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An Evolutionary Finance Model with Short Selling and Endogenous Asset Supply

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Abstract Evolutionary Finance focuses on questions of "survival and extinction" of investment strategies (portfolio rules) in the market selection process. It analyzes stochastic dynamics of financial markets in which asset prices are determined endogenously by a short-run equilibrium between supply and demand. Equilibrium is formed in each time period in the course of interaction of portfolio rules of competing market participants. A comprehensive theory of evolutionary dynamics of this kind has been developed for models in which short selling is not allowed and asset supply is exogenous. The present paper

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extends the theory to a class of models with short selling and endogenous asset supply.

Keywords Evolutionary finance · Survival portfolio rules · Market games · Stochastic games

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1 Introduction

The purpose of this work is to develop a version of the Evolutionary Finance (EF) models [1,3,22] taking into account possibilities of short selling and endogenous asset supply. The model we propose describes a market with short-lived assets that live one period, yield random payoffs at the end of it, and then are re-born at the beginning of the next period. At every stage, investors (traders) reinvest their wealth obtained at the previous stage into the traded assets. The fundamental goal of the analysis is to identify strategies that make it possible for an investor to "survive" in the market selection process. Survival means a possibility of keeping with probability one a strictly positive, bounded away from zero share of total market wealth over an infinite time horizon. The main results obtained in this area show that in the models at hand there exists a portfolio rule guaranteeing unconditional survival irrespective of the strategies of the competing market participants. Moreover, it is shown that such a portfolio rule is asymptotically unique and can be described by a simple explicit formula amenable for quantitative investment analysis.

The main focus of EF is on investors' objectives of an *evolutionary* nature: *survival* (especially in crisis environments), *domination* in a market segment, fastest capital *growth*, etc. By and large, these objectives are *relative*: they are stated in terms of criteria comparing the performance of one market participant with the performance of the others.

An important characteristic feature of EF models is that they do not assume that investors' behaviour is fully rational and can be described, as in the classical theory, by well-defined and precisely stated constrained optimization problems. They admit that market actors may be boundedly rational and their behaviour might be determined by their individual psychology. Investors' strategies may involve, for example, mimicking, satisficing, rules of thumb based on experience, etc. Strategies might be interactive: what one is doing might depend on what the others do.

To deal with bounded rationality and behavioural diversity of market players, EF relies upon mathematical theory, rather than empirical methods. The mathematical approach, as it is commonly understood and as it is employed here, aims at obtaining rigorous results in the most general settings. In the present context this means including into consideration *all* possible kinds of market behaviour. EF does not restrict analysis to the classical von Neumann-Morgenstern utilities or their generalizations defined in terms of Choquet integrals (Denneberg [19])—the approach attracting nowadays considerable attention; see, e.g., De Giorgi and Hens [16], De Giorgi et al. [17,18], and Zhou [60].

Various approaches to EF distinct from that in the present work were developed in the studies by Blume and Easley [7], Farmer and Lo [26], Farmer [25], Brock et al. [13], Lo [39–42], Lo et al. [43], Zhang et al. [59], Sciubba [53, 54], Coury and Sciubba [15], Flåm [27], Bottazzi et al. [10–12], Bottazzi and Dindo [8,9], and Tarnaud [58]. In those studies, for the most part different models were considered and different goals pursued.

As a starting point for this work we used the model [3] dealing with short-lived (one-period) assets. In this model, short selling is ruled out and the total amount (the number of "physical units") $V_{k,t}$ of each asset $k = 1, ..., K$, depending on the moment of time t and on the random situation in the market, is given exogenously. One unit of asset k issued at the beginning of a time period $[t, t+1]$ yields the random payoff $A_{t+1,k}$ by the end of it.

An *investment strategy*, or a *portfolio rule*, Λ of an investor specifies the proportions $\lambda_{t,k}$ according to which the available budget is allocated across assets $k = 1, ..., K$ at each moment of time t, depending on the current state of the world and the previous history of the market. In [3] it is shown that in the present context the strategy Λ^* guaranteeing survival in the market selection process has the following simple structure. It prescribes to distribute wealth between assets $k = 1, ..., K$ according to the proportions

$$\lambda^*_{k,t} := E_t R_{t+1,k} \ ,$$

where

$$R_{t+1,k} := \sum_k \frac{A_{t+1,k} V_{t,k}}{\sum_l A_{t+1,l} V_{t,l}},$$

and $E_t(\cdot)$ stands for the conditional expectation given the information available by time t. Here, $R_{t+1,k}$ are the *relative payoffs* of the assets, that are obtained by weighing the absolute payoffs $A_{t+1,k}$ with the weights

$$\phi_{t+1,k} := V_{t,k} / \sum_l A_{t+1,l} V_{t,l},$$

so that $\sum_k R_{t+1,k} = 1$.

The above portfolio rule is akin to the investment in the *market portfolio* (e.g. [24], Ch. 7). However, instead of the capitalization weights, we use here the weights $\phi_{t+1,k}$ defined in terms of the asset payoffs, rather than their equilibrium prices. This approach is usually referred to as *fundamental indexing* (Arnott et al. [5]).

In the model considered in this paper, market participants can construct portfolios not only with long, but also with short positions. Long positions are described, as before, in terms of vectors of investment proportions specifying how the investors allocate their budgets across the traded assets. To create short positions, a market participant issues "replicas" of the original assets that have the same prices (formed in equilibrium) and the same payoffs. The newly issued assets are sold on the market at the equilibrium prices, which yields the *short-selling income* for the one who has issued them. This income increases the investment budget which is spent for purchasing other assets (creating long portfolio positions). On the other hand, each unit of asset sold short implies the obligation of the seller to pay to the buyer the same payoff as the original asset. Furthermore, short selling leads to an increase of the exogenously given total number $V_{t,k}$ of each asset k in the market, which of course influences the equilibrium asset prices. Thus the consequences of short selling depend on the combination of all these factors, and the decisions made by the short-sellers should consider a trade-off between them all.

When analyzing this new model, we are primarily interested in the fundamental questions of existence and uniqueness (in an asymptotic sense) of a survival strategy, similar to those considered in all EF models. First of all, we ask the following question: does the strategy Λ^* (with no short selling) guarantee survival in a market where the rivals of the Λ^*-investor can sell short?

Our findings are as follows. The answer to the last question is affirmative. Yes, the strategy Λ^*, which does not involve short selling, indeed guarantees survival in a market where short sales are allowed. What about uniqueness? Are there strategies involving short selling that also guarantee survival? If so, are they asymptotically distinct from Λ^*? The answers to the last two questions are negative. The following result gives a key for an understanding of the answers to the above questions.

If an investor i sells short at some moment of time t with strictly positive probability, then the group of i's rivals can construct a "spiteful" strategy that drives investor i out of the market (leads to i's bankruptcy) at time $t+1$ with a strictly positive probability.

We do not think that this finding is surprising. On the contrary, it is the fact that a survival strategy exists in the conventional EF setting without short selling—this is what might seem surprising, especially from the perspective of traditional Financial Economics. It may be difficult to believe that there is a strategy protecting a market player almost surely against the coordinated spiteful actions of the pool of other investors. If short selling is not allowed, and since asset payoffs (not to be confused with asset returns!) are non-negative, no investor's wealth can ever become strictly negative, i.e. bankruptcy is excluded. Thus a market player cannot be driven out of the market in a finite time. This can happen of course if the time horizon is infinite: a market share of an unsuccessful investor may vanish in the limit. But if short sales are allowed, then, as has been said, the rivals of a short selling trader can form a coalition whose strategy will lead to the bankruptcy of this trader in a finite time. A case of this sort happened in Switzerland in 2002 when a famous private investor ran into liquidity problems and offered part of his portfolio to banks. Knowing (or guessing) his trading strategy, they traded against him so that he had to surrender and offer his portfolio at a minimum price. However, in normal circumstances reliable information about other investors' strategies is lacking, so that it is difficult to collude against them. In view of that our results should by no means be interpreted as an expression of the idea of total irrelevance of short selling.

