups that are not in Kropholler's hierarchy.

Question 8.2.11. *a) The two filtrations mentioned in Remark 8.2.10, except the last equality where we have one of the subcategories in the filtration being equal to the whole derived unbounded category, hold for any arbitrary group. Now, let Γ be Thompson's group given by $\langle x_0, x_1, x_2, .. : x_k^{-1} x_n x_k = x_{n+1}$ for $k < n\rangle$. We know this group is not in $H\mathscr{F}$ (see [41]). Now, using the notations of Remark 8.2.10 with $\mathscr{X} = \mathscr{F}$, does any of the filtrations $\mathrm{Loc}_{\mathscr{T}}\text{-}\langle \Lambda^{[0,1]} \rangle \subseteq \mathrm{Loc}_{\mathscr{T}}\text{-}\langle \Lambda^{[1,2]} \rangle \subseteq ... \subseteq \mathrm{Loc}_{\mathscr{T}}\text{-}\langle \Lambda^{[n-1,n]} \rangle \subseteq ...$ and $\mathrm{Coloc}_{\mathscr{T}}\text{-}\langle \Lambda_0 \rangle \subseteq \mathrm{Coloc}_{\mathscr{T}}\text{-}\langle \Lambda_1 \rangle \subseteq ... \subseteq \mathrm{Coloc}_{\mathscr{T}}\text{-}\langle \Lambda_{n-1} \rangle \subseteq \mathrm{Coloc}_{\mathscr{T}}\text{-}\langle \Lambda_n \rangle \subseteq ...$ eventually stabilise where $\mathscr{T} = \mathcal{D}(Mod\text{-}A\Gamma)$ and A is a fixed commutative ring?*

b) Again, using the notation from Remark 8.2.10, is there an example of a group Γ for which one of the filtrations, as mentioned in (a), eventually stabilises but the other one does not?

c) In Theorem 8.2.5.d, we see that if $\Gamma \in H_n\mathscr{X}$, for any $\mathscr{X}$ and A is some commutative ring, then $\mathcal{D}^b(Mod\text{-}A\Gamma) = \langle \Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus} \rangle$. Are there examples of groups not in $H_n\mathscr{X}$, for some $\mathscr{X}$, satisfying this result? Also, are there examples of groups $\Gamma \notin H\mathscr{X}$, for some $\mathscr{X}$, such that $\mathcal{D}^b(Mod\text{-}A\Gamma) = \langle \Lambda(\Gamma, \mathscr{X})^{\oplus} \rangle$, for any commutative ring A, where $\Lambda(\Gamma, \mathscr{X}) := \{Ind_{\Gamma'}^{\Gamma}(M) : M \in Mod\text{-}A\Gamma', \Gamma' \in H\mathscr{X}\}$?

8.3 Ending comments on generation of derived categories

In Theorem 8.2.3 and Theorem 8.2.5, we have seen that with derived unbounded and derived bounded below categories, when we look to generate them with the smallest direct-sum closed class of modules induced up from subgroups lower down the hierarchy, we can go all the way down to the zeroth level whereas in the case of derived bounded above, we need additional conditions to get down even one level. So working with generating the derived bounded category seems like the best bet. Below, we prove an easy result showing that if we introduce a similar definition for steps of generation like we did for modules in Section 2.1, we see that the number of steps taken to generate anything in the n-th hierarchy from the 0-th hierarchy is dependent exponentially on n but linearly on the length of the chain complex that we are generating.

Theorem 8.3.1. *Define generation of chain complexes in the following way, similar to the way we defined generation of modules: a chain complex X^* is generated from a class of chain complexes $\mathscr{T}$ in 0 steps iff $X^* \in \mathscr{T}$ and if we have a short exact sequence of chain complexes $0 \to X_1^* \to X_2^* \to X_3^* \to 0$ where the chain complex X_i^* for any two i, say $i = j, k \in \{1, 2, 3\}$, is generated from $\mathscr{T}$ in a_i steps, then the third chain complex is generated from $\mathscr{T}$ in $a_j + a_k + 1$ steps.*

Let Γ be a group and $\mathscr{X}$ a class of groups. Assume that $t_{n,\mathscr{X}}(G)$ or alternatively $\mathscr{F}\Lambda_{n-1}(\Gamma, \mathscr{X})^{\oplus}$-dim$(A\Gamma)$ is finite, denote any of these by t. Then a bounded chain complex of length m, where each module is in $\Lambda_n(\Gamma, \mathscr{X})^{\oplus}$, can be generated from $\Lambda_0(\Gamma, \mathscr{X})^{\oplus}$ in $m(t+1)^n - 1$ steps.

