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PFAFFIAN DEFINITIONS OF WEIERSTRASS ELLIPTIC FUNCTIONS

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ABSTRACT. We give explicit definitions of the Weierstrass elliptic functions $\wp$ and ζ in terms of pfaffian functions, with complexity independent of the lattice involved. We also give such a definition for a modification of the Weierstrass function σ. Our work has immediate applications to Diophantine geometry and we answer a question of Corvaja, Masser and Zannier on additive extensions of elliptic curves. We also point out further applications, also in connection with Pila-Wilkie type counting problems.

1. INTRODUCTION

Khovanskii's theory of pfaffian functions provides zero estimates for real analytic functions satisfying certain differential equations. Gabrielov and Vorobjov then extended the theory to include, among other things, effective stratification results for varieties defined by pfaffian functions. Following the presentation in [GV], we say that a sequence $f_1, \ldots, f_l : U \to \mathbb{R}$ of analytic functions on an open set U in $\mathbb{R}^n$ is a *pfaffian chain* if, for $i = 1, \ldots, l$ and $j = 1, \ldots, n$ there are real polynomials $p_{i,j}$ in $n + i$ indeterminates such that

$$\frac{\partial f_i}{\partial x_j}(x) = p_{i,j}(x, f_1(x), \ldots, f_i(x))$$

on U. We say that a function f is *pfaffian* if there is a polynomial p such that f is $p(x, f_1(x), \ldots, f_l(x))$. Pfaffian functions come equipped with a notion of complexity. We say that f as above has *order l* and degree (α, β), where α is a bound on the maximum of the degrees of the $p_{i,j}$ and β is a bound on the degree of p. Khovanskii proved bounds on the number of connected components of the zero-set of f, in terms of the complexity of f, provided that the domain U is a sufficiently simple set, such as a product of open intervals.

Macintyre [Mac, Mac2] was the first to observe a connection between pfaffian functions and elliptic functions. He gave Pfaffian definitions (in the sense of first-order logic) of the Weierstrass functions $\wp$ and ζ associated to a fixed lattice. As Macintyre noted, his definitions are

not uniform in the lattice, and the question of whether there is an effective uniform definition remained open after Macintyre's work. We completely resolve this question by giving explicit definitions of the Weierstrass functions $\wp$ and ζ in terms of pfaffian functions. See Theorems 1 and 2 for precise statements. Our results immediately imply uniform explicit zero estimates (see the two Corollaries to Theorem 1) and also an effective form of a result due to Corvaja, Masser and Zannier (see the Corollary to Theorem 2). Macintyre and the first author observed several years ago that the Weierstrass σ function can also be defined in terms of pfaffian functions. However, in contrast to $\wp$ and ζ, this definition is necessarily non-uniform, as we show in the appendix. We do get uniformity for a modification φ of σ. As we discuss below, this is motivated by applications to multiplicative extensions of elliptic curves.

In our definitions we identify $\mathbb{C}$ with $\mathbb{R}^2$. We express our results without the language of logic (the proofs are also logic-free). But this does necessitate further definitions. First, we say that $U \subseteq \mathbb{R}^n$ is a *simple domain* if U is the image of a product of open intervals under an invertible affine transformation. Now let $X \subseteq \mathbb{R}^n$. Suppose that $U_i \subseteq \mathbb{R}^n$ are simple domains, for $i = 1, \ldots, L$, and that for each i we have pfaffian functions $f_{i,1}, \ldots, f_{i,m_i} : U_i \to \mathbb{R}$ with a common chain of order r and degree (α, β). And suppose that $m_i \leq M$ and that

$$X = \bigcup_{i=1}^{L} \{x \in U_i : f_{i,1}(x) = \cdots = f_{i,m_i}(x) = 0\}$$

Then we call X *piecewise pfaffian* (for want of a shorter phrase that hasn't been taken), of *format* $(r, \alpha, \beta, n, L, M)$. We will also need projections of these sets. So if X is as above, and $Y = \pi X$ where $\pi : \mathbb{R}^n \to \mathbb{R}^m$ is the projection onto the first m coordinates, then we call Y a *piecewise subpfaffian* set of *format* $(r, \alpha, \beta, n, L, M)$ (so the measure of complexity of Y is that of X).

Recall that given a lattice $\Omega \subseteq \mathbb{C}$ there is an associated Weierstrass $\wp$-function given by

$$\wp_\Omega(z) = \frac{1}{z^2} + \sum \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

where the sum is taken over nonzero ω in the lattice. We fix a pair of elements $\omega_1, \omega_2 \in \Omega$ that form a basis of Ω such that $\tau = \omega_2/\omega_1$ lies in the upper half plane and satisfies $|\Re(\tau)| \leq \frac{1}{2}$ and $|\tau| \geq 1$. We associate to Ω the set

$$\mathfrak{F}_\Omega = \{r_1\omega_1 + r_2\omega_2; r_1, r_2 \in [0,1), r_1^2 + r_2^2 \neq 0\},$$

which is a fundamental domain for Ω with 0 removed. We prove the following.

Theorem 1. *On $\mathfrak{F}_\Omega$, the graph of $\wp|_{\mathfrak{F}_\Omega}$ is a piecewise pfaffian set of format $(7, 9, 1, 4, 144503, 2)$.*

Applying Khovankii's theorem (in the form of [GV, Corollary 3.3]) and our result, we immediately obtain the following two corollaries.

Corollary. *Suppose that P is a polynomial in two variables, with complex coefficients, not identically zero and of total degree bounded by $T \geq 20$. Then, on $\mathfrak{F}_\Omega$, the function $P(z, \wp(z))$ has at most*

$$7.5373 \times 10^{14} T^{11}$$

zeroes.

This estimate is presumably far from optimal in its dependence on the degree, not to mention the constant. If the constant is allowed to depend on the lattice, then the 11 can be replaced with a 2. See for instance Exercise 22.21 in Masser's recent book [Mas2] for such an estimate. And see also the paper [BM] by Brownawell and Masser for further discussion on optimality of zero estimates (also for the exponential function).

Corollary. *Suppose that $l \geq 1$ and that P is a polynomial in $2l$ variables, with complex coefficients, not identically zero, and of total degree bounded by $T \geq 4$. Let $\Omega_1, \ldots, \Omega_l$ be lattices. Then on $\mathfrak{F}_{\Omega_1} \times \cdots \times \mathfrak{F}_{\Omega_l}$, the zero set of the function $P(z_1, \ldots, z_l, \wp_{\Omega_1}(z_1), \ldots, \wp_{\Omega_l}(z_l))$ has at most*

$$10^{21l+7} l^{7l} T^{11l}.$$

connected components.

In contrast to the previous corollary, as far as we are aware there was no effective result of this form even with a constant depending on the lattices chosen. Indeed, the only ineffective proof of which we are aware goes via pfaffian methods in combination with Peterzil and Starchenko's work [PS] on the o-minimality of $\wp$ as a function of two variables. And the flexibility of pfaffian methods means many other similar results follow immediately. For instance, we could bound the number of connected components of the zero set of a function of the form $P(z_1, z_2, \wp_{\Omega_1}(z_1 + z_2), \wp_{\Omega_2}(z_2))$, uniformly and explicitly in terms of the degree of P. Or, using the fact that exp is pfaffian on its usual fundamental domain, we could count connected components of of a function of the form $P(z_1, z_2, \exp(z_1), \wp_{\Omega_1}(z_2))$. And so on.

Beyond zero estimates, our result opens up the possibility of applying results on pfaffian functions (for a survey, see [GV]) to elliptic functions in an effective manner. So, for instance, the second author has recently found [S2] a new proof of polynomial Galois bounds for torsion on elliptic curves, a result previously established by Masser [Mas] and David [D]. This proof combines our work here with ideas of Pila [P], [P2] and

the first author and Thomas [JT] on counting problems for pfaffian functions. Our work also potentially extends the applicability of the methods used by Binyamini and Novikov in their recent breakthrough on Wilkie's conjecture [BN]. Their methods allow for a certain uniformity and in combination with the results of this paper this might lead to a counting result for sets definable (by suitably simple formulas) in an expansion of the real ordered field by restrictions of Weierstrass elliptic functions that is effective and uniform in the lattice. We mention some other possible applications below.

Another important function associated to a lattice is the Weierstrass zeta function ζ_Ω which is the unique odd (meromorphic) function whose derivative is equal to $-\wp_\Omega$. Consequently it is not periodic but quasi-periodic with respect to the lattice. That is

$$\zeta_\Omega(z + \omega) = \zeta_\Omega(z) + \eta(\omega), \ \omega \in \Omega$$

where η is a group homomorphism from Ω to $\mathbb{C}$. Even so restricting ζ_Ω to a fundamental domain for the action of Ω is sufficient to describe the graph of ζ_Ω. We prove the following for this restriction.

Theorem 2. *On $\mathfrak{F}_\Omega$, the graph of $\zeta|_{\mathfrak{F}_\Omega}$ is a piecewise subpfaffian set of format* $(9, 9, 1, 6, 144503, 4)$.

As with $\wp$ above, this leads immediately to a uniform explicit zero estimate for ζ on $\mathfrak{F}_\Omega$, again presumably far from optimal (compare with Exercises 22.22 and 22.23 in [Mas2]). We omit this, and instead give the application which began our interest in the problems considered here. This is motivated by a result of Corvaja, Masser and Zannier [CMZ, Theorem 1]. Suppose that E is an elliptic curve over the complex numbers and that G is an extension of E by $\mathbb{G}_a$ with no nonconstant regular functions. Let $\mathfrak{K}$ be the maximal compact subgroup of G. Corvaja, Masser and Zannier proved that if $\mathcal{C}$ is a curve in G then the intersection $\mathcal{C} \cap \mathfrak{K}$ is finite. In fact they showed that the size of the intersection is bounded by a function of the degree of the curve $\mathcal{C}$. They asked about what shape this function $c(d)$ could take. Using Khovanskii's theorem together with our results above leads to the following.

Corollary. *We can take the function $c(d)$ above to be*

$$1.333 \times 10^{25} d^{19}$$

for $d \geq 3$.

So the bound is completely independent of the curve E. As Corvaja, Masser and Zannier noted, their result can be interpreted as a sharpening of Manin-Mumford for additive extensions of elliptic curves, and the above gives an explicit bound for Manin-Mumford that is independent of G. To the best of our knowledge, this is the first instance of such a uniform and explicit Manin-Mumford result. For more details,

and an extension to extensions of products of elliptic curves, see [JS].

Finally the function highest in the hierarchy of Weierstrass function is the sigma function σ_Ω. It vanishes at all points of Ω and its logarithmic derivative equals ζ. As mentioned above, σ can be defined from pfaffian functions, but not uniformly (see the appendix). To get uniformity, we instead investigate the function φ_Ω defined by

$$(1) \qquad \varphi_\Omega = \exp(-\frac{1}{2}z^2\eta_1/\omega_1 + \pi i z/\omega_1)\sigma_\Omega,$$

where $\eta_1 = \eta(\omega_1)$. This function is periodic in ω_1 and transforms as follows with translations by ω_2

$$\varphi_\Omega(z + \omega_2) = -\exp(-2\pi i z)\varphi_\Omega$$

[L, p.246, Theorem 3'] and by iteration

$$(2) \quad \varphi_\Omega(z + n\omega_2) = (-1)^n \exp(-2\pi i n z/\omega_1 - \pi i n(n-1)\omega_2/\omega_1)\varphi_\Omega(z)$$

for an integer $n \geq 0$ and then by translation

$$(3) \quad \varphi_\Omega(z - n\omega_2) = (-1)^n \exp(2\pi i n z/\omega_1 - \pi i n(n+1)\omega_2/\omega_1)\varphi_\Omega(z)$$

Hence giving a Pfaffian definition of the (graph of the) function on $\mathfrak{F}_\Omega$ leads to an understanding of its graph as a function on $\mathbb{C}$ and we prove the following.

Theorem 3. *The graph of φ restricted to $\mathfrak{F}_\Omega$ is a piecewise subpfaffian set of format* $(17, 9, 6, 10, 114565235503, 8)$.

This φ is again connected with extensions of elliptic curves, this time by the multiplicative group. And in future work, we will apply our theorems, together with recent work by Margaret Thomas and the first named author [JT3] to prove certain effective instances of relative Manin-Mumford for semi-constant families of multiplicative extensions of a fixed elliptic curve as in [BMPZ]. And again, there is potential for uniformity here, at least in the case that the elliptic curve has complex multiplication.