This work combines modeling features from evolutionary games and finance on the one hand, and from stochastic dynamic games (as pioneered by Shapley [55]) on the other hand. Our framework postulates a dynamic non-cooperative market game, in which the mechanisms of short-term price formation and market clearing follow those of one-shot strategic market games (see Shapley and Shubik [56] and Amir et al. [4]).[6] In addition, the present approach is reminiscent of games of survival, first considered by Milnor and Shapley [47] as a constant-sum stochastic game analog of the well-known gambler's ruin decision problem. Two players play a zero-sum matrix game at each stage of an infinite time horizon, the outcome of which determines their wealth dynamics (as a state variable). The ultimate outcome of the game is either bankruptcy of one player or a draw. In a similar vein, Shubik and Whitt [57] consider a dynamic market game with one unit of a durable good per period, and a fixed total wealth distributed across the players in exogenous fixed shares. Each player can bid part or all of his current wealth on the durable good, of which he obtains an amount in proportion to his bid. The total bid is then redistributed to the players according to their fixed shares, and play proceeds to the next period. In contrast to the present setting, each player's objective is to maximize the discounted sum of utilities of consumption. Finally, Giraud and Stahn [30] extend the basic Shapley-Shubik model to a two-period financial economy with uncertainty, allowing short selling by traders (as we do here).

The paper is organized as follows. Section 2 describes the model. Section 3 contains the statements of the main results of the paper. Sections 4 to 6 provide proofs of the results. Section 7 concludes.

2 The model

We consider an asset market influenced by random factors modeled in terms of an exogenous stochastic process $s_1, s_2, ...$, where s_t is a random element in a measurable space S_t. There are $K \geq 2$ *risky assets* (*securities*) traded in the market at dates $t = 0, 1, ...$ The total amount (the number of units) of asset k available at date t is given by $V_{t,k} = V_{t,k}(s^t) > 0$, where $s^t = (s_1, ..., s_t)$ is the history of the process of the states of the world s_t. If $t = 0$, then $V_{t,k}$ (as well as all the other functions of s^t) is constant. Assets live for one period: they are traded at the beginning of the period and yield payoffs at the end of it. One unit of asset k pays $A_{t,k} = A_{t,k}(s^t) \geq 0$ at the end of the time period $t-1, t$. It is assumed that $A_{t,k}(s^t)$, $t = 1, 2, ...$, are measurable and satisfy

$$\sum_{k=1}^{K} A_{t,k}(s^t) > 0. \tag{1}$$

[6] The class of dynamic games of industry competition with entry and exit introduced by Erickson and Pakes [21] also features short term (within-period) market competition and long run strategic interaction via investment decisions.

There are $N \geq 2$ *investors* (*traders*) acting in the market. Each investor $i = 1, ..., N$ at each time $t \geq 0$ has some wealth $w_t^i = w_t^i(s^t)$, which depends on s^t. For $t = 0$ (non-random) initial endowments $w_0^i > 0$, $i = 1, ..., N$ are given. The wealth dynamics of each investor depends on the strategies of this and other investors, as will be described below.

At every date $t \geq 0$ investor i purchases $x_{t,k}^i \geq 0$ units of asset $k = 1, ..., K$ and/or sells short at this date $y_{t,k}^i \geq 0$ units of this asset. The payoff investor i receives at the end of the time period $t, t+1$ from $x_{t,k}^i$ units of asset k will be $A_{t+1,k}x_{t,k}^i = A_{t+1,k}(s^{t+1})x_{t,k}^i$. If at the initial date t of this time period, investor i sells short $y_{t,k}^i$ units of asset k, then he has to pay the amount $A_{t+1,k}(s^{t+1})y_{t,k}^i$ at the end of the period.

Denote the vector of market prices of the securities by $p_t = (p_{t,1}, ..., p_{t,K})$, where $p_{t,k}$, $k = 1, ..., K$, is the price of one unit of asset k. The prices $p_{t,k} = p_{t,k}(s^t)$ depend on the history s^t of states of the world prior to time t. The prices $p_{t,k}$, $k = 1, ..., K$, are determined *endogenously* by an equilibrium condition. The *market equilibrium* is reached when total supply of each asset k is equal to its total demand (i.e., the market clears):

$$V_{t,k} + \sum_{i=1}^{N} y_{t,k}^i = \sum_{i=1}^{N} x_{t,k}^i, \ k = 1, ..., K. \tag{2}$$

As has been said, the opening of a short position $y_{t,k}^i$, $k = 1, ..., K$, at the beginning of the time period $[t, t+1]$ leads to an obligation to pay the amount $A_{t+1,k}(s^t)y_{t,k}^i$ at the end of it. On the other hand, the amount $y_{t,k}^i$ of asset k that the investor has sold short allows him to increase the budget available for further investments by an amount equal to $p_{t,k}y_{t,k}^i$ (the short-selling income). The total *investment budget* $w_t^i + v_t^i$ of trader i at date t consists of the trader's wealth w_t^i and the *short selling income*

$$v_t^i := \sum_{k=1}^{K} p_{t,k}y_{t,k}^i. \tag{3}$$

At each time t, every investor spends the entire available investment budget $w_t^i + v_t^i$ for buying assets:

$$w_t^i + v_t^i = \sum_{k=1}^{K} p_{t,k}x_{t,k}^i.$$

The wealth w_{t+1}^i of investor i at the end of the time period $[t, t+1]$ can be calculated as follows:

$$w_{t+1}^i := \sum_{k=1}^{K} A_{t+1,k}x_{t,k}^i - \sum_{k=1}^{K} A_{t+1,k}y_{t,k}^i = \sum_{k=1}^{K} A_{t+1,k}\left(x_{t,k}^i - y_{t,k}^i\right). \tag{4}$$

The volumes $x_{t,k}^i \geq 0$, $k = 1, ..., K$ of the assets purchased by investor i form the vector $x_t^i := \left(x_{t,1}^i, ..., x_{t,K}^i\right)$. The quantities $y_{t,k}^i \geq 0$, $k = 1, ..., K$, of the assets sold short by investor i form the vector $y_t^i := \left(y_{t,1}^i, ..., y_{t,K}^i\right)$. The *portfolio* of investor i is given by the pair of vectors $(x_t^i, -y_t^i)$.

For each $t \geq 0$, each trader $i = 1, 2, ..., N$ selects a vector of investment proportions $\gamma_t^i := (\gamma_{t,1}^i, ..., \gamma_{t,K}^i)$ according to which he plans to distribute the available investment budget between assets. Vectors γ_t^i belong to the unit simplex

$$\gamma_t^i \in \Delta^K := \{(a_1, ..., a_K) \in \mathbb{R}^K : a_1 + ... + a_K = 1, \, a_k \geq 0, \, k = 1, ..., K\}. \quad (5)$$

When selling short $y_{t,k}^i$ units of asset $k = 1, ..., K$ at date t, investor i issues and sells $y_{t,k}^i$ "replicas" of asset k, which have the same price and guarantee the same payoff for the buyer at the next date $t+1$ as the original asset k. Though this operation increases agent i's investment budget by $p_{t,k} y_{t,k}^i$ at date t, it leads to an obligation to pay $A_{t+1,k} y_{t,k}^i$ at date $t+1$. Formally, in this model investor i's *decision* (or *action*) at date t is specified by a pair of vectors $\xi_t^i = (\gamma_t^i, y_t^i)$, where γ_t^i is the vector of investment proportions and y_t^i is the vector whose coordinates define the short positions. Note that long positions of a portfolio are specified in terms of *investment proportions*, while its short positions are defined in terms of *units* of assets!