Proof. We proceed by induction on m. When $m = 1$, we have one module from the class $\Lambda_n(\Gamma, \mathscr{X})^{\oplus}$ in one degree and zero in every other degree. The result follows from Theorem 8.1.5 and Remark 8.1.7. Assume that for all $k \leq m$, all bounded chain complexes, where each module is in $\Lambda_n(\Gamma, \mathscr{X})^{\oplus}$, of length k are generated from $\Lambda_0(\Gamma, \mathscr{X})^{\oplus}$ (note that we are considering $\Lambda_0(\Gamma, \mathscr{X})^{\oplus}$ to be a class of complexes by considering all of the modules contained in it as chain complexes concentrated in degree zero) in $k(t+1)^n - 1$ steps - this is our induction hypothesis. Now let X_* be a bounded chain complex of length $m+1$. Like in the proof of Lemma 8.2.2, we can put X_* in the middle of a short exact where the other two terms are bounded chain complexes of lengths m and 1 respectively, so those other complexes are generated from $\Lambda_0(\Gamma, \mathscr{X})^{\oplus}$ in $m(t+1)^n - 1$ and $(t+1)^n - 1$ steps respectively by our induction hypothesis. Thus, X_* can be generated from $\Lambda_0(\Gamma, \mathscr{X})^{\oplus}$ in $m(t+1)^n - 1 + (t+1)^n - 1 + 1 = (m+1)(t+1)^n - 1$ steps. This ends our induction. $\square$

In the diagram below, we have gathered some generation results in the derived category and their analogous results in the module category. Note that, for derived bounded above categories here, what we have is cogeneration because we cannot generate in that case without closing our generating classes under products. The diagram below is informal, so we are a little loose with the word "generates" for every case. The inclusion symbols on the first two columns on the right denote triangulated inclusion provided, for the terms those symbols are connecting, we consider the smallest triangulated subcategory of the relevant derived category containing those terms. Also, it is worth noting that when we are talking about generating a derived category on $\Lambda_n(\Gamma, \mathscr{X})$, we are talking about generating the smallest triangulated subcategory of the relevant whole derived category.

Module Category	Derived Category
Anything in $\Lambda_n(\Gamma, \mathscr{X})^\oplus$ can be generated from $\Lambda_0(\Gamma, \mathscr{X})^\oplus$ in finite (given by $(1+t_0)(1+t_1)..(1+t_n)-1$) steps with $t_0, t_1, .., t_n < \infty$	$\mathcal{D}^+(\Lambda_n(\Gamma, \mathscr{X})^\oplus)$ $\veebar$ $\vdots \; \Uparrow$ (generates) $\veebar$ $\mathcal{D}^+(\Lambda_0(\Gamma, \mathscr{X})^\oplus)$ with $t_0, t_1, .., t_n < \infty$
$\Lambda_n(\Gamma, \mathscr{X})^\oplus \subseteq [\Lambda_{n-1}(\Gamma, \mathscr{X})^\oplus]_\infty \subseteq [[\Lambda_{n-2}(\Gamma, \mathscr{X})^\oplus]^\oplus_\infty]_\infty \subseteq \dots \subseteq \underbrace{[[\dots[}_{n \text{ times}}\Lambda_0(\Gamma, \mathscr{X})\underbrace{{}^\oplus]^\oplus_\infty]_\infty \dots{}^\oplus]_\infty}_{n \text{ times}}$	$\mathcal{D}^-(\Lambda_n(\Gamma, \mathscr{X})^\oplus)$ $\veebar$ $\vdots \; \Uparrow$ (generates) $\veebar$ $\mathcal{D}^-(\Lambda_0(\Gamma, \mathscr{X})^\oplus)$
$\Lambda_n(\Gamma, \mathscr{X}) \subseteq [\Lambda_{n-1}(\Gamma, \mathscr{X})^\oplus]$	$\Lambda_{n-1}(\Gamma, \mathscr{X})^\oplus$ generates $\mathcal{D}^b(\Lambda_n(\Gamma, \mathscr{X}))$
$\Lambda_n(\Gamma, \mathscr{X}) \subseteq \underbrace{[[\dots[}_{n \text{ times}}\Lambda_0(\Gamma, \mathscr{X})\underbrace{{}^\oplus]^\oplus]\dots{}^\oplus]}_{n \text{ times}}$	$\Lambda_0(\Gamma, \mathscr{X})$ generates $\mathcal{D}(\Lambda_n(\Gamma, \mathscr{X}))$ with products and coproducts

8.4 Generation in stable module categories of infinite groups

Our results on the generation of the derived categories for groups in Kropholler's hierarchy (Theorem 8.2.3.d, Theorem 8.2.5.d) can be used to comment on the generation of stable module categories for a large family of infinite groups. We need to provide some background material on this first, and we start with the stable module categories of finite groups.