Let $\wp = \wp_\Omega$ be as above, and let e_1, e_2 and e_3 be the zeros of the associated polynomial $4w^3 - g_2 w - g_3$ where $g_2(\Omega) = 60 \sum \omega^{-4}$ and $g_3(\Omega) = 140 \sum \omega^{-6}$ with the summations over nonzero periods. Macintyre proved that on a simply connected open set U in $\mathbb{C} \setminus \{e_1, e_2, e_3\}$ the real and imaginary parts of any fixed branch of the inverse of $\wp$ are pfaffian. In order to give a uniform pfaffian definition of the graph of $\wp$ on $\mathfrak{F}_\Omega$, we would like to use Macintyre's result as follows. Start with a small disc in $\mathbb{C} \setminus \{e_1, e_2, e_3\}$ and fix a branch of the inverse of $\wp$ on the disc that takes values in $\mathfrak{F}_\Omega$. Now continue this branch of the inverse to a half-plane bounded by a horizontal or vertical line through some e_i, or if there isn't such a continuation then instead continue to a strip bounded by two of these lines. (For instance, if

$e_1 = 0, e_2 = 1$ and e_3 is near 0 with positive imaginary part, then we could continue from a small disc between 0 and e_3 to the strip bounded by horizontal lines through 0 and e_3, whereas if we started with a small disc above e_3 we could continue to the halfplane bounded by the horizontal line through e_3.) Then repeat this process until we've covered $\mathbb{C} \setminus \{e_1, e_2, e_3\}$ with half-planes and strips on which we have branches of the inverse. Clearly the number of domains needed is uniform in Ω. And by Macintyre's result, all the functions involved have pfaffian real and imaginary parts. We would like to then define $\wp|_{\mathfrak{F}_\Omega}$ by translating the values of the inverse back into $\mathfrak{F}_\Omega$ whenever they happen to fall outside $\mathfrak{F}_\Omega$. But this introduces a potential lack of both effectivity and uniformity, for we don't know in advance how many translations we will need (or even if the number of translations needed is finite). This is the problem we solve, at least in a sufficiently general special case. We do not work directly with the $\wp$ functions as above, rather we work with the functions associated to the Legendre curves E_λ defined by $Y^2 = X(X - 1)(X - \lambda)$, for complex numbers $\lambda \neq 0, 1$. Let $\Gamma = \{\lambda \in \mathbb{C} \setminus \{0, 1\}; |\lambda| \leq 1, |1 - \lambda| \leq 1\}$. Then for λ in the interior of Γ we can define analytic functions

$$\pi F(\lambda), \ i\pi F(1 - \lambda)$$

where $F(\lambda) = F(1/2, 1/2, 1; \lambda)$ is a classical hypergeometric function

$$F(\lambda) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n!^2} \lambda^n$$

$((\frac{1}{2})_n = \frac{1}{2}(\frac{1}{2} + 1) \ldots (\frac{1}{2} + n - 1))$. It is known (see for example [Hu, page 184]) that $\omega_1 = \pi F(\lambda)$ and $\omega_2 = i\pi F(1 - \lambda)$ form a basis of the lattice associated to the differential $\frac{dX}{2Y}$ on E_λ. For our inverse function we start with

$$(4) \qquad z(\lambda, \xi) = \int_\xi^{-\infty} \frac{dX}{2\sqrt{X(X - 1)(X - \lambda)}}$$

where ξ lies in the open interval $(-\infty, 0)$. With these definition we can introduce the Betti coordinates (following Bertrand's terminology, see page 1016 of [CMZ2]) as follows:

$$(5) \qquad\qquad\qquad b_1(\lambda, \xi) = \frac{\bar{\omega_2} z - \omega_2 \bar{z}}{A}$$

$$(6) \qquad\qquad\qquad b_2(\lambda, \xi) = \frac{\omega_1 \bar{z} - \bar{\omega_1} z}{A}$$

where we set

$$A(\lambda) = \omega_1 \bar{\omega_2} - \omega_2 \bar{\omega_1}.$$

These Betti coordinates are the real coordinates of z in terms of the basis ω_1, ω_2 of $\mathbb{C}$. Another way of phrasing this would be to state that for an open disc U in $\mathbb{C} \setminus \{0, 1\}$ they induce a canonical isomorphism (after quotienting by $\mathbb{Z}^2$) $E_\lambda \times_\lambda U \cong \mathbb{R}^2/\mathbb{Z}^2 \times U$. For our discussion it is

important that they increase as we pass through further fundamental domains, and our main technical result for $\wp$ is an explicit bound on $|b_1|$ and $|b_2|$, with ξ as above and $\lambda \in \Gamma$ also such that $\Re(\lambda) \leq 1/2$. This can then be extended to bounds that hold for all ξ, via a topological argument on suitably continuing z. To bound $|b_1|$ and $|b_2|$ we first produce a lower bound for $|A|$, establishing an explicit lower bound of the form $|A| \gg \log|\lambda|^{-1}$ for small $\lambda \in \Gamma$. This may be of some independent interest, also because of the connection of A to the Faltings height of E_λ (for algebraic λ). To bound the Betti coordinates we then bound the numerators in (5), (6). Standard estimates would give upper bounds of order $(\log|\lambda|^{-1}|)^2$, where we stay with small λ in Γ. In order to bound $|b_1|, |b_2|$ by an absolute constant, we show that if we develop the elliptic logarithm $z(\xi, \lambda)$ of E_λ as a power series in λ, where ξ is the X-coordinate of E_λ and stays on the negative real line, and consider the numerator of b_1 in (5) in fact some cancellation occurs in the singular expansion and we can remove a power of the logarithm. (By singular expansion here we mean a power expanison in $\lambda, \overline{\lambda}$ with some logarithmic terms). Because of convergence reasons this only works when λ is large compared to ξ but when λ is small compared to ξ we can estimate directly. The arguments to obtain Proposition 5 go along similar lines. We refer the reader to the corresponding sections 5,7 and 8 for the technical details.

Once we have established bounds on $|b_1|$ and $|b_2|$ we can proceed more or less as in the sketch above to give a definition for the function $\wp_\lambda$ associated to E_λ, at least for those λ for which our bounds hold. But we can then extend easily to the general case, as every elliptic curve over $\mathbb{C}$ is isomorphic to E_λ for such a λ. We can then give definitions for ζ, using elliptic integrals of the second kind (again following Macintyre).

Betti maps of this type were introduced in a paper by Masser and Zannier [MZ], and the terminology comes from a paper by Bertand, Masser, Pillay and Zannier [BMPZ]. In these papers the maps are a tool in the study of unlikely intersection problems. Very recently, the maps themselves have been studied, for instance in a paper by Corvaja, Masser and Zannier [CMZ2], a paper by Voisin [V], and work by André, Corvaja and Zannier [AMZ]. These maps were also implicitly used in older work, in particular in Manin's famous proof of the Mordell conjecture over function fields [Man].

We now return to the Weierstrass functions. We cannot directly handle $\varphi = \varphi_\Omega$ but have to pass to its logarithm whose derivative is given by an expression involving ζ. Then we can use the definition of ζ to locally define the logarithm and compose with the exponential function to define φ locally. The main new technical problem here is that as

we continue the logarithm we might pass through many fundamental domains of the exponential function; again a potential threat to both uniformity and effectivity. We give an explicit absolute bound on the imaginary part of this continuation for φ_λ associated to E_λ.

In addition to the potential for diophantine applications, there is some possibility of using our work to study elliptic functions from the logical point of view. Macintyre has (see [Mac]) established decidability for the theory of the expansion of the real field by a restricted $\wp$-function, assuming suitable transcendence conjectures. Pfaffian ideas play an important role in Macintyre's proof and it could be interesting to revisit the proofs in search of uniformity.

This is how the article is organized. In the next section we investigate the functions we are working with more closely and state the propositions. Then in section 3 we prove several crucial lower bounds. In section 4 we reduce Theorem 1,2 and 3 to Propositions 4, 5 stated in section 2. In section 5 we prove the necessary upper bounds for the conclusion of Proposition 4 in section 6. Then in section 7 and 8 we prove the upper bounds necessary to conclude Proposition 5 in section 9. Finally, in the appendix we point out some limits of how far our results can be pushed. And then we conclude with some further remarks on the Betti map.

2. Analytic continuation

The definition of the inverse in (4) is ambiguous and it is important for us to have fixed definitions of the functions involved in the Betti-coordinates for our explicit estimates. So for $\lambda \in \Gamma$ we introduce the set $X_\lambda = \mathbb{C} \setminus ((-\infty, 0] \cup L_\lambda \cup [1, \infty))$ where L_λ is a straight closed line joining 0 and λ and we will define the inverse on this simply connected set.

We will also need definitions of ω_1 and ω_2 as closed elliptic integrals for the estimates in section 3, and we begin by giving these. They are

equal to

$$(7) \quad \omega_1 = \int_1^\infty \frac{dX}{\sqrt{X(X-1)(X-\lambda)}}, \ \omega_2 = \int_0^{-\infty} \frac{dX}{\sqrt{X(X-1)(X-\lambda)}}$$

where, for both, we take the integral over the real line. These equalities are proven in [Hu]. However, we also want to make the choice of a square-root clear. For ω_1 we chose the standard square-root on the complex plane sliced along the negative real axis such that $\sqrt{1} = 1$ and for ω_2 we chose the square-root on the complex plane sliced along the positive real axis such that $\sqrt{-1} = i$. By restricting to $\lambda \in (0,1)$ we see that we have made the right choice. For our investigations we also need the following two equalities, which are proven below

$$(8) \quad -\omega_1 = \int_0^\lambda \frac{dX}{\sqrt{X(X-1)(X-\lambda)}}, \ \omega_2 = \int_\lambda^1 \frac{dX}{\sqrt{X(X-1)(X-\lambda)}}.$$

Here we pick the same square-root for ω_1, ω_2 as in (7) and the contours for the integrals are straight lines.

We write Ω_λ for the lattice spanned by ω_1, ω_2. It is well-known that the corresponding Weierstrass-invariants are

$$g_2 = \frac{4}{3}(\lambda^2 - \lambda + 1), \ g_3 = \frac{4}{27}(\lambda - 2)(\lambda + 1)(2\lambda - 1).$$

We denote by $\wp_\lambda, \zeta_\lambda$, and φ_λ the Weierstrass functions associated to Ω_λ. It is known that (4) is a local inverse for $\wp_\lambda + \frac{1}{3}(\lambda + 1)$ (see for instance [MZ, page 1683]. We choose the square root for (4) to be such that $\sqrt{-1} = i$ (the same as for ω_2) and for each fixed λ we continue $z(\xi) = z(\lambda, \xi)$ as an analytic function of ξ to X_λ by continuing it north on the complex plane from $(-\infty, 0)$. The function $z(\lambda, \xi)$ has well-defined limits $z(\lambda, 1), z(\lambda, 0)$ and from (7) we deduce that

$$z(\lambda, 0) = \omega_2/2, \ z(\lambda, 1) = \omega_1/2.$$

In order to prove (8) we note that from the above follows that $\wp_\lambda(\omega_1/2) + \frac{1}{3}(\lambda + 1) = 1$, $\wp_\lambda(\omega_2/2) + \frac{1}{3}(\lambda + 1) = 0$ and so consequently $\wp_\lambda((\omega_1 + \omega_2)/2) + \frac{1}{3}(\lambda + 1) = \lambda$. Further

$$(9) \qquad\qquad dz = -\frac{dX}{2\sqrt{X(X-1)(X-\lambda)}}$$

and so $\int_0^\lambda \frac{dX}{\sqrt{X(X-1)(X-\lambda)}} = -\int_{\omega_2/2}^{(\omega_1+\omega_2)/2} dz = -\omega_1/2$ and similarly for the other equality in (8).

With (7) and (4) we can continue ω_1, ω_2, and z for fixed $\xi \in X_\lambda$, as analytic functions of λ to a small neighbourhood of Γ in $\mathbb{C} \setminus \{0, 1\}$ and call them analytic on Γ. From this point on we will work with the so established analytic function $z(\lambda, \xi)$ of two variables on the fibred

product $\Gamma \times_\lambda X_\lambda$. But for our purposes it actually suffices to work on $\mathcal{F} \times_\lambda X_\lambda$ where

$$\mathcal{F} = \{\lambda \in \Gamma; \Re(\lambda) \leq \frac{1}{2}\}.$$

We will deduce Theorem 1 from the following proposition.

Proposition 4. *Let ω_1, ω_2 and z be defined as above on $\mathcal{F} \times_\lambda X_\lambda$. Let b_1, b_2 be as in (5), (6). Then*

$$\max\{|b_1|, |b_2|\} \leq 42.$$

The number in the estimate is unlikely to be the best possible. But it does at least contain the answer to other fundamental questions.