In what follows, we will consider only those decisions that do not permit to open simultaneously a long and a short position for the same asset, i.e.,

$$\gamma_{t,k}^i y_{t,k}^i = 0, \; k = 1, ..., K, \; t \geq 0. \quad (6)$$

This property will be included into the definition of investors' decisions (actions).

The investment decisions ξ_t^i at each date $t \geq 0$ are selected by the N investors at the same time and independently (as in a simultaneous-move N-person dynamic game). For $t \geq 1$, this decision making usually depends on s^t and the *history of the game (*or the *history of the market)*

$$\xi^{t-1} := \left\{\xi_l^i, \; i = 1, ..., N, \; l = 0, ..., t-1\right\},$$

which contains information about all the actions of all the market participants in the past. A pair of vectors $\Xi_0^i = \left(\Gamma_0^i, Y_0^i\right) \in \Delta^K \times \mathbb{R}_+^K$ and a sequence of measurable functions

$$\Xi_t^i(s^t, \xi^{t-1}) = \left(\Gamma_t^i(s^t, \xi^{t-1}), Y_t^i(s^t, \xi^{t-1})\right), \; t = 1, 2, ...,$$

taking values in $\Delta^K \times \mathbb{R}_+^K$ form a *portfolio rule*, or an *investment (trading) strategy* Ξ^i of investor i, according to which player (investor) i makes the decision

$$\xi_t^i = \Xi_t^i\left(s^t, \xi^{t-1}\right) \quad (7)$$

at each date $t \geq 0$. This is a general game-theoretic definition of a pure strategy, assuming full knowledge of the history of the game and the previous states of the world. Among general portfolio rules, we will distinguish those for which Ξ_t^i depends only on s^t, and not on the game history ξ^{t-1}. We will call such portfolio rules $\Xi_t^i(s^t)$ *basic*.

Given a decision $\xi_t^i = (\gamma_t^i, y_t^i)$ of investor i, the long positions $x_{t,k}^i$ of i's portfolio $(x_t^i, -y_t^i)$ are computed according to the formulas

$$x_{t,k}^i = \frac{\gamma_{t,k}^i(w_t^i + v_t^i)}{p_{t,k}} = \frac{1}{p_{t,k}}\gamma_{t,k}^i\left(w_t^i + \sum_{m=1}^{K} p_{t,m}y_{t,m}^i\right),\ k = 1, ..., K, \tag{8}$$

and the short positions of this portfolio are specified by the vector $-y_t^i$, where $y_t^i = \left(y_{t,1}^i, ..., y_{t,K}^i\right)$.

In the system of equations (2), $x_{t,k}^i$ and v_t^i can be expressed by using formulas (8) and (3), respectively. This leads to the following system of equations from which we can determine the vector $p_t = (p_{t,1}, ..., p_{t,K})$ of *equilibrium asset prices*:

$$\sum_{i=1}^{N}\gamma_{t,k}^i\left(w_t^i + \sum_{m=1}^{K} p_{t,m}y_{t,m}^i\right) = p_{t,k}\left(V_{t,k} + \sum_{i=1}^{N} y_{t,k}^i\right),\ k = 1, ..., K. \tag{9}$$

Proposition 1. *Let the following conditions hold:*

$$w_t^i > 0,\ i = 1, ..., N;\ \sum_{i=1}^{N}\gamma_{t,k}^i w_t^i > 0,\ k = 1, ..., K. \tag{10}$$

Then the system of equations (9) has a unique strictly positive solution $p_t = (p_{t,1}, ..., p_{t,K})$, $p_{t,k} > 0$, *for each* k.

According to this proposition, if at date t the wealth w_t^i of each trader $i = 1, 2, ..., N$ is strictly positive and for each asset $k = 1, ..., K$ at least one of the traders selects a strictly positive investment proportion $\gamma_{t,k}^i > 0$, then the asset market has a unique equilibrium with strictly positive prices. Note that if the first inequality in (10) is satisfied and at least one of the investors has a strictly positive vector of investment proportions $\gamma_t^i = (\gamma_{t,1}^i, ..., \gamma_{t,K}^i)$ then the second inequality in (10) holds as well.

We conclude this section with remarks on the design of the model at hand.

Remark 1. The approach to short selling that involves "replicas" of assets with the same exogenous payoffs is quite common in mathematical models considered in Financial Economics (Magill and Quinzii [44]) and Mathematical Finance (Pliska [48], Ross [50], Föllmer and Schied [29], and Zierhut [61]). However, quite often it is not explicitly spelled out, since usually there is no need in a deeper analysis of the question. Here, we wish to discuss this approach in more detail, in particular, because of a certain asymmetry in our model, where long portfolio positions are specified in terms of investment proportions and the short ones in terms of "physical" units of assets. This asymmetry is conceptually important and has a clear meaning. It reflects the fact that the operations of creating long and short portfolio positions in the present context are substantially different. The former is concerned with purchasing available assets by allocating wealth across them according to the given investment strategy. The latter operation, understood as the creation of new one-period assets, replicas of the initially available ones, is nothing but *endogenous asset* *supply.* In the purely financial context, endogenous asset supply may be regarded as the analogue of *production* in models with real assets. (This analogy becomes especially transparent if we look at the creation of derivative securities, rather then identical replicas of the basic assets.) It should be noted that the liabilities related to the creation of new securities, copying the basic ones, can be precisely estimated only if one knows the number of the units issued: for every unit of asset k sold short at time t, the seller must later, at time $t+1$, pay to the buyer the amount denoted in our model by $A_{t+1,k}(s^{t+1})$. Furthermore, the total asset supply in the equilibrium pricing equations (2.9) must be also expressed in terms of units of assets, as long as its exogenous part $V_{t,k}$ is expressed in this way. These considerations justify the way of specifying short positions of investors' portfolios used in this paper. As regards the long ones, theoretically one can describe them both in terms of units of assets or in terms of their monetary values and investment proportions. The latter approach is traditional for classical capital growth theory (see, e.g., Evstigneev et al. [24], Ch. 17,18), and since EF may be regarded as an extension of this theory to the case of endogenous asset prices, it is natural to design the model in a way similar to the classical one in order to use, whenever possible, similar machinery, notation, etc.

For other models in capital growth theory and EF involving short selling and endogenous asset supply we refer the reader to the papers by Bucher and Woehrmann [14], Horváth and Urbán [36], and Schenk-Hoppé and Sokko [52], containing quite a few interesting modeling ideas.