8.4.1 Stable module categories of finite groups

Definition 8.4.1. *For a finite group G, and a field k whose characteristic divides the order of G, define the stable module category of G, denoted $StMod(kG)$, as having the same objects as $Mod\text{-}kG$ and its morphisms are given by quotienting out those module* *homomorphisms that factor through some projective kG-module.*

For finite groups, stable module categories are usually studied over fields of prime characteristic.

Theorem 8.4.2. *(see Theorem* 2.31 *of [15]) Let G be a finite group and let k be a field whose characteristic divides the order of G. Then, $StMod(kG)$ is triangulated, with the suspension given by Ω^{-1}.*

Note that since finite groups admit complete resolutions, we have the Ω^{-1} functor for finite groups, and also it follows from the definition of the stable module category that the Ω^{-1} functor is well-defined in the stable module category. It is now a standard fact that stable module categories of finite groups are compactly generated, i.e. generated by the compact objects in the stable module category which is a triangulated category. We provide a definition of compact objects of triangulated categories below.

Definition 8.4.3. *Let $\mathscr{T}$ be a triangulated category with coproducts. An object $C \in \mathscr{T}$ is called compact if $Hom_{\mathscr{T}}(C, ?)$ commutes with coproducts. We say $\mathscr{T}$ is compactly generated if the smallest localising subcategory of $\mathscr{T}$ containing all the compact objects is the whole of $\mathscr{T}$.*

Theorem 8.4.4. *(see Theorem* 2.31 *of [15]) Let G be a finite group and let k be a field whose characteristic divides the order of G. Then, $StMod(kG)$ is compactly generated and the class of compact objects is precisely the class of finitely generated modules.*

The localising subcategories of $StMod(kG)$ for finite groups has been classified by Benson, Iyengar and Krause in [14]. For infinite groups that admit complete resolutions, we can similarly define stable module categories.

8.4.2 Stable module categories of infinite groups

Now let Γ be an infinite group that admits complete resolutions over a commutative ring A of finite global dimension. Almost all of the material in this section up to Theorem 8.4.12 is from [60] which is connected to the paper [46].

Definition 8.4.5. *(see [60]) Define a category $ModProj(A\Gamma)$ in which the objects are the same objects as in $Mod\text{-}A\Gamma$. For any two objects M, N in $ModProj(A\Gamma)$, we* *define the Hom-sets of $ModProj(A\Gamma)$ in the following way.*

$$Hom_{ModProj(A\Gamma)}(M,N) = Hom_{Mod\text{-}A\Gamma}(M,N)/PHom_{Mod\text{-}A\Gamma}(M,N)$$

where $PHom_{Mod\text{-}A\Gamma}(M,N)$ is the class of all morphisms $f : M \to N$ such that f is the composition of two morphisms $g : M \to P$ and $h : P \to N$ for some projective $A\Gamma$-module P.

Remark 8.4.6. *Note that, in comparison with Definition 8.4.1, $ModProj(k\Gamma)$ as introduced in Definition 8.4.5 coincides with the stable module category of Γ when Γ is finite, with k being a field whose characteristic divides the order of Γ.*

In $ModProj(A\Gamma)$, if we have a morphism $f : M \to N$, the syzygy functor, Ω, induces a map between $\Omega(M)$ and $\Omega(N)$. It is clear that for any object M in $ModProj(A\Gamma)$, $\Omega(M)$ is well-defined upto isomorphism. The following is clear now.

Lemma 8.4.7. *Ω is a functor from $ModProj(A\Gamma)$ to itself.*

Definition 8.4.8. *We define the stable module category of $A\Gamma$-modules, written $Stab(A\Gamma)$ (to distinguish from the way we write stable module categories for finite groups), by stating it has the same objects as $Mod\text{-}A\Gamma$ and for any two objects $M, N \in Stab(A\Gamma)$,*

$$Hom_{Stab(A\Gamma)}(M,N) = \varinjlim_{\Omega} Hom_{ModProj(A\Gamma)}(\Omega^n(M), \Omega^n(N))$$

.