We show in the appendix that the bound necessarily depends on the choice of the fundamental domain.

We now turn to the definition of φ. With ω_1 and ω_2 defined above we define φ_λ by

$$\varphi_\lambda = \exp\left(-\frac{1}{2}\eta_1\omega_1(z/\omega_1)^2 + \pi i z/\omega_1\right)\sigma_\lambda$$

where σ_λ is the Weierstrass sigma function associated to the lattice Ω_λ. We also set $\eta_1 = 2\zeta_\lambda(\omega_1/2)$ and $\eta_2 = 2\zeta_\lambda(\omega_2/2)$ to be the quasi-periods associated to ω_1 and ω_2, respectively. These satisfy the following relations

$$(10) \qquad \eta_1 = \frac{1}{3}(1 - 2\lambda)\omega_1 + 2\lambda(1 - \lambda)\omega_1'$$

$$(11) \qquad \eta_2 = \frac{1}{3}(1 - 2\lambda)\omega_2 + 2\lambda(1 - \lambda)\omega_2'$$

(see for example [S]). These will play a crucial role in the investigations of φ_λ and, in the appendix, of σ_λ. The function φ_λ itself satisfies the following differential equation

$$(12) \qquad \varphi_\lambda'/\varphi_\lambda = -z\eta_1/\omega_1 + \pi i/\omega_1 + \zeta_\lambda.$$

We can write $\zeta_\lambda(z) - \eta_1/2 = -\int_{\omega_1/2}^z \wp_\lambda(t)dt$ and as $\wp_\lambda$ has no residues this integral is independent of the choice of the path. Setting $t = z(\lambda, X)$ and using (9) we obtain

$$\zeta_\lambda(z(\lambda, \hat{X})) = \int_1^{\hat{X}} \frac{(X - \frac{1}{3}(\lambda + 1))dX}{2\sqrt{X(X - 1)(X - \lambda)}} + \eta_1/2$$

for $\hat{X} \in X_\lambda$. Similarly, we have

$$z(\lambda, \hat{X}) = -\int_1^{\hat{X}} \frac{dX}{2\sqrt{X(X - 1)(X - \lambda)}} + \omega_1/2$$

and so with (12) we deduce that

$$\varphi_\lambda'/\varphi_\lambda(z(\lambda, \hat{X})) = \eta_1/\omega_1 \int_1^{\hat{X}} \frac{dX}{2\sqrt{X(X-1)(X-\lambda)}} + \int_1^{\hat{X}} \frac{(X - \frac{1}{3}(\lambda+1))dX}{2\sqrt{X(X-1)(X-\lambda)}} + \pi i/\omega_1.$$

We can't define $\log(\varphi_\lambda(z)) - \log(\varphi_\lambda(\omega_1/2))$ by the integral $\log(\varphi_\lambda(z)) - \log(\varphi_\lambda(\omega_1/2)) = \int_{\omega_1/2}^z \varphi_\lambda'/\varphi_\lambda(t)dt$ independently of the path of integration as $\varphi_\lambda'/\varphi_\lambda$ has residue 1 at every point of the lattice Ω_λ. However these are the only points where it has non-zero residue. The image of X_λ by z does not contain any lattice points. Thus an integral of φ_λ'/φ along a closed path in that image vanishes. This allows us to define

$$(13) \quad \log(\varphi_\lambda(z(\lambda, \xi))) - \log(\varphi_\lambda(\omega_1/2)) = -\int_1^\xi \frac{\varphi_\lambda'/\varphi_\lambda(z(\lambda, \hat{X}))d\hat{X}}{2\sqrt{X(X-1)(X-\lambda)}}$$

unambiguously for any $\xi \in X_\lambda$ where, except for the endpoint 1, the path of integration lies entirely in X_λ. To ease notation we define $\mathcal{L}$ to be equal to the left hand side of the equation, as defined by the right hand side. Clearly $\exp(\mathcal{L})$ is equal to φ_λ times a constant independent of ξ. We will find a pfaffian definition of the graph φ_λ in this manner with the help of the following proposition.

Proposition 5. *Let $\mathcal{L}$ be defined as above on $\mathcal{F} \times_\lambda X_\lambda$. Then*

$$|\Im(\mathcal{L})/2\pi| \le 384.$$

3. Bounding the area from below

In this section we investigate properties of the periods ω_1 and ω_2 and the area (up to $\pm 2i$) A. We prove a lower bound for $|A|$ and show that the quotient ω_2/ω_1 indeed lies in the standard fundamental domain if λ is restricted to $\mathcal{F}$. This latter fact is of course known, but we couldn't find a suitable reference and so have included a proof. In what follows we first assume that $\lambda \in \Gamma$.

We recall that the j-invariant of E_λ is

$$(14) \qquad\qquad j = 256 \frac{(\lambda^2 + \lambda - 1)^3}{(\lambda-1)^2\lambda^2}.$$

The discriminant D of E_λ is equal to $g_2^3 - 27g_3^2 = 16\lambda^2(1-\lambda)^2$ and it is well-known that the following relation holds

$$(15) \qquad\qquad D = \left(\frac{2\pi}{\omega}\right)^{12} \Delta(\tau)$$

where

$$\Delta(\tau) = q \prod_{n=1}^\infty (1 - q^n)^{24}$$

with

$$q = \exp(2\pi i \tau).$$

Here ω, τ are such that $\omega, \omega\tau$ span the lattice of E_λ and τ lies in the standard fundamental domain of the action of $SL_2(\mathbb{Z})$, so $|\tau| \geq 1$ and $|\Re(\tau)| \leq \frac{1}{2}$ and in particular $\Im(\tau) \geq \sqrt{3}/2$. As $|A|$ is invariant under linear changes of (ω_1, ω_2) by $GL_2(\mathbb{Z})$ we can replace (ω_1, ω_2) by the pair $(\omega, \omega\tau)$ and we see that

$$(16) \qquad\qquad |A| = 2|\omega|^2 |\Im(\tau)|.$$

Now we are ready to prove the following Lemma.

Lemma 6. *If $\lambda \in \Gamma$ then*

$$|A| \geq \max\left\{\frac{4}{\pi}(\log\max\{|\lambda|^{-1}, |1-\lambda|^{-1}\} - \log(11)), 2\sqrt{3}\right\}.$$

Proof. We first prove a lower bound for $|\omega|$. In view of the equality (15) it is enough to prove a lower bound for $|\Delta/D|$.

The product $\prod_{n=1}^{\infty}(1 - q^n)^{24}$ is bounded uniformly from below as follows.

$$(17) \qquad |\prod_{n=1}^{\infty}(1 - q^n)^{24}| \geq \prod_{n=1}^{\infty}(1 - \exp(-n\pi\sqrt{3}))^{24} \geq 9/10,$$

where for the last inequality one could use standard estimates or a simple numerical computation. We also need the following inequality

$$(18) \qquad\qquad \frac{|q|^{-1}}{2080} \leq \max\{1, |j(\tau)|\} \leq 2080|q|^{-1}.$$

which can be derived from the Fourier expansion of the j-function [BMZ, Lemma 1]. From (18) we see that

$$(19) \qquad\qquad |q| \geq \frac{1}{\max\{1, |j(\tau)|\}2080}.$$

This with (15) and (17) leads to the lower bound $|\omega|^2 \geq 6/\delta$, where

$$\delta = \max\{1, 256\frac{|\lambda^2 - \lambda + 1|^3}{|\lambda(\lambda-1)|^2}\}^{1/6}|\lambda(1-\lambda)|^{1/3}.$$

If the maximum on the right hand side of the above equality is attained at 1, then δ is equal to $|\lambda|^{\frac{1}{3}}|1-\lambda|^{\frac{1}{3}} \leq 1$. If the maximum is not attained at 1 then

$$\delta = 2^{\frac{4}{3}}|\lambda^2 - \lambda + 1|^{\frac{1}{2}} = 2^{\frac{4}{3}}|(\lambda - \zeta)(\lambda - \overline{\zeta})|^{\frac{1}{2}} \leq 2^{\frac{4}{3}} \leq 3$$

where $\zeta = \frac{1}{2} + i\sqrt{3}/2$. For the first inequality we used the fact that if we write $\lambda = r + it$ for real r, t then $|r + it - \zeta| = |r - 1/2 + i(t - \sqrt{3}/2)|$ is maximal for varying r if λ lies on the boundary of Γ and the same holds for $\overline{\zeta}$. So we may assume that $|\lambda| = 1$ and $t^2 = 1 - r^2$. A short calculation shows that then $|(\lambda - \zeta)(\lambda - \overline{\zeta})|^2 = 4(r - \frac{1}{2})^2$ which is maximal at $r = 1$ (as r now lies between $1/2$ and 1). Thus

$$(20) \qquad\qquad |\omega|^2 \geq 2$$

and we can already deduce from (16) that

$$(21) \qquad\qquad |A| \geq 2\sqrt{3}.$$

However we can go a little further and note that from (18) we have

$$(22) \qquad\qquad \Im(\tau) \geq \frac{1}{2\pi}(\log\max\{1, |j|\} - \log 2080).$$

Now we will bound $\log\max\{1, |j|\}$ from below.

We denote by $\zeta = \frac{1}{2} + \frac{\sqrt{-3}}{2}$ a root of $\lambda^2 - \lambda + 1$. We can check that for $\lambda \in \Gamma$ and $\min\{|\lambda|, |1 - \lambda|\} \leq \frac{1}{2}$ we have

$$|\lambda^2 - \lambda + 1| = |\lambda - \zeta||\lambda - \overline{\zeta}| \geq \sqrt{3}/4$$

and so

$$|j| \geq 18\max\{|\lambda|^{-2}, |1 - \lambda|^{-2}\}, \text{ for } \min\{|\lambda|, |1 - \lambda|\} \leq \frac{1}{2}.$$

With (22) we get that

$$\Im(\tau) \geq \frac{1}{\pi}(\log\max\{|\lambda|^{-1}, |1 - \lambda|^{-1}\} - \log(11)) \text{ for } \min\{|\lambda|, |1 - \lambda|\} \leq \frac{1}{2},$$

which, with (16) and (21), completes the proof. $\qquad\square$

We now prove some standard facts about $\mathcal{F}$. As mentioned above, these are well-known but we include proofs as we were unable to find a reference.

It is well-known that the symmetric group S_3 acts on $\mathbb{C} \setminus \{0, 1\}$ by the transformations generated by $\lambda \to 1/\lambda$, $\lambda \to 1 - \lambda$. The orbit O_λ of an element $\lambda \in \mathbb{C} \setminus \{0, 1\}$ is given by

$$O_\lambda = \{\lambda, 1/\lambda, 1 - \lambda, 1/(1 - \lambda), \lambda/(\lambda - 1), (\lambda - 1)/\lambda\}.$$

We first observe that $\mathcal{F} \setminus \mathcal{A}$, where $\mathcal{A}$ is the set

$$\mathcal{A} = \{\lambda \in \mathbb{C} \setminus \{0, 1\}; |1 - \lambda| = 1, \Re(\lambda) \leq \frac{1}{2}, \Im(\lambda) < 0\} \cup \{\lambda \in \mathbb{C}; \Re(\lambda) = \frac{1}{2}, \Im(\lambda) < 0\}$$

is a fundamental domain for the action of S_3. Below we write $\mathcal{A}^*$ for the image of $\mathcal{A}$ under complex conjugation.

Lemma 7. *The sets $\mathcal{F} \setminus \mathcal{A}$ and $\mathcal{F} \setminus \mathcal{A}^*$ are fundamental domains for the action of S_3 on $\mathbb{C} \setminus \{0, 1\}$.*

Proof. We already noted that $\mathcal{F}$ contains at least one element of each orbit of S_3. The transformation that sends λ to $\lambda/(\lambda - 1)$ acts like complex conjugation on the circle $|1 - \lambda| = 1$ so also $\mathcal{F} \setminus \mathcal{A}$ and $\mathcal{F} \setminus \mathcal{A}^*$ contain a fundamental domain of S_3. Now it remains to check that $\mathcal{F} \setminus \mathcal{A}$ does not contain two distinct elements of one orbit. We leave the details to the reader. $\qquad\square$

We write $\mathcal{S}$ for the (closure of the) standard fundamental domain in the upper half plane:

$$\mathcal{S} = \{\tau \in \mathbb{H}; |\Re(\tau)| \leq \frac{1}{2}, |\tau| \geq 1\}.$$

Lemma 8. *The set $\mathcal{S}$ is the image of $\mathcal{F}$ under the function ω_2/ω_1. In particular, for $\lambda \in \mathcal{F}$, we have $|\Re(\omega_2/\omega_1)| \leq 1/2$ and $|\omega_2/\omega_1| \geq 1$. Further for $\lambda \in \mathcal{F}$ we have $\min\{|\omega_1|, |\omega_2|\} \geq 1$.*

Proof. The strategy is to show that the boundary of $\mathcal{F}$ maps surjectively to the boundary of $\mathcal{S}$ under the map ω_2/ω_1. This shows with the previous Lemma that no point in the interior of $\mathcal{F}$ maps to the boundary of $\mathcal{S}$. We also show that at least one point in the interior of $\mathcal{F}$ maps to the interior of $\mathcal{S}$ and then conclude with the intermediate value theorem.