Remark 2. Some comments on the definition of a strategy we use are in order.[7] There are two general modeling approaches in discrete-time stochastic control theory—both in its conventional, single-agent version, and its game-theoretic setting, where decisions are made by several players with different objectives. Models of the first kind are described in terms of transition functions (stochastic kernels) specifying the distribution of the state of the system at time $t+1$ for each given state and control at time t; see e.g. Shapley [55], Bertsekas and Shreve [6] and Dynkin and Yushkevich [20]. In models of the second kind (such as the one in the present work), random factors influencing the system are described in terms of an exogenous random process of states of the world, the distribution of which does not depend on the actions of the players. This approach is often associated with the term "stochastic programming" (e.g. Rockafellar and Wets [49]). Although theoretically both approaches are in many cases equivalent (see Dynkin and Yushkevich [20], Section 2.2), in different contexts one is more natural and convenient than the other. Generally, the latter is preferable when the model possesses properties of convexity, which is characteristic for economic and financial applications. In a stochastic game setting, the second approach has been pursued in the work of A. Haurie and coauthors; see e.g. Haurie et al. [33]. The focus in that line of work is primarily on strategies that depend only on the exogenous states of the world.

[7] A detailed discussion of game-theoretic aspects of EF models is given in [3], pp. 123-125.

Haurie et al. [33] call them $\mathcal{S}$-adapted (adapted to the given filtration $\mathcal{S}$); we call them basic in this paper.

3 The main results

The focus of this study is on the analysis of the dynamics of investors' wealth depending on their strategies. We would like to define stochastic dynamics of the vectors $w_t = (w_t^1, ..., w_t^N)$, where w_t^i is the wealth of investor i at date t. Let $\Xi = (\Xi^1, ..., \Xi^N)$ be a strategy profile of the N investors. The dynamics of w_t will be defined recursively, step by step from t to $t+1$. The initial state (for $t = 0$) is the vector $w_0 = (w_0^1, ..., w_0^N)$, where $w_0^i > 0$ is the given initial endowment of trader i. Suppose $w_0, w_1, ..., w_t$ are defined for some $t \geq 0$. Assume that the following condition holds:

(**A**) The vector $w_t = (w_t^1, ..., w_t^N)$ and the investment proportions $\gamma_{t,k}^i$ (generated the strategy profile Ξ) satisfy (10).

Then according to Proposition 1, there exists a unique strictly positive vector $p_t = (p_{t,1}, ..., p_{t,K})$ of equilibrium asset prices, in terms of which we can express the wealth w_{t+1}^i of each investor i:

$$w_{t+1}^i = \sum_{k=1}^K A_{t+1,k} x_{t,k}^i - \sum_{k=1}^K A_{t+1,k} y_{t,k}^i =$$

$$\sum_{k=1}^K A_{t+1,k} \frac{1}{p_{t,k}} \gamma_{t,k}^i \left(w_t^i + \sum_{m=1}^K p_{t,m} y_{t,m}^i \right) - \sum_{k=1}^K A_{t+1,k} y_{t,k}^i =$$

$$\sum_{k=1}^K A_{t+1,k} \left(\frac{1}{p_{t,k}} \gamma_{t,k}^i \left(w_t^i + \sum_{m=1}^K p_{t,m} y_{t,m}^i \right) - y_{t,k}^i \right). \tag{11}$$

Definition 1. If in the course of this dynamical process, condition (**A**) happens to hold almost surely (a.s.) for all $t \in [0, T)$ ($T \leq \infty$), we say that the given strategy profile Ξ is *admissible* for the time interval $[0, T)$.

Remark 3. Assume that one of the traders, e.g., trader 1, employs a *fully diversified* portfolio rule, which prescribes investing into all assets in strictly positive proportions $\gamma_{t,k}^1 > 0$ for all $k = 1, ..., K$ and all $t \geq 0$. Then, given that no investor goes bankrupt during the time interval $[0, T)$, the strategy profile is admissible for the time interval $[0, T)$.

In this work, we will identify random variables coinciding almost surely, and we will often omit "a.s." if this does not lead to ambiguity. In fact the random variables under consideration may be defined not everywhere, but only almost everywhere: with probability one. If not otherwise stated, all relations between them (equalities, inequalities, etc.) will be supposed to hold almost surely.

By summing up the above equations over $i = 1, ..., N$ and taking into account the pricing equations, we obtain the following formula for the *total market wealth* W_{t+1}:

$$W_{t+1} := \sum_{i=1}^{N} w_{t+1}^{i} = \sum_{i=1}^{N} \sum_{k=1}^{K} A_{t+1,k} \left(\frac{1}{p_{t,k}} \gamma_{t,k}^{i} \left(w_{t}^{i} + \sum_{m=1}^{K} p_{t,m} y_{t,m}^{i} \right) - y_{t,k}^{i} \right) =$$

$$\sum_{k=1}^{K} A_{t+1,k} \left(\frac{1}{p_{t,k}} \sum_{i=1}^{N} \left[\gamma_{t,k}^{i} \left(w_{t}^{i} + \sum_{m=1}^{K} p_{t,m} y_{t,m}^{i} \right) \right] - \sum_{i=1}^{N} y_{t,k}^{i} \right) =$$

$$\sum_{k=1}^{K} A_{t+1,k} \left(V_{t,k} + \sum_{i=1}^{N} y_{t,k}^{i} - \sum_{i=1}^{N} y_{t,k}^{i} \right) = \sum_{k=1}^{K} A_{t+1,k} V_{t,k}. \tag{12}$$

Denote by

$$r_{t+1}^{i} := \frac{w_{t+1}^{i}}{W_{t+1}}$$

the *relative wealth* (*market share*) of investor i and put

$$r_{t+1} := \left(r_{t+1}^{1}, ..., r_{t+1}^{N} \right).$$

Definition 2. We say that a strategy Ξ^i employed by investor i can be *driven out of the market at a (finite) time* $T < \infty$ if there exists a strategy profile $(\Xi^1, ..., \Xi^N)$ including the strategy Ξ^i and admissible for $t \in [0, T)$ such that $P\left\{ w_T^i \leq 0 \right\} > 0$.

Definition 3. We say that a strategy Ξ^i employed by investor i *can be driven out of the market in an infinite time* if there exists a strategy profile $(\Xi^1, ..., \Xi^N)$ including the strategy Ξ^i and admissible for $t \in [0, \infty)$ such that $P\left\{ \inf_{t \geq 0} r_t^i = 0 \right\} > 0$.

Definition 4. We call a strategy Ξ *a survival strategy* if for any number $N \geq 2$ of agents acting in the market an investor using Ξ cannot be driven out of the market in any (finite or infinite) time.

Define the *relative payoffs* by

$$R_{t+1,k} := \frac{A_{t+1,k} V_{t,k}}{\sum_{l=1}^{K} A_{t+1,l} V_{t,l}} \tag{13}$$

and put $R_{t+1} := (R_{t+1,1}, ..., R_{t+1,K})$. Consider the investment strategy $\Xi^* = (\xi_t^*)_{t=0}^{\infty} = (\gamma_t^*, y_t^*)_{t=0}^{\infty}$ for which $y_t^* = 0$ and $\gamma_t^*(s^t) := (\gamma_{t,1}^*(s^t), ..., \gamma_{t,K}^*(s^t))$, where

$$\gamma_{t,k}^*(s^t) := E_t R_{t+1,k}(s^{t+1}). \tag{14}$$

We denote by $E_t(\cdot) = E(\cdot|s^t)$ the conditional expectation given s^t (if $t = 0$, then $E_0(\cdot) = E(\cdot)$).