Recall that since Γ admits complete resolutions and since A is of finite global dimension, all $A\Gamma$-modules admit complete resolutions (see Theorem 4.1.3). In [46], the following was shown.

Theorem 8.4.9. *(Theorem 3.9 of [46]) Any complex in $\mathcal{D}^b(Mod\text{-}A\Gamma)$ admits a complete resolution.*

We now need to expand a bit on $\Omega^0(M)$ for a given $A\Gamma$-module M.

Definition 8.4.10. *Fix an $A\Gamma$-module M. Take a complete resolution (F_*, d_*) of M, and define $\Omega^t(M) := \mathrm{Ker}(d_{t-1})$. Note that t can be negative.*

Remark 8.4.11. *Note that with the notations of Definition 8.4.10, $\Omega^0(M)$ need not be the same as M in the module category, however $\Omega^0(M)$ and M are isomorphic in the stable category. We have a natural map $f : \Omega^0(M) \to M$ such that $\Omega^{>>0}(f) = id$.*

We now provide the following characterisation of the stable module category of Γ in terms of other known triangulated categories. It was proved in [46] with Γ being of type Φ over A, but the proof works fine if it is just known that Γ admits complete resolutions over A.

Theorem 8.4.12. *(Theorem 3.7 of [60], also see Theorem 3.10 of [46]) The following categories are equivalent as triangulated categories. Here, $\mathcal{D}^b(Proj\text{-}A\Gamma)$ denotes the derived category of bounded complexes of projective $A\Gamma$-modules.*

a) $Stab(A\Gamma)$.

b) $\mathcal{D}^b(Mod\text{-}A\Gamma)/\mathcal{D}^b(Proj\text{-}A\Gamma)$.

c) The category of acyclic complexes of projectives with the morphisms being given by chain homotopies.

d) The category of Gorenstein projective $A\Gamma$-modules with the morphisms being given by the morphisms of $ModProj(A\Gamma)$.

Here, the $(a) \to (b)$ map is given by considering modules as complexes concentrated in degree zero; the $(b) \to (c)$ map is given by complete resolutions (see Theorem 8.4.9), the $(c) \to (d)$ map is given by Ω^0, and the $(d) \to (a)$ map is given by inclusion, and the composition of these maps in this order is isomorphic to the identity map.

We are now well settled to state the main result of this section. To do so, we need to first go through some technical definitions all of which are from [51].

Definition 8.4.13. *If we have a triangulated category $\mathscr{T}$ that admits coproducts of small sets of objects, for any regular cardinal α, we define α-localising subcategories of $\mathscr{T}$ to be triangulated subcategories of $\mathscr{T}$ that are closed under taking fewer than α many coproducts. The α-localising subcategory of $\mathscr{T}$ generated by $\mathscr{U}$, written $\langle \mathscr{U} \rangle^\alpha$, where $\mathscr{U}$ is a class of objects in $\mathscr{T}$, is the smallest triangulated subcategory of $\mathscr{T}$ closed under taking fewer than α many coproducts containing $\mathscr{U}$.*

Of course, if $\alpha = \aleph_0$, then taking fewer than α many coproducts means taking finite coproducts. The following lemma, which provides a nice application of this terminology, is standard knowledge in localising categories.

Lemma 8.4.14. *Let $\mathscr{T}$ be a triangulated category that admits arbitrary coproducts. Then, α-localising subcategories are thick if $\alpha > \aleph_0$.*

Proof. Let $\mathscr{U}$ be an α-localising subcategory of $\mathscr{T}$ where $\alpha > \aleph_0$. Let $X \oplus Y \in \mathscr{U}$. Then, since $(X \oplus Y) \oplus (X \oplus Y) \oplus ..$ is isomorphic to $X \oplus (Y \oplus X) \oplus (Y \oplus X) \oplus ..$, we have a triangle $X \to (X \oplus Y)^{(\mathbb{N})} \to (X \oplus Y)^{(\mathbb{N})} \to \Sigma X$ where Σ is the suspension. Here, $(X \oplus Y)^{(\mathbb{N})} \in \mathscr{U}$ because $\mathscr{U}$ is an α-localising subcategory of $\mathscr{T}$ where $\alpha > \aleph_0$. Thus, X is in $\mathscr{U}$.