To begin, note that $\omega_2/\omega_1(\frac{1}{2}) = i$. From the series expansion of $F(\lambda)$ we see that $\omega_2/(i\omega_1)$ is a real increasing function on the interval $(0, \frac{1}{2}]$ approaching $+\infty$ as λ approaches 0. Hence if $\lambda \in (0, \frac{1}{2})$ then $\omega_2/\omega_1(\lambda)$ lies in the interior of $\mathcal{S}$. We also observe that $\omega_2(1 - \lambda) = i\omega_1(\lambda)$ and since the series defining $F(\lambda)$ has real coefficients $\overline{\omega_2}(\lambda) = -\omega_2(\overline{\lambda})$ and $\overline{\omega_1}(\lambda) = \omega_1(\overline{\lambda})$. So for $\Re(\lambda) = \frac{1}{2}$ we have $|\omega_2/\omega_1| = 1$. From (14) we read off that $j = 0$ for $\lambda \in \{1/2 + i\sqrt{3}/2, 1/2 - i\sqrt{3}/2\}$ and as $j(1/2 \pm i\sqrt{3}/2) = 0$ we deduce that $\omega_2/\omega_1(1/2 \pm i\sqrt{3}/2) \in \{1/2 + i\sqrt{3}/2, 1/2 - i\sqrt{3}/2\}$. We also deduce from the previous Lemma that ω_2/ω_1 is injective on the line $\{\lambda \in \mathcal{F}; \Re(\lambda) = \frac{1}{2}\}$. Hence ω_2/ω_1 maps the line $\{\lambda \in \mathcal{F}; \Re(\lambda) = \frac{1}{2}\}$ bijectively to the arc $\{\tau; |\tau| = 1, |\Re(\tau)| \leq \frac{1}{2}\}$.

We set $\omega_2/\omega_1(1/2 + i\sqrt{3}/2) = \tau_0$ and $\omega_2/\omega_1(1/2 - i\sqrt{3}/2) = \tau_1$, where τ_0, τ_1 are distinct elements of $\{1/2 + i\sqrt{3}/2, 1/2 - i\sqrt{3}/2\}$. We will show now that the image of $\mathcal{A}$ is the line $L_0 = \{\tau; \Re(\tau) = \Re(\tau_0), |\tau| \geq 1\}$ and of $\mathcal{A}^*$ is the line $L_1 = \{\tau; \Re(\tau) = \Re(\tau_1), |\tau| \geq 1\}$. We first remark that $j(\tau)$ is real for $\tau \in \mathcal{S}$ if and only if $\Re(\tau) \in \frac{1}{2}\mathbb{Z}$ or $|\tau| = 1$. As complex conjugation acts like a transformation of S_3 on the circle $|1 - \lambda| = 1$, j is real for λ in $\mathcal{A} \cup \mathcal{A}^*$. We argue only for $\mathcal{A}$. The arguments for $\mathcal{A}^*$ are exactly analogous.

We have that $j(\tau_0) = j'(\tau_0) = j''(\tau_0) = 0$ and $j'''(\tau_0) \neq 0$. Hence the set $\Im(j) = 0$ locally at τ_0 consists of the image of at most three simple real analytic curves intersecting in τ_0. One of these curves is $\{\Re(\tau) = \Re(\tau_0)\}$ and the other two are $\{|\tau| = 1\}$ and $\{|\tau \pm 1| = 1\}$ where the sign $\pm$ depends on the value of τ_0. It is well-known that the derivative of j vanishes only on the sets $\mathbb{Z} + \tau_0$ and $\mathbb{Z} + i$. In particular the set $\Im(j) = 0$ is locally the image of a smooth analytic curve anywhere else (does not "branch"). Thus if for some $\lambda \in \mathcal{A}$, $\omega_2/\omega_1(\lambda)$ lies on one of those curves and satisfies $\Im(\omega_2/\omega_1(\lambda)) < \sqrt{3}/2$ then since $\mathcal{F} \setminus \mathcal{A}^*$ is

a fundamental domain for S_3 all $\lambda \in \mathcal{A}$ satisfy $\Im(\omega_2/\omega_1(\lambda)) < \sqrt{3}/2$. From the singular expansion of ω_2/ω_1 at 0, which is $\omega_2/\omega_1 = \frac{i}{\pi} \log \lambda + v$ for an analytic function v (see the discussion around (28)), we can read off that $\Im(\omega_2/\omega_1)$ tends to infinity as λ approaches 0 so we get a contradiction. Further, again as $\mathcal{F} \setminus \mathcal{A}^*$ is a fundamental domain $\omega_2/\omega_1(\lambda)$ also can not lie on any of the circles of radius 1 centred at an integer and satisfy $\Im(\omega_2/\omega_1(\lambda)) > \sqrt{3}/2$. Thus $\omega_2/\omega_1(\lambda)$ lies on L_0 for all $\lambda \in \mathcal{A}$. As $\Im(\omega_2/\omega_1(\lambda))$ tends to infinity as λ approaches 0 ω_2/ω_1 restricted to $\mathcal{A}$ surjects onto L_0. Now if there were some λ in the interior of $\mathcal{F}$ that does not map to $\mathcal{S}$, then by the intermediate value theorem there would be another λ in the interiour that maps to the boundary of $\mathcal{S}$. This concludes the proof of the first part of the Lemma.

From the discussion above follows that we can take $\omega_1 = \omega$ and $\omega_2 = \omega_1 \tau$ in Lemma 6 and then we can read off the proof of Lemma 6 that $|\omega_1| \geq 1$ and as $|\omega_2/\omega_1| \geq 1$ the second part of the present Lemma follows. $\qquad\square$

4. Reduction to the propositions

In this section we deduce Theorem 1, 2 and 3 from the propositions. The proof goes by constructing the definitions explicitly.

We start with $\wp$. Fix $\lambda \in \mathcal{F}$, and define $z(\xi) = z(\lambda, \xi)$ on X_λ as in section 2. We assume that the imaginary part of λ is not negative. The definition in the other case is completely analogous. Let Ω_λ be the lattice generated by $\omega_1(\lambda)$ and $\omega_2(\lambda)$ and $\mathfrak{F}_\lambda$ be the fundamental domain $\mathfrak{F}_{\Omega_\lambda}$ spanned by those two periods. Let $\wp$ be the $\wp$-function associated to this lattice.

Let V_1 be the half-plane north of the line $\Im(\xi) = \Im(\lambda)$. Let V_2 and V_3 be the pieces of the strip $0 < \Im(\xi) < \Im(\lambda)$ lying west and east of the line L_λ, respectively. And let V_4 be the half-plane with negative imaginary part. Let V_5 be the horizontal line extending west from λ, and V_6 be the horizontal line extending east from λ. And let V_7, V_8 and V_9 be the three lines removed to make X_λ. Finally let V_{10} be the interval $(0, 1)$. On each of these sets we consider z as above, and also its other branch $-z$. We denote the real and imaginary parts of the two branches by u_-, v_- and u_+, v_+ (we suppress the dependence on the domain here). Below we write these as functions of one complex number rather than two reals, except in the following lemma.

Lemma 9. *Let $j \leq 10$ be a positive integer. On the domain V_j the real functions u_-, v_- are collectively pfaffian of order 7 and degree $(9, 1)$. The same statement holds for u_+, v_+.*

Proof. This lemma is due to Macintyre [Mac2]. We give some of the details, following [JT2]. We just write u, v with the choice of branch fixed. For ξ varying in V_j we write $\xi = s + it$ with real s and t. Then we have

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial s} = \frac{\Re\left(\sqrt{g_\Omega(\xi)}\right)}{|g_\Omega(\xi)|}$$

$$\frac{\partial u}{\partial t} = -\frac{\partial v}{\partial s} = \frac{\Im\left(\sqrt{g_\Omega(\xi)}\right)}{|g_\Omega(\xi)|}$$

where $g_\Omega(\xi) = \xi(\xi-1)(\xi-\lambda)$. Let A_Ω and B_Ω be the real and imaginary parts of the polynomial g_Ω. Then

$$\Re\sqrt{g_\Omega} = \frac{1}{\sqrt{2}}\sqrt{\sqrt{A_\Omega^2 + B_\Omega^2} + A_\Omega}$$

$$\Im\sqrt{g_\Omega} = \frac{1}{\sqrt{2}}\sqrt{\sqrt{A_\Omega^2 + B_\Omega^2} - A_\Omega}.$$

Here the inner square root is taken positive, and the outer one has the sign that makes the equations for the partials above hold. We then take

$$f_1 = \frac{1}{\sqrt{A_\Omega^2 + B_\Omega^2}}$$

$$f_2 = \frac{1}{\sqrt{\sqrt{A_\Omega^2 + B_\Omega^2} + A_\Omega}}$$

$$f_3 = \frac{1}{\sqrt{\sqrt{A_\Omega^2 + B_\Omega^2} - A_\Omega}}$$

$$f_4 = \sqrt{\sqrt{A_\Omega^2 + B_\Omega^2} + A_\Omega}$$

$$f_5 = \sqrt{\sqrt{A_\Omega^2 + B_\Omega^2} - A_\Omega}.$$

And then with $f_6 = u$ and $f_7 = v$ we have a pfaffian chain of order 7 and degree $(9, 1)$. $\qquad\square$

Here is our first main result.

Theorem 10. *Suppose that Ω is a lattice in the complex numbers, and that $\wp$ is the associated $\wp$-function. Then on a fundamental domain $\mathfrak{F}_\Omega$ for Ω, the graph of $\wp|_{\mathfrak{F}_\Omega}$ is a piecewise pfaffian set of format $(7, 9, 1, 4, 144503, 2)$.*

Proof. Let $\lambda \in \mathcal{F}$ be such that $c\Omega = \Omega_\lambda$ for some complex number c, where Ω_λ is the lattice generated by $\omega_1(\lambda)$ and $\omega_2(\lambda)$. Let $\wp_\lambda$ be the associated $\wp$-function. Since

$$(z, \xi) \in \text{graph}(\wp_\Omega) \text{ if and only if } (cz, c^{-2}\xi) \in \text{graph}(\wp_\lambda)$$

it is sufficient to show that the graph of $\wp_\lambda$, restricted to the fundamental domain $\mathfrak{F}_\lambda$ given by $\omega_1(\lambda)$ and $\omega_2(\lambda)$, is a piecewise pfaffian set of the format claimed.

For $j = 1, \ldots, 10$, we let $V_j^* = \{z - \frac{1}{3}(\lambda + 1) : z \in V_j\}$ (note that this is still a simple domain). We define a chain on V_j^* by composing the chain from the previous lemma with $z + \frac{1}{3}(\lambda + 1)$. This doesn't change complexities. In particular, we will write $u_-^*(z)$ for the function $u_-(z + \frac{1}{3}(\lambda + 1))$ and similarly for the other branch, and for the branches of the imaginary part. For $j = 1, \ldots, 10$ and integers m, n with absolute value at most 42 we put

$$Y_{\wp,j,m,n}^\pm = \left\{(x, y, f_\wp, g_\wp) \in \mathfrak{F}_\lambda \times V_j^* : u_\pm^*(f_\wp + ig_\wp) + iv_\pm^*(f_\wp + ig_\wp) = x + iy + m\omega_1 + n\omega_2\right\}.$$

We then have

$$\text{graph}(\wp_\lambda|_{\mathfrak{F}_\lambda}) = \bigcup_{j=1,\ldots,10, |m|,|n|\leq 42} Y_{\wp,j,m,n}^\pm$$

$$\cup \left\{\left(\frac{\omega_1}{2}, \frac{\lambda - 2}{3}\right), \left(\frac{\omega_2}{2}, -\frac{\lambda + 1}{3}\right), \left(\frac{\omega_1 + \omega_2}{2}, \frac{1 - 2\lambda}{3}\right)\right\}.$$

That this union does indeed give all the graph follows from Proposition 4. And it is easy to see that the format is that claimed. □

Theorem 11. *Suppose that Ω is a lattice in the complex numbers, and that ζ is the associated ζ-function. Then on a fundamental domain $\mathfrak{F}_\Omega$ for Ω, the graph of $\zeta|_{\mathfrak{F}_\Omega}$ is a piecewise subpfaffian set of format $(9, 9, 1, 6, 144503, 4)$.*

Proof. As before, we take $\lambda \in \mathcal{F}$ such that $c\Omega = \Omega_\lambda$ for some complex number c, where Ω_λ is the lattice generated by $\omega_1(\lambda)$ and $\omega_2(\lambda)$. Then $\zeta_\Omega(z) = c\zeta_\lambda(cz)$ for ζ_λ the ζ-function associated to Ω_λ. And so it is enough to show the result for ζ_λ.