Throughout the paper, we will assume that for each $k = 1, ..., K$,

$$E \ln E_t R_{t+1,k}(s^{t+1}) > -\infty. \tag{15}$$

This assumption implies that the conditional expectations $E_t R_{t+1,k} = E(R_{t+1,k}|s^t)$ $(k = 1, ..., K)$ are strictly positive (a.s.). Consequently, we can choose their versions $\gamma^*_{t,k}(s^t)$ which are strictly positive for all s^t. In what follows, the notation $\gamma^*_{t,k}(s^t)$ will refer to such versions of the above conditional expectations.

The central results are as follows.

Theorem 1. *The portfolio rule Ξ^* is a survival strategy.*

It is important to note that the portfolio rule Ξ^* is basic: the investor's decisions $\xi^*_t(s^t)$ depend only on the history s^t of the process of states of the world and do not depend on the history of the market. Furthermore, it does not involve short selling, but it survives in competition with *all*, not necessarily basic, strategies with short selling.

The results we formulate below show that in the class of basic strategies, the survival portfolio rule Ξ^* is (at least asymptotically) unique.

Theorem 2. *If a basic strategy prescribes to sell short at least one asset at some moment of time with strictly positive probability, then it can be driven out of the market in a finite time.*

Thus basic survival portfolio rules can exist only in the class of basic strategies that do not involve short selling (a.s.). It should be emphasized that a short seller can be driven out of the market in a finite time by a *basic* strategy profile of the rivals. This will be shown in the course of the proof of Theorem 2.

Theorem 3. *If Ξ is a basic survival strategy defined by a sequence of decisions $(\gamma_t(s^t), y_t(s^t))$, $t = 0, 1, 2...$, with $y_t(s^t) = 0$ (a.s.), then*

$$\sum_{t=0}^{\infty} ||\gamma^*_t - \gamma_t||^2 < \infty \text{ (a.s.).} \tag{16}$$

This theorem (pertaining in fact to a version of the present model without short selling) follows easily from Theorem 2 in [3]—see Section 6. Notice that vectors of investment proportions γ^*_t coincide with vectors λ^*_t generated by the strategy Λ^* considered in [3]. In that model (where no short-selling is allowed), Λ^* is also an asymptotically unique basic survival strategy.

Proofs of Proposition 1 and Theorems 1, 2 and 3 are given in the next three sections.

It is worth making some comments on the modeling of short-run equilibrium in this work. The dynamics of the asset market described above are similar to the dynamics of the commodity market as outlined in the classical treatise by Alfred Marshall [46] (Book V, Chapter II "Temporary Equilibrium of Demand and Supply"). Marshall's ideas were introduced into formal economics by Samuelson [51], pp. 321–323. As it was noticed by Samuelson [51], in order to study the process of market dynamics by using the Marshallian "temporary equilibrium method," one needs to distinguish between at least two sets of economic variables changing with different speeds. Then the set of variables changing slower (in our case, the set of investors' portfolios) can be temporarily fixed, while the other (in our case, the asset prices) can be assumed to rapidly reach the unique state of partial equilibrium.

The above concept of temporary, or moving, equilibrium was first introduced in economics by Marshall. However, in the last decades the term "temporary equilibrium" has been by and large understood differently. For the most part it was associated with a different notion suggested by the studies of Lindahl [38] and Hicks [34]. That notion was developed in formal settings by Hildenbrand, Grandmont, and others (see, e.g., Grandmont and Hildenbrand [31], Grandmont [32], and Magill and Quinzii [45]). The characteristic feature of the Lindahl-Hicks temporary equilibrium is its formulation in terms of *forecasts* or *beliefs* of market participants about the future states of the world. Mathematically, beliefs of economic agents are represented by stochastic kernels (transition functions) conditioning the distributions of future states of the world upon the agents' private information.

The model studied in the present work does not use information about individual utilities, beliefs and other unobservable agents' characteristics. What matters is the investment strategy as such, rather than the data and the logic on which its choice is based. The results obtained are stated in the form of recommendations for investment that use only some fundamental information about the market, in the same spirit as, for example, in the well-known principles of derivative securities pricing (Black, Scholes, Merton, and others, see e.g. [24], Part II).

4 Short-run equilibrium

Proof of Proposition 1. Let us fix t and omit it in the notation. For all $k = 1, ..., K$ define

$$u_k := p_k \left(V_k + \sum_{i=1}^{N} y_k^i \right), \ \sigma_k := \frac{\sum_{i=1}^{N} y_k^i}{V_k + \sum_{i=1}^{N} y_k^i},$$

$$\theta_k^i := \begin{cases} \dfrac{y_k^i}{\sum_{j=1}^{N} y_k^j}, \text{ if } \sum_{j=1}^{N} y_k^j > 0, \\ N^{-1}, \qquad \text{otherwise.} \end{cases}$$

Note that $0 \le \sigma_k < 1$ because $V_k > 0$, and we have $\sum_{i=1}^{N} \theta_k^i = 1$ and $\theta_k^i \ge 0$. If $\sum_{j=1}^{N} y_k^j > 0$, then for each $m = 1, ..., K$ the following identity holds:

$$y_m^i = \left(V_m + \sum_{j=1}^{N} y_m^j \right) \frac{(\sum_{j=1}^{N} y_m^j) y_m^i}{(V_m + \sum_{j=1}^{N} y_m^j) \cdot \sum_{j=1}^{N} y_m^j} = \left(V_m + \sum_{j=1}^{N} y_m^j \right) \sigma_m \theta_m^i,$$

which yields

$$p_m^i y_m^i = u_m \sigma_m \theta_m^i, \ m = 1, ..., K.$$

If $\sum_{j=1}^{N} y_k^j = 0$, the above equality holds as well, because in that case $y_m^i = \sigma_m = 0$. Hence, the system of equations (9) can be written as

$$\sum_{i=1}^{N} \gamma_k^i w^i + \sum_{i=1}^{N} \gamma_k^i \sum_{m=1}^{K} \theta_m^i \sigma_m u_m = u_k, \ k = 1, ..., K. \tag{17}$$

A vector $u = (u_1, ..., u_K)$ solves (17) if and only if u is a fixed point of the operator

$$F(u) := \left(\sum_{i=1}^{N} \gamma_k^i w^i + \sum_{i=1}^{N} \gamma_k^i \sum_{m=1}^{K} \theta_m^i \sigma_m u_m \right)_{k=1}^{K}$$

that is determined by the left-hand side of (17). This operator transforms the cone

$$\mathbb{R}_+^K = \{u \mid u_k \geq 0, \ k = 1, ..., K\}.$$

of non-negative K-dimensional vectors into itself, and in order to show that it has a unique fixed point in $\mathbb{R}_+^K$ it sufficient to prove that it is contracting in the norm $\|u\| = \sum_{k=1}^{K} |u_k|$. This follows from the chain of relations:

$$\|F(u) - F(u')\| = \sum_{k=1}^{K} |\sum_{i=1}^{N} \gamma_k^i \sum_{m=1}^{K} \theta_m^i \sigma_m (u_m - u'_m)| \leq$$

$$(\max_m \sigma_m) \sum_{i=1}^{N} (\sum_{k=1}^{K} \gamma_k^i) \sum_{m=1}^{K} \theta_m^i |u_m - u'_m| =$$

$$(\max_m \sigma_m) \sum_{m=1}^{K} |u_m - u'_m| \sum_{i=1}^{N} \theta_m^i = (\max_m \sigma_m) \|u - u'\|,$$

where the equalities hold since $\sum_{k=1}^{K} \gamma_k^i = 1$ and $\sum_{i=1}^{N} \theta_k^i = 1$. Here, $\max_m \sigma_m < 1$ because $\sigma_m < 1$ for all m, and so the operator F is contracting. Thus, system (17) has a unique solution $u = (u_1, ..., u_K) \geq 0$. Moreover, if condition (10) is satisfied then