□

For any regular cardinal α, there is a process of associating to a triangulated category $\mathscr{T}$ admitting coproducts of small sets of objects a canonically defined α-localising subcategory $\mathscr{T}^\alpha$ (see Chapter 1 of [51] for details). We are not going to provide the definition of $\mathscr{T}^\alpha$ here. However, if $\mathscr{T}$ is "generated" by a class of objects $\mathscr{U}$ in our definition which is the strongest notion of generation (in our Definition 8.2.4.d., we didn't directly use the word "generates" for the derived bounded category; so, to clarify: in our definition, $\mathscr{U}$ generates $\mathscr{T}$ if the smallest triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$ is $\mathscr{T}$, and in general we write $\langle \mathscr{U} \rangle$ for the smallest triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$), then $\langle \mathscr{U} \rangle^\alpha$, meaning the canonically associated α-localising subcategory of $\langle \mathscr{U} \rangle$, coincides with the α-localising subcategory of $\mathscr{T}$ generated by $\mathscr{U}$ as in Definition 8.4.13 (see Definition 1.12 and Lemma 1.13 of [51]). In [51], Neeman uses the generation sign $\langle\rangle$ to mean $\bigcup_\alpha \langle\rangle^\alpha$, but again our notion of generation is stronger so we need not worry.

Now, assume that in addition to admitting coproducts of small sets of objects, $\mathscr{T}$ also has small Hom-sets. Now, if for some regular cardinal α, $\mathscr{T}^\alpha$ is essentially small and $\mathscr{T}^\alpha$ generates $\mathscr{T}$ in Neeman's sense, then $\mathscr{T}$ is said to be *well-generated* (Definition 1.15 of [51]). Now if we take $\mathscr{T} = \mathcal{D}^b(Mod\text{-}A\Gamma)$ where Γ is in $H_n\mathscr{F}$, then $\mathscr{T}$ satisfies all these properties (due to Theorem 8.4.12), with $\alpha = \aleph_0$ and $\mathscr{T} = \langle \Lambda_{n-1}(G, \mathscr{F})^\oplus \rangle$ (we are using the symbol $\langle\rangle$ in our sense which is stronger than the sense in which it is used by Neeman so we are fine). So, $\mathscr{T}^\alpha = \langle \Lambda_{n-1}(G, \mathscr{F})^\oplus \rangle^\alpha = \mathscr{T}$ (the last equality follows from the fact that the α-localising subcategory generated by $\Lambda_{n-1}(G, \mathscr{F})^\oplus$ is $\mathscr{T}$ - see Theorem 8.2.5.d), and it follows that $\mathcal{D}^b(Mod\text{-}A\Gamma)$ is well-generated, and using the main theorem in the introduction of [40], we get that $\mathcal{D}^b(Mod\text{-}A\Gamma)/\mathcal{D}^b(Proj\text{-}A\Gamma)$ is well-generated. This gives us the following theorem using Theorem 8.4.12.

Theorem 8.4.15. *Let Γ be an $H_n\mathscr{F}$-group that admits complete resolutions over a* *commutative ring A of finite global dimension. Then, $Stab(A\Gamma)$ is well-generated.*

Remark 8.4.16. *Well-generation is a generalized version of compact generation (see Chapter 1 of [51]). So, Theorem 8.4.15 generalizes the analogous result for finite groups, Theorem 8.4.4.*

We seem to be seeing a combination of two classes of groups in the statement of Theorem 8.4.15. One being $H_n\mathscr{F}$ and the other being groups that admit complete resolutions over A with A being a commutative ring of finite global dimension. It is important to note that these two classes are not disconnected as we have already seen in Chapter 5 - it is known that all groups in $H\mathscr{F}$ that admit complete resolutions over any commutative ring A of finite global dimension are of type Φ over A (see Proposition 5.1.12) and although $H_1\mathscr{F} \subseteq \mathscr{F}_{\phi,A}$ is known (see Lemma 5.1.7), whether $\mathscr{F}_{\phi,\mathbb{Z}} = H_1\mathscr{F}$ is still open (see Conjecture 4.1.12 or Conjecture 5.1.13).

We end with the following question. The case of finite groups was handled in [14].

Question 8.4.17. *For any fixed commutative ring A of finite global dimension, classify the localizing subcategories of $Stab(A\Gamma)$, where Γ is an $H_n\mathscr{F}$-group, for some positive integer n, that admits complete resolutions over A.*

8.4.3 Stable module categories with a finiteness property

It is well-known that for a finite group G and a field k, the class of finitely generated kG-modules forms a triangulated subcategory of $St.Mod(kG)$, usually denoted $st.mod(kG)$.