For each $j = 1, \ldots, 10$ pick a point $a_j \in V_j$. Define

$$\tilde{G}_j(\hat{X}) = \int_{a_j}^{\hat{X}} \frac{X - \frac{1}{3}(\lambda + 1)}{2\sqrt{X(X - 1)(X - \lambda)}} dX,$$

a function on V_j. Then we have

$$\tilde{G}_j(\hat{X}) = -\zeta_\lambda(z(\hat{X})) + \zeta_\lambda(z(a_j)).$$

Adding the real and imaginary parts of $\tilde{G}_j$ to the chain from the lemma above we get a chain of length 9 and degree still $(9, 1)$. We then shift again, defining

$$G_j(\hat{X}) = -\tilde{G}_j(\hat{X} + \frac{1}{3}(\lambda + 1)) + \zeta_\lambda(z(a_j)).$$

on V_j^*. And then we have

$$G_j(\wp_\lambda(z)) = \zeta_\lambda(z),$$

with real and imaginary parts occurring in a chain of length 9, together with u_-^*, v_-^*. Let

$$Y_{\zeta,j,m,n}^{\pm} = \{(x, y, f_\wp, g_\wp, f_\zeta, g_\zeta) \in \mathfrak{F}_\lambda \times V_j \times \mathbb{R}^2 : (x, y, f_\wp, g_\wp) \in Y_{\wp,j,m,n}^{\pm}$$

$$\text{and } f_\zeta + ig_\zeta = G_j(f_\wp, g_\wp) - \eta(m\omega_1 + n\omega_2)\}$$

Then

$$\text{graph}(\zeta_\lambda|_{\mathfrak{F}_\lambda}) = \pi\left(\bigcup_{j=1,\ldots,10,|m|,|n|\leq 42} Y_{\zeta,j,m,n}^{\pm}\right.$$

$$\left.\cup \left\{\left(\frac{\omega_1}{2}, 1, \frac{\eta_1}{2}\right), \left(\frac{\omega_2}{2}, 0, \frac{\eta_2}{2}\right), \left(\frac{\omega_1 + \omega_2}{2}, \lambda, \frac{\eta_1 + \eta_2}{2}\right)\right\}\right)$$

where π is the projection which omits the middle two coordinates (that is, the $\wp$-coordinates).

$\square$

Now we turn to φ_λ. We define

$$\hat{H}_{V_j}(\hat{X}) = -\frac{\eta_1}{\omega_1}z(\hat{X}) + \frac{\pi i}{\omega_1} + G_{V_j}(\hat{X})$$

on V_j and then define the function $\mathcal{L}_{V_j}$ by

$$\mathcal{L}_{V_j}(\xi) = -\int_1^\xi \frac{\hat{H}_{V_j} d\hat{X}}{2\sqrt{\hat{X}(\hat{X}-1)(\hat{X}-\lambda)}}.$$

If we define the real and imaginary part of $\mathcal{L}_{V_j}$ to be f_{10}, f_{11} and add them to the chain formed by $f_1, \ldots, f_9$ we get a chain of order 11 and degree still $(9, 1)$.

We also need to define a chain for the exponential function restricted to the set $\mathfrak{F}_e$ of complex numbers whose imaginary part $\Im$ satisfies $-\pi \leq \Im < \pi$. We write

$$\exp(x + iy) = \exp(x)(\cos(y) + i\sin(y))$$

and note that the functions $\exp(x), \tan(y/3), \cos(y/3)$ form a chain of order 3 and degree 2 on the interval $(-3\pi/2, 3\pi/2)$. And with that chain, $\sin(y/3)$ has degree $(2, 2)$. Then using the fact that $\sin(y)$ and $\cos(y)$ are polynomials of $\sin(y/3)$ and $\cos(y/3)$, respectively, of degree 3 we find that the real and imaginary part of $\exp(x + iy)$ is a Pfaffian function of order 3 and degree $(2, 6)$ on the simple domain $\mathbb{R} \times [-\pi, \pi)$.

From the discussions in section 2, in particular (12), (13), we have

$$\varphi_\lambda(\omega_1/2)\exp(\mathcal{L}_{V_j}(\xi)) = \varphi_\lambda(z(\xi)).$$

However as we have to deal with translations by periods we also note that if $\tilde{z}$ is the translate of z that lies in the fundamental domain $\mathfrak{F}_\lambda$ then by Proposition 4 $z = \tilde{z} + m\omega_1 + n\omega_2$ where $|m|, |n| \leq 42$ and

$$(-1)^n \exp(\psi_n(\tilde{z}))\varphi_\lambda(\tilde{z}) = \varphi_\lambda(z).$$

where

$$\psi_n = -2\pi i n\tilde{z}/\omega_1 - \pi i n(n-1)\omega_2/\omega_1$$

if $n \geq 0$ and

$$\psi_n = 2\pi i n\tilde{z}/\omega_1 - \pi i n(n+1)\omega_2/\omega_1$$

otherwise. Using Lemma 8 and Proposition 4 we compute that

$$|\Im(\psi_n(\tilde{z}))/(2\pi)| \leq 515.$$

Now we continue with the definition of the graph of φ_λ. We define

$$Y_{\varphi,j,m,n,k,l}^\pm = \Big\{(x, y, x_\varphi, y_\varphi, f_\mathcal{L}, g_\mathcal{L}, f_\wp, g_\wp, f_\varphi, g_\varphi) \in \mathfrak{F}_\lambda \times \mathfrak{F}_e^2 \times V_j \times \mathbb{R}^2;$$

$$(x, y, f_\wp, g_\wp) \in Y_{\varphi,j,m,n}^\pm, f_\mathcal{L} + ig_\mathcal{L} + 2k\pi i = \mathcal{L}_{V_j}(f_\wp, g_\wp), x_\varphi + iy_\varphi + 2l\pi i = \psi_n(x + iy)$$

$$\text{and } (-1)^n \exp(x_\varphi + iy_\varphi)(f_\varphi + ig_\varphi) = \varphi_\lambda(\omega_1/2)\exp(f_\mathcal{L} + ig_\mathcal{L})\Big\}.$$

Then by Proposition 5, we have that

$$\text{graph}(\varphi_\lambda|_{\mathfrak{F}_\lambda}) = \pi_{x,y,f_\varphi,g_\varphi}\left(\bigcup_{j=1,\ldots,10,|m|,|n|\leq 42,|k|\leq 384,|l|\leq 515} Y_{\varphi,j,m,n,k,l}^\pm\right)$$

$$\cup\left\{\left(\frac{\omega_1}{2}, \varphi_\lambda\left(\frac{\omega_1}{2}\right)\right), \left(\frac{\omega_2}{2}, \varphi_\lambda\left(\frac{\omega_2}{2}\right)\right), \left(\frac{\omega_1+\omega_2}{2}, \varphi_\lambda\left(\frac{\omega_1+\omega_2}{2}\right)\right)\right\}$$

is a piecewise subpfaffian set of format $(17, 9, 6, 10, 114565235503, 8)$. For general Ω we pick $\lambda \in \mathcal{F}$ such that $c\Omega = \Omega_\lambda$. We have

$$\varphi_\Omega(z) = c^{-1}\varphi_\lambda(cz).$$

So we can define φ_Ω on $\mathfrak{F}_\Omega$, by

$$(z, \xi) \in \text{graph}(\varphi_\Omega) \text{ if and only if } (cz, c\xi) \in \text{graph}(\varphi_\lambda).$$

Thus, assuming the propositions we have proved the following.

Theorem 12. *Let Ω be a lattice in $\mathbb{C}$, and let φ be the function associated to Ω as in (1). Then on the fundamental domain $\mathfrak{F}_\Omega$ the graph of the function $\varphi|_{\mathfrak{F}_\Omega}$ is a piecewise subpfaffian set of format at most $(17, 9, 6, 10, 114565235503, 8)$.*

5. Bounding the numerator from above

In this section we establish a logarithmic upper bound for the "numerators" $B_1 = Ab_1$ and $B_2 = Ab_2$ of the Betti-coordinates on $(-\infty, 0] \cup L_\lambda \cup [1, \infty)$. We recall that they are given by

$$(23) \qquad\qquad B_1 = \overline{\omega_2}z - \omega_2\overline{z}$$

$$(24) \qquad\qquad B_2 = \omega_1\overline{z} - \overline{\omega_1}z.$$

It would be relatively straightforward to get a bound of the order of $\log^2$ by estimating each term but in order to get a bound with the right growth we have to aim for some cancellation in the sum. Because of the way we've set things up, the main obstacle lies in estimating B_1 on $(-\infty, 0]$. Before we address that problem we prove some inequalities that can be achieved by rather standard estimates but are nevertheless important for us. We define $z = z(\lambda, \xi)$ on $\Gamma \times (-\infty, 0]$ as in the introduction by (4).

Lemma 13. *If* (λ, ξ) *is in* $\Gamma \times (-\infty, 0]$ *then*

$$|\omega_2| \leq \sqrt{2}\log(|\lambda|^{-1}) + 5,$$

and

$$|z| \leq \log(|\lambda|^{-1}) + \frac{5}{2}.$$

And if λ *is in* $\mathcal{F}$ *then*

$$|\omega_1| \leq 5.$$

Proof. We set $X = -t$ and note that as $\Re(\lambda) \geq 0$ standard inequalities yield

$$(25) \qquad\qquad |t + \lambda| \geq \frac{1}{2}(t + |\lambda|),$$

for $\lambda \in \Gamma$. We use the integral expression (7) for ω_2 and get that

$$|\omega_2| \leq \sqrt{2}\left(\int_0^1 \frac{dt}{\sqrt{t(t+1)(t+|\lambda|)}} + \int_1^\infty \frac{dt}{\sqrt{t(t+1)(t+|\lambda|)}}\right)$$

$$\leq \sqrt{2}\left(\int_0^1 \frac{dt}{\sqrt{t(t+|\lambda|)}} + \int_1^\infty \frac{dt}{t\sqrt{t+1}}\right) \leq \sqrt{2}\left(2\log(\sqrt{2}+1) + 177/100 + \log(|\lambda|^{-1})\right)$$

$$\leq \sqrt{2}\log(|\lambda|^{-1}) + 5$$

where for the second inequality we have used that

$$(26) \qquad \frac{1}{2}\int \frac{dX}{\sqrt{X(X+|\lambda|)}} = \log\left(\sqrt{X+|\lambda|} + \sqrt{X}\right).$$

We can perform the same estimates for z but note that the integrand for z is one-half of that for ω_2. This provides us with the inequality for

z.

Now it remains to prove the inequality for ω_1. We again use the integral expression (7). There we set $X = 1 + t$ and note that

$$|X(X-1)(X-\lambda)| \geq \max\{\tfrac{1}{2}|t|, |t|^3\}, \text{ for } \lambda \in \mathcal{F}.$$

Thus we have

$$|\omega_1| \leq \sqrt{2} \int_0^1 \frac{dt}{\sqrt{t}} + \int_1^\infty \frac{dt}{t^{\frac{3}{2}}} \leq 5.$$

$\square$

In order to estimate B_1 we develop z as a series in λ at 0. We first write

$$z^{(n)}(\xi) = \left(\frac{1}{2}\right)_n \int_\xi^{-\infty} \frac{dX}{2X^{n+1}\sqrt{X-1}}$$

so that

$$(27) \qquad z(\lambda, \xi) = \sum_{n=0}^\infty \frac{z^{(n)}(\xi)}{n!} \lambda^n$$

when this series converges in a neighbourhood of 0. We can then write $z = z^{(0)} + \tilde{z}$. We have a similar expansion for ω_2, given by

$$(28) \qquad \omega_2(\lambda) = i\frac{\omega_1}{\pi} \log(\lambda) + u(\lambda)$$

where u is an analytic function at 0 and log is the canonical branch of the logarithm. We can give a series expansion for u [WW, p.299]

$$(29) \qquad u = i \sum_{n=0}^\infty \frac{(\frac{1}{2})_n^2 (4 \log 2 - 4\gamma_n)}{(n!)^2} \lambda^n$$

where $\gamma_n = 1 - \frac{1}{2} + \cdots - \frac{1}{2n}, (\gamma_0 = 0)$. We also quickly recall that the Taylor expansion of ω_1 at 0 is

$$\omega_1 = \pi \sum_{n=0}^\infty \frac{(\frac{1}{2})_n^2}{n!^2} \lambda^n.$$

We write $\omega_1 = \pi + \tilde{\omega}$. As we will be dealing with log as a real analytic function it is also convenient to set

$$\log(\lambda) = \log(|\lambda|) + i \arg(\lambda)$$

where $-\frac{\pi}{2} \leq \arg \leq \frac{\pi}{2}$ (as we are working on Γ). The next Lemma, which is the main estimate of this section, deals with the main issue connected to the singularity at 0.