$$u_k = \sum_{i=1}^{N} \gamma_k^i w^i + \sum_{i=1}^{N} \gamma_k^i \sum_{m=1}^{K} \theta_m^i \sigma_m u_m \geq \sum_{i=1}^{N} \gamma_k^i w^i > 0$$

and system (9) also has the unique strictly positive solution

$$p = (p_1, ..., p_K), \ p_k = \frac{u_k}{V_k + \sum_{i=1}^{N} y_k^i}$$

and $p_k > 0$ for each k. □

5 Survival portfolio rule

Proof of Theorem 1. Consider the market with $N \geq 2$ investors. Assume that player 1 uses the strategy $(\gamma_t^*, 0) = \big(E_t R_{t+1}(s^{t+1}), 0\big)$. Agent 1 using the strategy Ξ^* cannot be driven out of the market at any finite time $0 < T < \infty$ since assumption (15) guarantees that investor 1's portfolio is fully diversified (a.s.), hence, $w_T^1 > 0$ (a.s.). We shall prove that ξ_t^* cannot be driven out of the market over the time interval $[0, \infty)$.

Consider a strategy profile $\Xi = (\Xi^1, ..., \Xi^N)$ admissible for $[0; \infty)$. For this profile, $w_t^i > 0$ a.s. for all $t \geq 0$ and $i = 1, ..., N$. Let $(\xi_t^1(s^t), ..., \xi_t^N(s^t))_{t=0}^{\infty}$ be the set of investors' decisions generated by Ξ. For all $t \geq 0$, $k = 1, ..., K$, $i = 1, ..., N$, define the numbers

$$\lambda_{t,k}^i = \lambda_{t,k}^i\left(s^t\right) = \frac{w_t^i + \sum_{m=1}^K p_{t,m} y_{t,m}^i}{w_t^i}\gamma_{t,k}^i - \frac{p_{t,k} y_{t,k}^i}{w_t^i} \tag{18}$$

and the vectors $\lambda_{t,k} = \left(\lambda_{t,k}^1, ..., \lambda_{t,k}^N\right)$ and $\lambda_t^i = \left(\lambda_{t,1}^i, ..., \lambda_{t,K}^i\right)$. Note that $\lambda_t^1 = \gamma_t^*$ because $\xi_t^1 = (\gamma_t^*, 0)$.

We shall derive a system of equations describing the dynamics of the market shares r_t^i in terms of the sequence of vectors $(\lambda_t^1, ..., \lambda_t^N)$. We have

$$p_{t,k} V_{t,k} = \langle \lambda_{t,k}, w_t \rangle,\ k = 1, ..., K,$$

where $\langle \lambda_{t,k}, w_t \rangle := \sum_{i=1}^N \lambda_{t,k}^i w_t^i$. Indeed,

$$\sum_{i=1}^N \lambda_{t,k}^i w_t^i = \sum_{i=1}^N (w_t^i + \sum_{m=1}^K p_{t,m} y_{t,m}^i)\gamma_{t,k}^i - \sum_{i=1}^N p_{t,k} y_{t,k}^i = p_{t,k} V_{t,k}$$

by virtue of (18) and (9). Proposition 1 and the definition of a strategy profile admissible for the time interval $[0; \infty)$ guarantee that

$$p_{t,k} = \frac{\langle \lambda_{t,k}, w_t \rangle}{V_{t,k}} > 0 \text{ (a.s.)},\ k = 1, ..., K. \tag{19}$$

In view of (18), (8) and (8), we get

$$\frac{\lambda_{t,k}^i w_t^i V_{t,k}}{\langle \lambda_{t,k}, w_t \rangle} = \frac{\lambda_{t,k}^i w_t^i}{p_{t,k}} = \frac{w_t^i + \sum_{m=1}^K p_{t,m} y_{t,m}^i}{p_{t,k}}\gamma_{t,k}^i - \frac{p_{t,k} y_{t,k}^i}{p_{t,k}} = x_{t,k}^i - y_{t,k}^i \tag{20}$$

($k = 1, ..., K$). Thus, the wealth w_{t+1}^i of investor i defined by (4) can be expressed as follows:

$$w_{t+1}^i = \sum_{k=1}^K A_{t+1,k}\left(x_{t,k}^i - y_{t,k}^i\right) = \sum_{k=1}^K A_{t+1,k} V_{t,k} \frac{\lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t \rangle}. \tag{21}$$

By summing up these equations over $i = 1, ..., N$, we obtain

$$W_{t+1} = \sum_{k=1}^K A_{t+1,k} V_{t,k} \frac{\sum_{i=1}^N \lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t \rangle} = \sum_{k=1}^K A_{t+1,k} V_{t,k}. \tag{22}$$

Dividing the left-hand side of (21) by W_{t+1}, the right-hand side of (21) by

$$\sum_{m=1}^{K} A_{t+1,m} V_{t,m},$$

and using (13), we arrive at the system of equations

$$r_{t+1}^i = \sum_{k=1}^{K} R_{t+1,k} \frac{\lambda_{t,k}^i r_t^i}{\langle \lambda_{t,k}, r_t \rangle},\ i = 1, ..., N, \tag{23}$$

Since we consider a strategy profile $(\Xi^1, ..., \Xi^N)$ admissible for the time interval $[0; \infty)$ all the market shares are strictly positive:

$$r_t^i > 0,\ i = 1, ..., N,\ t \geq 0. \tag{24}$$

Observe that

$$\sum_{k=1}^{K} \lambda_{t,k}^i = \frac{w_t^i + \sum_{m=1}^{K} p_{t,m} y_{t,m}^i}{w_t^i} \sum_{k=1}^{K} \gamma_{t,k}^i - \frac{\sum_{k=1}^{K} p_{t,k} y_{t,k}^i}{w_t^i} = 1,$$

and so

$$\sum_{k=1}^{K} \lambda_{t,k}^i = 1,\ t \geq 0. \tag{25}$$

From (19) we conclude that

$$\langle \lambda_{t,k}, r_t \rangle > 0,\ k = 1, ..., K \tag{26}$$

for every $t \geq 0$.

To complete the proof we shall use Proposition 2 from which we conclude that $\inf_{t\geq 0} r_t^1 > 0$ (a.s.) and investor 1 cannot be driven out of the market at moment $T = \infty$. So Ξ^* is a survival strategy. □

Proposition 2. *If* $\lambda_t^1(s^t) = E_t R_{t+1}(s^{t+1})$ *and dynamical system (23) satisfies the conditions (24)-(26) then*

$$\inf_{t\geq 0} r_t^1 > 0 \text{ (a.s.)}.$$

Proof. It is sufficient to prove the statement in the case of $N = 2$. Consider the given random dynamical system and define for all $k = 1, .., K$

$$\tilde{\lambda}_{t,k}^2(s^t) = \begin{cases} (\lambda_{t,k}^2 r_t^2 + ... + \lambda_{t,k}^N r_t^N)/(1 - r_t^1) & \text{if } r_t^1 < 1, \\ 1/K & \text{if } r_t^1 = 1. \end{cases}$$