In this section, we take an $LH\mathscr{F}$-group Γ that admits complete resolutions over a commutative ring A of finite global dimension, and consider the class of all $A\Gamma$-modules of type FP_∞, denoted $\mathscr{F}\mathscr{P}$. Then, we look at the smallest triangulated subcategory of $Stab(A\Gamma)$ containing $\mathscr{F}\mathscr{P}$, denoted $stab(A\Gamma)$, and prove a generation property admitted by it. Stable module categories of infinite groups with some additional finiteness properties on the modules were partially considered in [12], however we are not using the definitions of the stable category used in [12].

Again, we fix a commutative ring A that is of finite global dimension. Note that, the objects in $stab(A\Gamma)$ can be given an easy characterisation:

Lemma 8.4.18. *Let $\mathscr{M}$ be the class of all $A\Gamma$-modules M which are eventually of type FP_∞ in the module category, i.e. there exists a non-negative integer n such that $\Omega^n(M)$ is of type FP_∞. Then,*

a) In $Stab(A\Gamma)$, $\mathscr{M}$ is a triangulated subcategory of $Stab(A\Gamma)$. Note that when we consider $\mathscr{M}$ as a class of modules in $Stab(A\Gamma)$, $\mathscr{M}$ contains all modules that are stably isomorphic to modules which are eventually of type FP_∞ in the module category.

b) As triangulated subcategories of $Stab(A\Gamma)$, $\mathscr{M} = stab(A\Gamma)$.

Proof. a) Consider $\mathscr{M}$ as a class of modules in $Stab(A\Gamma)$. Since Ω^{-1} is the suspension functor of $Stab(A\Gamma)$, we need to show that $M \in \mathscr{M}$ forces $\Omega^{-1}(M) \in \mathscr{M}$ and that for any short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ where two of the three modules are in $\mathscr{M}$, the third one is in $\mathscr{M}$ as well.

Let $M \in \mathscr{M}$. We have that $\Omega^n(M) \in \mathscr{M}$, for some n, then for $N = \Omega^{-1}(M)$, $\Omega^{n+1}(N) \in \mathscr{M}$.

Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence where 2 of M_1, M_2, M_3 are of eventually type FP_∞. So, in the module category, two of M_1, M_2 and M_3 admit projective resolutions that are of eventually finite type, and therefore the third module does as well.

b) Any module M of type FP_∞ is in $\mathscr{M}$ as $\Omega^0(M)$ is isomorphic to M in $Stab(A\Gamma)$. Therefore, the smallest triangulated subcategory of $Stab(A\Gamma)$ containing all modules that are of type FP_∞ in the module category is contained in $\mathscr{M}$, i.e. $stab(A\Gamma) \subseteq \mathscr{M}$.

Now, take a module $M \in \mathscr{M}$. Then, for some non-negative n, $\Omega^n(M) \in stab(A\Gamma)$. Since $stab(A\Gamma)$ is a triangulated subcategory of $Stab(A\Gamma)$, by repeated applications of the suspension functor Ω^{-1}, we get that $\Omega^0(M) \in stab(A\Gamma)$. Thus, $M \in stab(A\Gamma)$ as M is isomorphic to $\Omega^0(M)$ in the stable category. □

Before going forward, we need to define two classes of modules - *completely finitary* modules and *polybasic* modules.

Definition 8.4.19. *(defined over $A = \mathbb{Z}$ in Definition 2.1 of [33]) Let Γ be a group. An $A\Gamma$-module M is called completely finitary if the functors $\widehat{Ext}^*_{A\Gamma}(M, ?)$ commute with all filtered colimit systems of coefficient modules.*

Definition 8.4.20. *(made over $\mathbb{Z}$ in Definition 2.6 of [33]) Let Γ be a group. An $A\Gamma$-module is said to be basic if it is of the form $U \otimes_{AG} A\Gamma$ where G is a finite subgroup* *of Γ and U is a completely finitary, Benson's cofibrant AG-module. An $A\Gamma$-module M is called polybasic if there's a filtration $0 = M_0 \subseteq M_1 \subseteq .. \subseteq M_n = M$ where each M_i/M_{i-1} is a basic $A\Gamma$-module.*

Remark 8.4.21. *In the notations of Definition 8.4.19, if M is an FP_∞ module, i.e. if there is a projective resolution of M with finitely generated projectives, then by the characterisation of FP_∞ modules in terms of Ext-functors, we have that the functors $\mathrm{Ext}^*_{A\Gamma}(M, ?)$ commute with filtered colimits of coefficient modules, and by Result 4.1 of [41], it follows that M is completely finitary.*

The following result will be crucial for us.