Lemma 14. *If $(\lambda, \xi) \in \mathcal{F} \times (-\infty, 0]$ then*

$$\max\{|B_1|, |B_2|\} \leq 14 \log |\lambda|^{-1} + 36.$$

Proof. We first get the estimate for B_2 out of the way. It suffices to use the triangle inequality for the terms in the expression (24) and plug in the estimates from Lemma 13.

For B_1 we first address the convergence of (27). As $(\frac{1}{2})_n/n! \leq 1$ we have

$$|z^{(n)}|/n! \leq |\int_0^\infty \frac{dt}{2(t-\xi)^{n+1}\sqrt{t-\xi+1}}| \leq |\int_0^\infty \frac{dt}{(t-\xi)^{n+1}}| \leq |\xi|^{-n}.$$

Hence the series (27) converges absolutely whenever $|\xi| \geq C|\lambda|$ for some $C > 1$. We set $C = 2$ and, until we explicitly say otherwise, assume that $|\lambda| \leq |\xi|/2$. Thus we are now in the case in which ξ is near to $-\infty$.

Now we plug the expansions

$$z = z^{(0)}(\xi) + \tilde{z}, \ \omega_2 = i\log(|\lambda|) - \arg(\lambda) + \frac{i\tilde{\omega}}{\pi}\log(\lambda) + u$$

into (23). This leads to $B_1 = M + R$ with

$$(30) \qquad M = -i(z^{(0)} + \overline{z^{(0)}})(\xi)\log(|\lambda|) + (\overline{z^{(0)}} - z^{(0)})(\xi)\arg(\lambda).$$

But we will be more precise in our definition of the integral involved here. We set $X = \xi - t$ and plugging this into the integral for $z^{(0)}$ we get

$$(31) \qquad\qquad z^{(0)} = i\int_0^\infty \frac{dt}{2(t-\xi)\sqrt{t+1-\xi}}$$

where we made the choice $\sqrt{-1} = i$. Note that the integral above is an honest real integral now and we see that

$$z^{(0)} + \overline{z^{(0)}} = 0$$

(independently of our choice for the square-root).

This is the cancellation that we hoped for. We now get back to estimating.

First $z^{(0)}(\xi)$, which appears in M. If $|\xi| \geq 1$ then

$$|z^{(0)}(\xi)| = \int_0^\infty \frac{dt}{2(t-\xi)\sqrt{t+1-\xi}} \leq \int_0^\infty \frac{dt}{2(t-\xi)^{\frac{3}{2}}} \leq 1.$$

When $|\xi| < 1$ we note that

$$|z^{(0)}(\xi)| = \int_0^1 \frac{dt}{2(t-\xi)\sqrt{t+1-\xi}} + \int_1^\infty \frac{dt}{2(t-\xi)\sqrt{t+1-\xi}}$$

and

$$\int_1^\infty \frac{dt}{2(t-\xi)\sqrt{t+1-\xi}} \leq \int_1^\infty \frac{dt}{2(t-\xi)^{\frac{3}{2}}} \leq 1$$

while with a similar argument as above

$$\int_0^1 \frac{dt}{2(t-\xi)\sqrt{t+1-\xi}} \leq \int_0^1 \frac{dt}{2(t-\xi)} \leq \frac{1}{2}\log(|\xi|^{-1}) + \log(2)/2.$$

We deduce from the above that

$$(32) \qquad |z^{(0)}(\xi)| \leq \begin{cases} \frac{1}{2}\log(|\xi|^{-1})| + 1 & \text{if } \xi \leq 1 \\ \frac{1}{2} & \text{else.} \end{cases}$$

Now we are going to treat R. From now on we also assume that $|\lambda| \leq \frac{1}{2}$ unless we say otherwise.

Unravelling the terms of R by first decomposing $\omega_1 = \pi + \tilde{\omega}$ collecting terms and then applying the triangle inequality we find that

$$(33) \qquad |R| \leq 2\left|\left(\frac{i\tilde{\omega}}{\pi}\log(\lambda) + u\right)z^{(0)}\right| + 2|\omega_2 \tilde{z}|$$

We first estimate u and note that $\frac{1}{2} \leq \gamma_n \leq \log 2$ so

$$4|\log 2 - \gamma_n| \leq 4\log 2 \leq 3.$$

By majorising $u, \omega_1/\pi, \frac{\tilde{\omega}}{\pi\lambda}$ and $\tilde{z}$ by a geometric series we obtain

$$|u| \leq 3, \ \left|\frac{\tilde{\omega}}{\pi\lambda}\right| \leq 2, \ |\omega_1/\pi| \leq 2, \ |\tilde{z}| \leq 1$$

(for $|\lambda| \leq \frac{1}{2}$). From considerations of the graph of the (continuous) function $t\log(t)$ for $t \in [0, 1/2]$ we deduce that

$$|\lambda\log(|\lambda|)| \leq \exp(-1) \leq \frac{1}{2},$$

(for $|\lambda| \leq \frac{1}{2}$) and that

$$|\log(\lambda)| \leq \log(|\lambda|^{-1}) + \pi/2.$$

And from (32) we have

$$(34) \qquad |z^{(0)}(\xi)| \leq \frac{1}{2}\log(|\lambda|^{-1}) + 1.$$

Using the triangle inequality for (33) and plugging in all of the above inequalities we find that

$$|R| \leq 12\log|\lambda|^{-1} + 30.$$

Using the cancellation in M together with (32) we find that

$$|M| \leq \frac{\pi}{2}\log|\lambda|^{-1} + \pi$$

and finally

$$(35) \qquad |B_1| \leq 14|\log|\lambda|^{-1}| + 33$$

for $|\lambda| \leq \frac{1}{2}$ and $|\lambda| \leq |\xi|/2$.

Now we assume that $|\lambda| \geq \frac{1}{2}$. Then we can use the bounds in Lemma 13 to directly deduce that

$$|B_1| \leq 36.$$

Finally we assume that $|\xi| < 2|\lambda|$ with no other restrictions on (λ, ξ) so we are in the case where ξ is near to 0. We replace z by $\hat{z} = z - \frac{1}{2}\omega_2$ which, as can be seen from the definition, does not change B_1. The integral we consider is now $\hat{z} = -\int_0^\xi \frac{dX}{\sqrt{X(X-1)(X-\lambda)}}$. We will estimate $|\hat{z}|$ independently of ξ and λ.

We set $X = -t$ and compute

$$|\hat{z}| = \left| \int_0^{|\xi|} \frac{dt}{\sqrt{t(t+1)(t+\lambda)}} \right| \leq \int_0^{|\xi|} \frac{dt}{\sqrt{t(t+1)(|t+\lambda|)}}.$$

Clearly $t + 1 \geq 1$ and with (25) we may estimate as follows

$$|\hat{z}| \leq 2^{\frac{1}{2}} \int_0^{|\xi|} \frac{dt}{\sqrt{t(t+|\lambda|)}}.$$

Note that this, again, is an honest real integral.

So finally, using (26) and the fact that the integrand is positive and $|\xi| < 2|\lambda|$ we get

$$|\hat{z}| \leq 2^{\frac{1}{2}} \int_0^{2|\lambda|} \frac{dt}{\sqrt{t(t+|\lambda|)}} = 2^{\frac{3}{2}} \log\left(\sqrt{3} + \sqrt{2}\right).$$

This, with Lemma 13 leads (very crudely) to

$$|B_1| \leq 7 \log |\lambda|^{-1} + 35.$$

for $|\xi| < 2|\lambda|$ and concludes the proof.

$\square$

Now we treat the line L_λ. For this we define z on $\mathcal{F} \times_\lambda L_\lambda$ by

$$(36) \qquad z(\lambda, \xi) = \int_\xi^{-\infty} \frac{dX}{2\sqrt{X(X-1)(X-\lambda)}}$$

where we take the integral along $(-\infty, 0) \cup L_\lambda$.

Lemma 15. *If (λ, ξ) is in $\mathcal{F} \times_\lambda L_\lambda$ then*

$$\max\{|B_1|, |B_2|\} \leq 13 \log(|\lambda|^{-1}) + 65.$$

Proof. As in Lemma 14 we replace z by $\hat{z} = z - \int_\lambda^{-\infty} \frac{dX}{2\sqrt{X(X-1)(X-\lambda)}}$. From our discussion of analytic continuations, in particular (7) and (8),

we see that $\hat{z} = z - \frac{1}{2}(\omega_1 + \omega_2)$ which replaces B_1, B_2 by $B_1 \pm \frac{1}{2}A, B_2 \pm \frac{1}{2}A$. But first note that

$$\hat{z} = \int_\xi^\lambda \frac{dX}{2\sqrt{X(X-1)(X-\lambda)}}$$

where we take the integral along L_λ. We set $X = \xi + t(\lambda - \xi)$ and using $|X - 1| \geq \frac{1}{2}, |X - \lambda| \geq |\xi - \lambda|(1 - t)$ and $|X| \geq t|\lambda - \xi|$ get

$$|\hat{z}| \leq \int_0^1 \frac{dt}{\sqrt{t(1-t)}} = \pi.$$

Now we can read off (23), (24) that $|B_1| \leq 2\pi|\omega_2| + |\omega_1\omega_2|$ and $|B_2| \leq 2\pi|\omega_1| + |\omega_1\omega_2|$. If we plug in the estimates in Lemma 13 we deduce the present Lemma. $\qquad\square$

In the final lemma of this section we treat the interval $[1, \infty)$. We define z on $\mathcal{F} \times [1, \infty)$ by

$$(37) \qquad z(\lambda, \xi) = \int_\xi^\infty \frac{dX}{2\sqrt{X(X-1)(X-\lambda)}}$$

were we take the integral along the real line.

Lemma 16. *Let z be defined as above for (λ, ξ) in $\mathcal{F} \times [1, \infty)$. Then*

$$\max\{|B_1|, |B_2|\} \leq 5\log(|\lambda|^{-1}) + 25.$$

Proof. We can prove using the same computations for ω_1 at the end of Lemma 2 that

$$|z| \leq 5/2.$$

Then using the estimates in Lemma 2 the present Lemma follows. $\quad\square$

6. PROOF OF PROPOSITION 4

We recall that z is defined as the continuation of (4) to the north. We also note that the continuation of z to L_λ from the north is given by (36) where we can take the integral along $(-\infty, 0) \cup L_\lambda$. Using a homotopy argument we see that the continuation of z as a function of ξ to $[1, \infty)$ from the north is equal to (37), where we again take the integral along the real line. However we do not keep track of the sign of the square-root anymore. If we continue z to $[1, \infty)$ from the south and call this continuation z_S while setting z_N for the continuation from the north, then $z_N + z_S$ is an integral of $\frac{dX}{\sqrt{X(X-1)(X-\lambda)}}$ from 1 to ∞. Thus we get $z_N + z_S = \pm\omega_1$ (where the sign $\pm$ depends on the square-root) and so we can write z_S as

$$(38) \qquad z_S = -\int_\xi^\infty \frac{dX}{2\sqrt{X(X-1)(X-\lambda)}} \pm \omega_1.$$

We can then continue z to $(-\infty, 0) \cup L_\lambda$ from the south using (38) and another homotopy argument show that the continuation is of the form

$$(39) \qquad z = -\int_\xi^{-\infty} \frac{dX}{2\sqrt{X(X-1)(X-\lambda)}} \pm \omega_1$$

where we take the integral along $(-\infty, 0) \cup L_\lambda$. In fact, though we won't need this, by continuing z as given by (36) along a small loop around λ and then taking the limit as $\xi \to \lambda$ we can check that $\pm\omega_1 = \omega_1$ and so the square root in (38) is the same as in (7) for ω_1.