Then we have

$$\lambda_{t,k}^2 r_t^2 + ... + \lambda_{t,k}^N r_t^N = (1 - r_t^1)\tilde{\lambda}_{t,k}^2,$$

$$\langle \lambda_{t,k}, r_t \rangle = r_t^1 \lambda_{t,k}^1 + (1 - r_t^1)\tilde{\lambda}_{t,k}^2,$$

and so

$$r_{t+1}^1 = \sum_{k=1}^K R_{t+1,k} \frac{\lambda_{t,k}^1 r_t^1}{r_t^1 \lambda_{t,k}^1 + (1 - r_t^1)\tilde{\lambda}_{t,k}^2}. \tag{27}$$

By summing up equations (23) over $i = 2, ..., N$, we obtain

$$1 - r_{t+1}^1 = \sum_{k=1}^K R_{t+1,k} \frac{\tilde{\lambda}_{t,k}^2 (1 - r_t^1)}{r_t^1 \lambda_{t,k}^1 + (1 - r_t^1)\tilde{\lambda}_{t,k}^2}. \tag{28}$$

Thus the sequence $(r_t^1(s^t))$ generated by the original N-dimensional system (23) is the same as the analogous sequence generated by the two-dimensional system (27)–(28) corresponding to the game with two investors $i = 1, 2$ whose investment proportions are $\lambda_{t,k}^1(s^t)$ and $\tilde{\lambda}_{t,k}^2(s^t)$, respectively. Notice that all assumptions of the proposition hold for the reduced system.

Consider $N = 2$ and $\lambda_{t,k}^1 = \lambda_{t,k}^*$. Putting $\kappa_t = \kappa_t(s^t) := r_t^1(s^t)$, we obtain from (23) with $N = 2$:

$$\kappa_{t+1} = \sum_{k=1}^K R_{t+1,k} \frac{\lambda_{t,k}^1 \kappa_t}{\lambda_{t,k}^1 \kappa_t + \lambda_{t,k}^2 (1 - \kappa_t)}.$$

Observe that the process $\ln \kappa_t$ is a submartingale. Indeed, we have

$$E_t \ln \kappa_{t+1} - \ln \kappa_t = E_t \ln \sum_{k=1}^K R_{t+1,k} \frac{\lambda_{t,k}^1}{\lambda_{t,k}^1 \kappa_t + \lambda_{t,k}^2 (1 - \kappa_t)} \geq$$

$$E_t \sum_{k=1}^K R_{t+1,k} \ln \frac{\lambda_{t,k}^1}{\lambda_{t,k}^1 \kappa_t + \lambda_{t,k}^2 (1 - \kappa_t)} = \sum_{k=1}^K \lambda_{t,k}^1 \ln \frac{\lambda_{t,k}^1}{\lambda_{t,k}^1 \kappa_t + \lambda_{t,k}^2 (1 - \kappa_t)} =$$

$$\sum_{k=1}^K \lambda_{t,k}^1 \ln \lambda_{t,k}^1 - \sum_{k=1}^K \lambda_{t,k}^1 \ln[\lambda_{t,k}^1 \kappa_t + \lambda_{t,k}^2 (1 - \kappa_t)] \geq 0 \text{ (a.s.)}.$$

We used here Jensen's inequality for the concave function $\ln x$ and the elementary inequality

$$\sum_{k=1}^K a_k \ln a_k \geq \sum_{k=1}^K a_k \ln b_k \ [\ln 0 := -\infty]$$

holding for any vectors $(a_1, ..., a_K) > 0$ and $(b_1, ..., b_K) \geq 0$ with $\sum a_k = \sum b_k = 1$.

Further,

$$\kappa_{t+1} = \kappa_t \sum_{k=1}^K R_{t+1,k} \frac{\lambda_{t,k}^1}{\lambda_{t,k}^1 \kappa_t + \lambda_{t,k}^2 (1 - \kappa_t)} \geq$$

$$\kappa_t \sum_{k=1}^K R_{t+1,k} (\min_m \lambda_{t,m}^1) = \kappa_t (\min_m \lambda_{t,m}^1).$$

Since $E\min_m \ln\lambda^1_{t,m} > -\infty$ by virtue of assumption (15) and κ_0 is a strictly positive non-random number, each of the random variables $0 < \kappa_t \le 1$ satisfies $E|\ln\kappa_t| < \infty$.

The non-positive submartingale $\ln\kappa_t$ has a finite limit a.s., and so $\kappa_t \to \kappa_\infty$ (a.s.), where κ_∞ is a strictly positive random variable. Consequently, the sequence $\kappa_t > 0$ is bounded away from zero with probability one. □

6 Asymptotic uniqueness of a survival strategy

In this section we prove Theorems 2 and 3.

Proof of Theorem 2. Put $\bar{K} = \{1, ..., K\}$ and consider a market with $N = 2$ investors. Suppose investor 2 uses a strategy Ξ^2 prescribing to open with strictly positive probability a short position for some asset at some moment of time. Let $T \ge 0$ be the smallest among such moments of time. Then there exists a non-random M-element subset $\bar{M} \subset \bar{K}$ such that $M \ge 1$ and the event

$$\bar{S}^T := \{s^T : y^2_{T,k}\left(s^T\right) > 0 \text{ for } k \in \bar{M} \text{ and } y^2_{T,k}\left(s^T\right) = 0 \text{ for } k \in \bar{K}\backslash\bar{M}\}$$

has a strictly positive probability. (Note that $k \in \bar{K}\backslash\bar{M} \neq \emptyset$ because investor 2 cannot sell short *all* the assets.)

Without loss of generality, we can assume that investor 2 cannot be driven out of the market at any time $t \le T$; otherwise, the theorem is proved. Under this condition, we will construct a *spiteful strategy* Ξ^1 of investor 1, i.e. a strategy driving investor 2 out of the market at time $T+1$. With the strategy Ξ^1, the strategy profile $\left(\Xi^1, \Xi^2\right)$ will be admissible for the time interval $[0, T+1)$ and investor 2's wealth w^2_{T+1} will be negative with strictly positive probability. The negativity of w^2_{T+1} with strictly positive probability will be established under the additional assumption that the initial wealth w^2_0 of investor 2 is sufficiently small comparative to the initial wealth w^1_0 of his rival.

Fix some $\mu \in \bar{M}$. For any $\delta > 0$ and $\delta' > 0$ denote by $\bar{S}^{T+1}\left(\delta, \delta'\right)$ the following event:

$$\bar{S}^{T+1}\left(\delta, \delta'\right) = \{s^{T+1} : s^T \in \bar{S}^T,\ V^{-1}_{T,\mu} R_{T+1,\mu}(s^{T+1}) > \delta',\ y^2_{T,\mu}\left(s^T\right) > \delta\}. \tag{29}$$

Since $P\{s^T \in \bar{S}^T\} > 0$ and $P\{V^{-1}_{T,\mu} R_{T+1,\mu} > 0 \mid s^T\} > 0$ a.s. (by virtue of assumption (15)), there exist $\delta > 0$ and $\delta' > 0$ such that $P\left\{s^{T+1} \in \bar{S}^{T+1}\left(\delta, \delta'\right)\right\} > 0$.