Theorem 8.4.22. *(done over $\mathbb{Z}$ in Proposition 2.13 of [33]) Let $\Gamma \in LH\mathscr{F}$. Take M to be an $A\Gamma$-module that is both completely finitary and Benson's cofibrant, then M is isomorphic to the summand of a polybasic module and a projective module.*

It is easy to note that the class of polybasics, as defined in Definition 8.4.20, allows us to just deal with basic modules and capture all polybasics by triangles in the the stable category:

Lemma 8.4.23. *Let Γ be a group that admits complete resolutions over A. Then, any triangulated subcategory of $Stab(A\Gamma)$ containing all basic $A\Gamma$-modules contains all polybasic $A\Gamma$-modules.*

Definition 8.4.24. *Let $\mathscr{T}$ be a triangulated category. A thick subcategory of $\mathscr{T}$ is defined to be a full triangulated subcategory of $\mathscr{T}$, $\mathscr{S}$, such that given $M, N \in \mathscr{S}$, with $M \oplus N \in \mathscr{S}$, then M, $N \in \mathscr{S}$.*

For any class of objects $\mathscr{U}$ in $\mathscr{T}$ and any object $M \in \mathscr{T}$, we say M is properly generated by $\mathscr{U}$ in $\mathscr{T}$ if M is in the smallest thick subcategory of $\mathscr{T}$ containing $\mathscr{U}$.

Remark 8.4.25. *We have seen generation in triangulated categories using localising and colocalising subcategories. Generation using thick subcategories is also a very useful concept (one can consult [58] to see more about the theory surrounding this) in general - to be clear, in this concept, one can say a class of objects $\mathscr{U}$ in a triangulated subcategory $\mathscr{T}$ "generates" $\mathscr{T}$ iff the smallest thick subcategory of $\mathscr{T}$ containing $\mathscr{U}$ is all of $\mathscr{T}$.*

So, the definition of "proper" generation that we provide in Definition 8.4.24 is not very unnatural.

Take an $LH\mathscr{F}$-group that admits complete resolutions over A (such a group is in $\mathscr{F}_{\phi,A}$ by Proposition 5.1.12). Note that if we take any FP_∞ module M, some high enough syzygy of it, say $\Omega^n(M)$, is Gorenstein projective (= Benson's cofibrant in this case, see Remark 6.4.7), and also of type FP_∞. Recall that FP_∞ modules are completely finitary by Remark 8.4.21. Now, by Theorem 8.4.22, Lemma 8.4.23 and Definition 8.4.24, $\Omega^n(M)$ is in the smallest thick subcategory of $Stab(A\Gamma)$ containing the basics (note that projectives are isomorphic to zero in the stable category). Like we saw in the proof of Lemma 8.4.18.b., it is straightforward to note that whenever $\Omega^n(M)$ is in a triangulated subcategory $\mathscr{T} \subseteq Stab(A\Gamma)$, then, by repeated application of the suspension functor Ω^{-1}, $\Omega^0(M)$ (which is isomorphic to M in $Stab(A\Gamma)$) is in $\mathscr{T}$. Thus, we have the following result.

Theorem 8.4.26. *Let Γ be an $LH\mathscr{F}$-group that admits complete resolutions over a commutative ring A of finite global dimension, and let $\mathscr{B}$ be the class of all basic $A\Gamma$-modules. Then, in the language of Definition 8.4.24, every object in $stab(A\Gamma)$ is properly generated by $\mathscr{B}$ in $Stab(A\Gamma)$.*

8.5 Ending open problems

We end this chapter with a couple of open question on a few aspects and applications of generation in triangulated categories that we have not touched before. We need to first define tensor triangulated categories which were first introduced by Balmer [8].