We may continue b_1, b_2 using (5), (6) in the same fashion as real analytic functions from $\mathcal{F} \times_\lambda X_\lambda$ to $\mathcal{F} \times_\lambda ((\infty, 0] \cup L_\lambda \cup [1, \infty))$. With the various Lemmas proven in the previous section we can estimate this continuation explicitly.

Lemma 17. *The continuations of b_1, b_2 from $\mathcal{F} \times_\lambda X_\lambda$ to $\mathcal{F} \times_\lambda ((-\infty, 0] \cup L_\lambda \cup [1, \infty))$ satisfy*

$$\max\{|b_1|, |b_2|\} \leq 41.$$

Proof. By Lemma 14, 15, 16 we see that for any fixed λ in $\mathcal{F}$ the continuation of B_1, B_2 as real analytic functions on X_λ to $(-\infty, 0] \cup L_\lambda \cup [1, \infty)$ from the north is bounded by

$$\max\{|B_1|, |B_2|\} \leq 14 \log |\lambda|^{-1} + 65.$$

Now first assume that $|\lambda| \geq \exp(-21/4)$. From Lemma 6 we have $|A| \geq 2\sqrt{3}$ and we can compute that $\max\{|b_1|, |b_2|\} \leq 40$. Now assume that $|\lambda| \leq \exp(-21/4)$. Then we can deduce from Lemma 6 that $|A| \geq (69/100) \log |\lambda|^{-1}$ and from the above inequality follows that $\max\{|B_1|, |B_2|\} \leq 27 \log |\lambda|^{-1}$. We deduce that $\max\{|b_1|, |b_2|\} \leq 40$ for all λ. Thus $\max\{|b_1|, |b_2|\} \leq 40$ for the continuation of b_1, b_2 to $\mathcal{F} \times_\lambda ((-\infty, 0] \cup L_\lambda \cup [1, \infty))$ from the north. By the discussion just before the statement of this Lemma, in particular (38), (39), the continuation of b_1, b_2 to $(-\infty, 0] \cup L_\lambda \cup [1, \infty)$ from the south differs from the continuation from the north by at most 1 in absolute value and we conclude the proof. $\square$

Equipped with Lemma 17 we can prove Proposition 4 using a topological argument.

Proof. (of Proposition 4) We fix λ in $\mathcal{F}$ and let I be the image of the two curves $f_1, f_2 : (0, 1) \to \mathbb{C}$

$$f_1(r) = \wp_\lambda(r\omega_1) + \frac{1}{3}(\lambda + 1), \ \ f_2(r) = \wp_\lambda(r\omega_2) + \frac{1}{3}(\lambda + 1).$$

The set $X_\lambda \setminus I$ might not be connected. But as f_1, f_2 are non-intersecting curves the boundary of each connected component of $X_\lambda \setminus I$ contains a point of $(\infty, 0] \cup L_\lambda \cup [1, \infty)$. So for each $\xi \in X_\lambda \setminus I$ we can continue (z, b_1, b_2) from ξ to $(\infty, 0] \cup L_\lambda \cup [1, \infty)$ along a path lying entirely in $X_\lambda \setminus I$. Now if $z(\xi, \lambda)$ lies in a fundamental domain for $\mathbb{C}$ modulo

$\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ then the continuation of z from ξ to $(-\infty, 0] \cup L_\lambda \cup [1, \infty)$ along such a path will lie in the closure of the same fundamental domain. Thus the Betti-coordinates of the continuation of z and $z(\xi)$ differ by at most 1 in absolute value. This concludes the proof. $\qquad\square$

7. BOUNDING $\Im(\mathcal{L})$ FOR LARGE ξ

We now begin working towards Proposition 5. First, by (10) we have $\eta_1/\omega_1 = 1/3 + \lambda(-2/3 + 2(1-\lambda)\omega_1'/\omega_1)$. Then on writing $X - (\lambda+1)/3 = -1/3 + (X - \lambda/3)$ we can expand $\varphi_\lambda'/\varphi_\lambda$ as

$$\varphi_\lambda'/\varphi_\lambda(z(\lambda, \hat{X})) = \int_1^{\hat{X}} \frac{(X - \lambda/3)dX}{2\sqrt{X(X-1)(X-\lambda)}} + \lambda R_\varphi + \pi i/\omega_1,$$

where

$$R_\varphi = (-2/3 + 2(1-\lambda)\omega_1'/\omega_1) \int_1^{\hat{X}} \frac{dX}{2\sqrt{X(X-1)(X-\lambda)}}.$$

Using (13) we then have

$$\mathcal{L} = -\int_1^\xi \left(\int_1^{\hat{X}} \frac{(X - \lambda/3)dX}{2\sqrt{X(X-1)(X-\lambda)}} \right) \frac{d\hat{X}}{2\sqrt{\hat{X}(\hat{X}-1)(\hat{X}-\lambda)}} -$$

$$\lambda \int_1^\xi \frac{R_\varphi d\hat{X}}{2\sqrt{\hat{X}(\hat{X}-1)(\hat{X}-\lambda)}} + z\pi i/\omega_1 - \pi i.$$

Now we note that

$$\left(\int_1^{\hat{X}} \frac{(X - \lambda/3)dX}{2\sqrt{X(X-1)(X-\lambda)}} \right) \frac{1}{2\sqrt{\hat{X}(\hat{X}-1)(\hat{X}-\lambda)}} =$$

$$\left(\int_1^{\hat{X}} \frac{(X - \lambda/3 - \sqrt{X(X-\lambda)})dX}{2\sqrt{X(X-1)(X-\lambda)}} \right) \frac{1}{2\sqrt{\hat{X}(\hat{X}-1)(\hat{X}-\lambda)}} + \frac{1}{2\sqrt{\hat{X}(\hat{X}-\lambda)}}.$$

With this we can write $\mathcal{L}$ as

$$(40) \qquad \mathcal{L} = -\int_1^\xi \frac{d\hat{X}}{2\sqrt{\hat{X}(\hat{X}-\lambda)}} - \mathcal{R} - \mathcal{R}_\varphi + z\pi i/\omega_1 - \pi i.$$

where

$$\mathcal{R} = \int_1^\xi \left(\int_1^{\hat{X}} \frac{(X - \lambda/3 - \sqrt{X(X-\lambda)})dX}{2\sqrt{X(X-1)(X-\lambda)}} \right) \frac{d\hat{X}}{2\sqrt{\hat{X}(\hat{X}-1)(\hat{X}-\lambda)}}$$

$$\mathcal{R}_\varphi = \lambda \int_1^\xi \frac{R_\varphi d\hat{X}}{2\sqrt{\hat{X}(\hat{X}-1)(\hat{X}-\lambda)}}.$$

We begin by bounding $\mathcal{R}$ and $\mathcal{R}_\varphi$. As with the Betti-coordinates we start with some rather elementary Lemmas.

Lemma 18. *For* $(\lambda, X) \in \mathcal{F} \times_\lambda X_\lambda$

$$|X - \lambda/3 - \sqrt{X(X - \lambda)}| \le 5|\lambda|.$$

Proof. For $|X| \le 2|\lambda|$, the bound follows from the triangle inequality. In the case that $|\lambda/X| \le \frac{1}{2}$, we have $|\frac{\frac{1}{2}(\frac{1}{2})_n}{n!}| \le \frac{1}{2}$. The bound then follows after considering the Taylor expansion of $\sqrt{X(X - \lambda)}$ at $\lambda = 0$, which gives

$$X - \lambda/3 - \sqrt{X(X - \lambda)} = \lambda/6 + \lambda \sum_{n=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2})_n}{(n+1)!} \frac{\lambda^n}{X^n}.$$

$\square$

Lemma 19. *We have*

$$\int_{|\xi|}^{\xi} \left| \frac{d\hat{X}}{2\sqrt{\hat{X}(\hat{X} - 1)(\hat{X} - \lambda)}} \right| \le \pi/\sqrt{|\lambda|}$$

for $|\lambda| \le \frac{1}{2}|\xi|$, *where we take the integral along the arc of a circle with radius* $|\xi|$.

Proof. We first note that as $|\xi - \lambda| \ge |\lambda|$ it is sufficient to prove that

$$(41) \qquad \int_{|\xi|}^{\xi} \left| \frac{d\hat{X}}{2\sqrt{\hat{X}(\hat{X} - 1)}} \right| \le \pi.$$

We set $\hat{X} = |\xi| \exp(2\pi i\theta)$ and obtain

$$\pi \int_0^{\theta_0} \left| \frac{d\theta|\sqrt{\xi}|}{\sqrt{||\xi| \exp(2\pi i\theta) - 1|}} \right| \le 2\pi \int_0^{\frac{1}{4}} \frac{d\theta}{\sqrt{|\sin(2\pi i\theta)|}} \le 4\pi \int_0^{\frac{1}{4}} \frac{d\theta}{\sqrt{\theta}} \le \pi.$$

$\square$

Lemma 20. *For* $\lambda \in \mathcal{F}$ *the following holds*

$$|-2/3 + 2(1 - \lambda)\omega_1'/\omega_1| \le 11.$$

Proof. Lemma 16 gives $|\omega_1| \ge 1$, thus it suffices to find an estimate for $|\omega_1'|$. We write

$$\omega_1' = \int_1^{\infty} \frac{dX}{2(X - \lambda)\sqrt{X(X - 1)(X - \lambda)}}$$

and note that $|X - \lambda| \ge \frac{1}{2}$. It then follows from the estimates at the end of Lemma 13 that $|\omega_1'| \le 5$, which is enough to complete the proof. $\square$

Lemma 21. *For* $|\xi| \ge 1, |\lambda/\xi| \le \frac{1}{2}$ *it holds that*

$$\max\{|\mathcal{R}(\xi)|, |\mathcal{R}_\varphi(\xi)|\} \le 132$$

Proof. We decompose the double integrals involved in the definition of $\mathcal{R}$ and $\mathcal{R}_\varphi$ (formally) as follows

$$\int_1^\xi \int_1^{\hat{X}} = \int_1^{|\hat{\xi}|} \int_1^{|\hat{X}|} + \int_{|\xi|}^\xi \int_1^{|\hat{X}|} + \int_{|\xi|}^\xi \int_{|\hat{X}|}^{\hat{X}} = I_1 + I_2 + I_3$$

where we take the integral along the real line and then along the arc of a circle. We first treat I_1. By Lemma 18 and 20 it is enough to bound $|\int_1^{|\hat{X}|} \frac{dX}{2\sqrt{X(X-1)(X-\lambda)}}|$ by an absolute constant. For this we can use Lemma 13 which gives

(42)
$$\int_1^{|\hat{X}|} \left| \frac{dX}{2\sqrt{X(X-1)(X-\lambda)}} \right| \leq \int_1^\infty \left| \frac{dX}{2\sqrt{X(X-1)(X-\lambda)}} \right| = |\omega_1|/2 \leq 5/2,$$

from which we deduce that $|I_1| \leq 74$.

For I_2 and I_3 we make a distinction between $|\lambda| \geq \frac{1}{2}$ and $|\lambda| \leq \frac{1}{2}$. For $|\lambda| \geq \frac{1}{2}$ we may deduce directly from Lemma 18, 20 and (42) that $\max\{|I_2|, |I_3|\} \leq 66$. For $|\lambda| \leq \frac{1}{2}$ we can use Lemma 18 and Lemma 20 as well as Lemma 19 (note the cancellation of $|\lambda|$) to estimate $\max\{|I_2|, |I_3|\} \leq 132$. □

Now we handle the case $|\xi| \leq 1$.

Lemma 22. *For $1 \geq |\hat{X}| \geq 2|\lambda|$ we have*

$$\int_1^{\hat{X}} \frac{|dX|}{|2\sqrt{X(X-1)(X-\lambda)}|} \leq \log|\lambda|^{-1} + 9,$$

where we integrate first along the real line and then along a circle with radius $|\hat{X}|$.