Fix a positive number $\varepsilon < \min\left(\delta'\delta/(M \cdot K), 1/K\right)$ and define the strategy Ξ^1 as follows. For each $t \in [0, T+1)$, put $y^1_t = 0$ (no short selling) and define:

$$e = (1, 1, ..., 1),$$

$$\gamma^1_t = \frac{1}{2}\gamma^2_t + \frac{1}{2K}e, \tag{30}$$

for $t < T$ and

$$\gamma_{T,k}^1 = \begin{cases} \varepsilon & \text{if } k \in \bar{M}, \\ \frac{1-M\varepsilon}{K-M} & \text{if } k \in \bar{K} \backslash \bar{M} \end{cases} \tag{31}$$

Since investor 2 cannot be driven out of the market at any time $t \le T$, we conclude that $w_t^2 > 0$ (a.s.) for $t \le T$. Further, observe that $\gamma_{t,k}^1 > 0$, which implies $w_t^1 > 0$ and $\sum_{i=1}^N \gamma_{t,k}^i w_t^i \ge \gamma_{t,k}^1 w_t^1 > 0$ for each $k \in \bar{K}$. Consequently, the strategy profile $\left(\Xi^1, \Xi^2\right)$ is admissible for the time interval $[0, T+1)$.

Note that $y_t^1 = y_t^2 = 0$ and $\gamma_t^1 \ge \frac{1}{2}\gamma_t^2$ (see (30)) for every $t < T$. Therefore, in view of (11) and (8), we have

$$\frac{w_{t+1}^2}{w_{t+1}^1} = \frac{\sum_{k\in\bar{K}} A_{t+1,k} x_{t,k}^2}{\sum_{k\in\bar{K}} A_{t+1,k} x_{t,k}^1} = \frac{\sum_{k\in\bar{K}} A_{t+1,k} \frac{\gamma_{t,k}^2 w_t^2}{p_{t,k}}}{\sum_{k\in\bar{K}} A_{t+1,k} \frac{\gamma_{t,k}^1 w_t^1}{p_{t,k}}} \le 2\frac{w_t^2}{w_t^1}, \quad t < T. \tag{32}$$

Now let us assume that the initial wealth w_0^2 of investor 2 is small enough compared to the initial wealth w_0^1 of investor 1, specifically,

$$\frac{w_0^2}{w_0^1} \le \frac{1}{2^T}\left(\frac{\delta'\delta}{K} - M\varepsilon\right). \tag{33}$$

Then the inequality in (32) yields

$$\frac{w_T^2}{w_T^1} \le 2^T \frac{w_0^2}{w_0^1} \le \frac{\delta'\delta}{K} - M\varepsilon. \tag{34}$$

For $s^{T+1} \in \bar{S}^{T+1}\left(\delta, \delta'\right)$, we have

$$\gamma_{T,k}^2 = 0 \text{ for } k \in \bar{M} \tag{35}$$

because $y_{T,k}^2 > 0$ for $k \in \bar{M}$. By virtue of (9), (35) and (31), the prices $p_{T,k}$ satisfy the following conditions:

$$p_{T,k} = \frac{w_T^1 \gamma_{T,k}^1 + \left(w_T^2 + v_T^2\right)\gamma_{T,k}^2}{V_{T,k} + y_{T,k}^2} = \frac{w_T^1 \varepsilon}{V_{T,k} + y_{T,k}^2}, \text{ if } k \in \bar{M}, \tag{36}$$

$$p_{T,k} \ge \frac{w_T^1 \cdot \gamma_{T,k}^1}{V_{T,k}} \ge \frac{w_T^1}{V_{T,k}}\frac{1}{K}, \text{ if } k \in \bar{K}\backslash\bar{M}. \tag{37}$$

By using relations (8), (36), (37) and (34) to estimate $x_{T,k}^2$, $k \in \bar{K}\backslash\bar{M}$, we find that

$$x_{T,k}^2 \le \frac{w_T^2 + \sum_{m\in\bar{M}} p_{T,m} y_{T,m}^2}{p_{T,k}} \le \left(w_T^2 + \sum_{m\in\bar{M}} \frac{w_T^1 \varepsilon \cdot y_{T,m}^2}{V_{T,m} + y_{T,m}^2}\right) \cdot \left(\frac{w_T^1}{V_{T,k}}\frac{1}{K}\right)^{-1} \le$$

$$\left(\frac{w_T^2}{w_T^1} + M\varepsilon\right) V_{T,k} K \le \delta'\delta V_{T,k}. \tag{38}$$

Let $s^{T+1} \in \bar{S}^{T+1}\left(\delta, \delta'\right)$. By using (29), (38), and (13), we arrive at the following sequence of inequalities for investor 2's wealth:

$$w_{T+1}^2\left(s^{T+1}\right) = -\sum_{k\in\bar{M}} A_{T+1,k}y_{T,k}^2 + \sum_{k\in\bar{K}\setminus\bar{M}} A_{T+1,k}x_{T,k}^2 \leq$$

$$-A_{T+1,\mu}\delta + \delta'\delta \sum_{k\in\bar{K}\setminus\bar{M}} A_{T+1,k}V_{T,k} + \delta'\delta \sum_{k\in\bar{M}} A_{T+1,k}V_{T,k} =$$

$$-A_{T+1,\mu}\delta + \delta'\delta \sum_{k\in\bar{K}} A_{T+1,k}V_{T,k} = \delta \cdot \sum_{k\in\bar{K}} A_{T+1,k}V_{T,k} \cdot (\delta' - \frac{R_{T+1,\mu}}{V_{T,\mu}}) \leq 0.$$

Consequently, if condition (33) holds and agent 1 uses the strategy defined by (31) and (30), then

$$P\left\{w_{T+1}^2 \leq 0\right\} \geq P\left\{s^{T+1} \in \bar{S}^{T+1}\left(\delta, \delta'\right)\right\} > 0,$$

which means that the strategy Ξ^2 does not survive. □

Proof of Theorem 3. Consider the basic strategy Ξ' defined by the sequence of decisions $(\gamma_t(s^t), 0)$ $(t = 0, 1, ...)$. Denote by w_t, $t = 0, 1, ...$ the wealth of investor employing Ξ and by w_t', $t = 0, 1, ...$ the wealth of investor employing Ξ'. Then $w_t' = w_t$ (a.s.), and Ξ' is a survival strategy. Strategies Ξ^* and Ξ' do not allow for short selling (not just a.s., but everywhere). By applying Theorem 2 from [3] we obtain (16). □

7 Conclusion

The conventional theory of asset pricing currently prevailing in Financial Economics is based on the Walrasian equilibrium paradigm going back to Leon Walras, one of the classics of economic thought of the 19th century. Equilibrium models of this kind typically describe the world of small investors who strive to maximize their individual utilities of consumption subject to budget constraints. Market equilibrium is understood as a situation in which the goals and interests of such economic agents are equilibrated by the market clearing prices (though see, e.g., Flåm [28]). In contrast with Evolutionary Finance, where equilibrium is defined in short-run terms, consecutively from time t to time $t + 1$, in the classical setting one deals with a long-run notion of equilibrium defined for the whole time horizon.

Evolutionary Finance depicts a world radically different from the Walrasian one—a world of large, even super large (primarily institutional) investors who may act on the global level, and whose fundamental objectives are of an evolutionary character: e.g. survival, domination and fastest growth. In fact, fastest growth is often related, and in our models is equivalent, to survival [8]. These

[8] See [3], Sect. 6

factors, rather than the utilities of individual consumption (one gets immeasurably more than one can consume!) come to the fore. In this framework, investment decisions made by each of the market players might substantially affect the equilibrium prices, in contrast with a variety of classical market models where the influence of every particular individual is negligible.

The problem of introducing short selling and endogenous asset supply in EF models has been open for quite a while. The present work suggests a solution to this problem for the EF framework with short lived assets. It would be of interest to extend the results obtained to another basic EF model, describing a market with long-lived dividend-paying assets [2, 23]. Up to now attempts to make progress in this direction faced serious difficulties. Work on this problem represents an interesting topic for further research.

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