Definition 8.5.1. *(Definition 3 of [8]) A tensor triangulated category $(\mathscr{T}, \otimes, u)$ is a triangulated category $\mathscr{T}$ equipped with a monoidal structure (see Chapter 7 of [45] for reference) $\mathscr{T} \times \mathscr{T} \xrightarrow{\otimes} \mathscr{T}$ with a unit object $u \in \mathscr{T}$. Both the functors $? \otimes x : \mathscr{T} \to \mathscr{T}$ and $x \otimes ? : \mathscr{T} \to \mathscr{T}$ are assumed to be exact for every $x \in \mathscr{T}$. For all $x, y \in \mathscr{T}$ and denoting the suspension functor of $\mathscr{T}$ by Σ, this involves natural isomorphisms $(\Sigma x) \otimes y \cong \Sigma(x \otimes y)$ and $x \otimes (\Sigma y) \cong \Sigma(x \otimes y)$, that we assume to be compatible in the sense that the two ways from $(\Sigma x) \otimes (\Sigma y)$ to $\Sigma^2(x \otimes y)$ only differ by a sign. In addition, we assume that, for all $x, y \in \mathscr{T}$, $x \otimes y \cong y \otimes x$.*

Example 8.5.2. *Stable module categories of finite groups over fields, stable module categories of not necessarily finite groups admitting complete resolutions over commutative rings of finite global dimension, derived unbounded categories of chain complexes as discussed earlier in this chapter are all examples of tensor triangulated categories.*

To state our open questions in this section, we need to define a very key concept related to tensor triangulated categories - the spectrum. We start with the definition of thick $\otimes$-ideals:

Definition 8.5.3. *(Definition 7 and Construction 8 of [8]) Let $(\mathscr{T}, \otimes, u)$ be a tensor triangulated category. $\mathscr{I}$ is called a thick $\otimes$-ideal of $\mathscr{T}$ iff $\mathscr{I}$ is a thick triangulated subcategory of $\mathscr{T}$ satisfying the following property: for any $y \in \mathscr{I}$ and any $x \in \mathscr{T}$, $x \otimes y \in \mathscr{I}$.*

A thick $\otimes$-ideal $\mathscr{P}$ is called prime if $u \notin \mathscr{P}$ and if, for any $x, y \in \mathscr{T}$, $x \otimes y \in \mathscr{P}$ implies $x \in \mathscr{P}$ or $y \in \mathscr{P}$. The Balmer spectrum of $\mathscr{T}$, $Spc(\mathscr{T})$, is defined as $\{\mathscr{P} \subsetneq \mathscr{T} : \mathscr{P}$ prime $\}$. Defining $U(x) := \{\mathscr{P} \in Spc(\mathscr{T}) : x \in \mathscr{P}\}$, an open basis of the topology of $Spc(\mathscr{T})$ is given by $\{U(x) : x \in \mathscr{T}\}$.

Remark 8.5.4. *The Balmer spectra of various important tensor triangulated categories that arise in algebraic geometry, geometric group theory and equivariant homotopy theory have been computed, like the Balmer spectrum of the equivariant homotopy category of a finite abelian group [10], the Balmer spectra of the category of finite G-spectra for a compact Lie group G [9], etc.*

Problem 8.5.5. *a) In [59], Steen and Stevenson study a notion of strong generation for tensor triangulated categories and show that if a non-zero proper thick tensor ideal of a tensor triangulated category $\mathscr{T}$ is strongly generated, then the Balmer spectrum of $\mathscr{T}$ is disconnected as a topological space(see Theorem 4.1 of [59]). The notions of generation that we looked into in Section 8.3 are different from this "strong generation", but it is still worth investigating whether we can apply one of our generation results like Theorem 8.2.5 or a variant of it to determine whether or not the derived categories of unbounded chain complexes of modules over groups in Kropholler's hierarchy have connected Balmer spectra.*

b) It was shown in [16] that the stable module category of finitely generated $\mathbb{Z}G$-modules, where G is a finite group with at least 2 elements, has a disconnected Balmer *spectrum; however, one gets a connected Balmer spectrum in this case upon replacing $\mathbb{Z}$ with a field. Do we get a similar result regarding the disconnectedness of the Balmer spectrum of the stable module category of finitely generated $A\Gamma$-modules where Γ is an infinite group that admits complete resolutions over A with A being a commutative ring of non-zero global dimension?*

A possible approach of attacking Question 8.5.5.b. would be to start with a couple of distinct thick tensor ideals of the stable module category (here again, there is a link to the Krull dimension problem discussed in Remark 8.2.10) and see if they can be fit into a localisation sequence and if they can be, that will give us a pair of fully faithful exact functors from the embeddings with their right adjoints and their left adjoints. These functors along with the *Spc*-functor can then be crucial in using some abstract support theory, which has already been formalised in the case of finite groups, to achieve a decomposition of the whole spectrum. This is why one may find it helpful if the more challenging problem of classifying localising tensor ideals has already been handled.

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