Proof. We note that $|\sqrt{X(X-1)(X-\lambda)}| = |X\sqrt{(X-1)(1-\lambda/X)}| \geq \sqrt{2}|X||\sqrt{X-1}|$. We first treat the integral along the real line. If $|\hat{X}| \geq \frac{3}{4}$ then

$$\int_1^{|\hat{X}|} \frac{|dX|}{|2\sqrt{X(X-1)(X-\lambda)}|} \leq \int_1^{\frac{3}{4}} \frac{|dX|}{2\sqrt{2}|X|\sqrt{X-1}} \leq \int_1^{\frac{3}{4}} \frac{|dX|}{|\sqrt{X-1}|} \leq 1.$$

If $|\hat{X}| \leq \frac{3}{4}$ then $|X-1| \geq \frac{1}{4}$ and we decompose the integral into two parts, the second being

$$\int_{\frac{3}{4}}^{|\hat{X}|} \frac{|dX|}{|2\sqrt{X(X-1)(X-\lambda)}|} \leq \frac{1}{\sqrt{2}} \int_{\frac{3}{4}}^{|\hat{X}|} \frac{dX}{X} \leq 1/\sqrt{2}\log|\lambda|^{-1} + |(\log(3/2)/\sqrt{2}| \leq \log|\lambda|^{-1} + 1$$

(where we used that $|\hat{X}| \geq 2|\lambda|$). This proves that

$$\int_1^{|\hat{X}|} \frac{|dX|}{|2\sqrt{X(X-1)(X-\lambda)}|} \leq \log|\lambda|^{-1} + 2.$$

For the integral along the arc of the circle we again first assume that $|\hat{X}| \geq \frac{3}{4}$. Then

$$\int_{|\hat{X}|}^{\hat{X}} \frac{|dX|}{|2\sqrt{X(X-1)(X-\lambda)}|} \leq \int_{|\hat{X}|}^{\hat{X}} \frac{dX}{|\sqrt{X-1}|} \leq \int_{|\hat{X}|}^{\hat{X}} \frac{dX}{|\sqrt{X(X-1)}|} \leq 2\pi$$

where in the last inequality we have used (42). Now for $|\hat{X}| \leq \frac{3}{4}$. Then $|X-1| \geq \frac{1}{4}$ and we obtain

$$\int_{|\hat{X}|}^{\hat{X}} \frac{|dX|}{|2\sqrt{X(X-1)(X-\lambda)}|} \leq \int_{|\hat{X}|}^{\hat{X}} \frac{|dX|}{|X|} \leq \pi.$$

$\square$

Lemma 23. *For* $|\xi| \leq 1, |\lambda/\xi| \leq \frac{1}{2}$ *we have*

$$\max\{|\mathcal{R}(\xi)|, |\mathcal{R}_\varphi(\xi)|\} \leq 1100.$$

Proof. By Lemma 18, 20 and 22 it suffices to bound $11|\lambda||\log|\lambda|^{-1} + 6|^2$. Using $\log|\lambda|^{-1} \leq |\lambda|^{-\frac{1}{4}}$ we have

$$11|\lambda||\log|\lambda|^{-1} + 9|^2 \leq 11||\lambda|^{\frac{1}{4}} + 9|\lambda|^{\frac{1}{2}}|^2 \leq 1100.$$

$\square$

Now we handle the term $\int_1^\xi \frac{dX}{2\sqrt{X(X-\lambda)}}$.

Lemma 24. *The following holds*

$$\Im\left(\int_1^\xi \frac{dX}{2\sqrt{X(X-\lambda)}}\right) \leq 7,$$

for $|\lambda/\xi| \leq \frac{1}{2}$.

Proof. As usual, we first integrate along the real line and then along the arc of a circle. For the integral along the arc of a circle we have

$$\left|\int_{|\xi|}^\xi \frac{dX}{2\sqrt{X(X-\lambda)}}\right| \leq \int_{|\xi|}^\xi \left|\frac{dX}{2X\sqrt{1-\lambda/X}}\right| \leq \int_{|\xi|}^\xi \frac{dX}{|X|} \leq \pi.$$

For the integral along the real line we first assume that $1 \leq |\xi| \leq 2$ then as $|X-\lambda| \geq \frac{1}{2}$ and $|X| \geq 1$ we get

$$\left|\int_1^{|\xi|} \frac{dX}{2\sqrt{X(X-\lambda)}}\right| \leq \int_1^2 \left|\frac{dX}{2\sqrt{X(X-\lambda)}}\right| \leq 1.$$

In order to treat the cases $|\xi| \geq 2$ and $|\xi| \leq 1$ we develop $\int_\varepsilon^\xi dX/(2\sqrt{X(X-\lambda)})$ into a Taylor series at $\lambda = 0$ where we set ε equal to 1 if $|\xi| \leq 1$ and equal to 2 if $|\xi| \geq 2$

$$\int_\varepsilon^\xi \frac{dX}{2\sqrt{X(X-\lambda)}} = \int_\varepsilon^\xi \frac{dX}{2X} + \frac{1}{2}\sum_{n=1}^\infty \frac{(\frac{1}{2})_n}{n!} \int_\varepsilon^\xi \left(\frac{\lambda^n dX}{X^{n+1}}\right)$$

and the infinite series converges whenever $|\lambda|/\varepsilon \leq \frac{1}{2}$ and $|\lambda|/|\xi| \leq \frac{1}{2}$. Now if $\varepsilon = 2$ these conditions are always satisfied and if $|\xi| \leq 1$ then $|\lambda| \leq \frac{1}{2}$ and these conditions are satisfied as well. We get the following for $n \geq 1$

$$\left|\frac{1}{2}\frac{(\frac{1}{2})_n}{n!} \int_\varepsilon^\xi \left(\frac{\lambda^n}{X^{n+1}}\right)\right| \leq \left(\frac{1}{2}\right)^n$$

and so

$$\left|\int_\varepsilon^\xi \frac{dX}{2X} - \int_\varepsilon^\xi \frac{dX}{2\sqrt{X(X-\lambda)}}\right| \leq 1.$$

For $|\xi| \leq 1$ or $|\xi| \geq 2$ we get

$$\int_1^\xi \frac{dX}{2\sqrt{X(X-\lambda)}} = \frac{1}{2}\log(\xi) + R_l$$

where $|R_l| \leq 2 + \log(2) \leq 3$ by first integrating from 1 to ε and then from ε to ξ. With the previous estimates for $1 \leq |\xi| \leq 2$ we obtain

$$\left|\Im\left(\int_1^\xi \frac{dX}{2\sqrt{X(X-\lambda)}}\right)\right| \leq \pi + 3 \leq 7.$$

$\square$

8. Bounding $\Im(\mathcal{L})$ for small ξ

In this section we assume that $|\xi| \leq 2|\lambda|$.

It is convenient to define $z(\lambda, \hat{X}), \zeta_\lambda(z(\lambda, \hat{X}))$ with another integral. We have

$$\zeta_\lambda(z(\lambda, \hat{X})) = \int_0^{\hat{X}} \frac{dX(X - \frac{1}{3}(\lambda + 1))}{2\sqrt{X(X-1)(X-\lambda)}} + \eta_2/2.$$

Similarly, we have

$$z(\lambda, \hat{X}) = -\int_0^{\hat{X}} \frac{dX}{2\sqrt{X(X-1)(X-\lambda)}} + \omega_2/2.$$

Now let

$$\tilde{\mathcal{L}} = \int_0^\xi \left(\int_0^{\hat{X}} \frac{dX(X - \frac{1}{3}(\lambda + 1))}{2\sqrt{X(X-1)(X-\lambda)}} \right) \frac{d\hat{X}}{2\sqrt{\hat{X}(\hat{X}-1)(\hat{X}-\lambda)}} -$$

$$(\eta_1/\omega_1) \int_0^\xi \left(\int_0^{\hat{X}} \frac{dX}{2\sqrt{X(X-1)(X-\lambda)}} \right) \frac{d\hat{X}}{2\sqrt{\hat{X}(\hat{X}-1)(\hat{X}-\lambda)}}.$$

Then using the Legendre relation

$$\omega_2 \eta_1 - \omega_1 \eta_2 = 2\pi i,$$

with (12) and the new integrals, we find that

(43) $$\tilde{\mathcal{L}} = \log(\varphi_\lambda(z(\xi))) - \log(\varphi_\lambda(\omega_2/2)).$$

Here we define log locally at $\varphi_\lambda(\omega_2/2) \neq 0$ and then continue $\log(\varphi_\lambda(z(\xi)))$ to the domain $\{|\xi| \leq 2|\lambda|\} \cap X_\lambda$. We note that on the circle $|\xi| = 2|\lambda|$ we have

$$\mathcal{L} - \tilde{\mathcal{L}} = \log(\varphi_\lambda(\omega_2/2)) - \log(\varphi_\lambda(\omega_1/2))$$

and then by analytic continuation this holds also on the whole domain where $\mathcal{L}$ is defined. We also note that

$$|\Im \left(\log(\varphi_\lambda(\omega_2/2)) - \log(\varphi_\lambda(\omega_1/2)) \right)| \leq 2\pi.$$

Now we need another Lemma

Lemma 25. *If* $|\hat{X}| \leq 2|\lambda|$ *then*

(44) $$\int_0^{\hat{X}} \left| \frac{dX}{2\sqrt{X(X-1)(X-\lambda)}} \right| \leq 12.$$

Proof. We first consider the case that $|\hat{X} - 1| \geq \frac{1}{4}$ and show that

$$\int_0^{\hat{X}} \left| \frac{dX}{\sqrt{X(X-\lambda)}} \right| \leq 6,$$

where we take the integral along a straight line joining 0 and $\hat{X}$. We set $X = \hat{X}t$ and, to begin with, assume that $|\lambda/\hat{X}| \leq 1$

$$\int_0^{\hat{X}} \left| \frac{dX}{\sqrt{X(X-\lambda)}} \right| \leq \int_0^1 \frac{dt}{\sqrt{|t(t - \lambda/\hat{X})|}} \leq \int_0^{|\lambda|/|\hat{X}|} \frac{dt}{\sqrt{t(|\lambda/\hat{X}| - t)}} + \int_{|\lambda|/|\hat{X}|}^1 \frac{dt}{\sqrt{t(t - |\lambda/\hat{X}|)}}.$$

We split the first integral on the right into two parts. The first can be estimated as follows

$$\int_0^{|\lambda|/|2\hat{X}|} \frac{dt}{\sqrt{t(|\lambda/\hat{X}| - t)}} \leq \sqrt{2|\hat{X}|/|\lambda|} \int_0^{|\lambda|/|2\hat{X}|} \frac{dt}{\sqrt{t}} \leq 2.$$

and the second as follows

$$\int_{|\lambda|/|2\hat{X}|}^{|\lambda|/|\hat{X}|} \frac{dt}{\sqrt{t(|\lambda/\hat{X}| - t)}} \le \sqrt{2|\hat{X}|/|\lambda|} \int_{|\lambda|/|2\hat{X}|}^{|\lambda|/|\hat{X}|} \frac{dt}{\sqrt{|\lambda/\hat{X}| - t}} \le 2.$$

For the second integral on the right hand side above, we use the fact that $\int \frac{dt}{\sqrt{t(t-|\lambda/\hat{X}|)}} = 2\log(\sqrt{t - |\lambda/\hat{X}|} + \sqrt{t})$ and obtain

$$\int_{|\lambda|/|\hat{X}|}^{1} \frac{dt}{\sqrt{t(t - |\lambda/\hat{X}|)}} \le \log(4) \le 2,$$

where we used that $|\lambda/\hat{X}| \ge \frac{1}{2}$. To complete the first case, note that if $|\lambda/\hat{X}| \ge 1$ then

$$\int_{0}^{\hat{X}} \left| \frac{dX}{\sqrt{X(X - \lambda)}} \right| \le \int_{0}^{1} \frac{dt}{\sqrt{t(1 - t)}} = \pi.$$

Now assume that $|\hat{X} - 1| \le \frac{1}{4}$. Then $|X - \lambda| \ge \frac{1}{4}$ and $|X| \ge \frac{3}{4}$ and we end up with

$$\int_{0}^{\hat{X}} \left| \frac{dX}{2\sqrt{X(X - 1)(X - \lambda)}} \right| \le \int_{0}^{\hat{X}} \left| \frac{dX}{\sqrt{X - 1}} \right|.$$

Here we decompose this last integral into an integral along the real line which is bounded by $1/2$ and an integral along the arc of a circle which by (41) is bounded by 2π. □

We can now bound $|\tilde{\mathcal{L}}|$ and then $\Im(\mathcal{L})$. We have seen at the end of the proof of Lemma 20 that $|\omega_1'| \le 5$. From this, we derive that $|\eta_1/\omega_1| \le 11$. Since $|X| \le 2|\lambda|$, we have $|X - \frac{1}{3}(\lambda + 1)| \le 3$. Using Lemma 25 repeatedly we then find that

$$|\tilde{\mathcal{L}}| \le 2016.$$

This leads to a bound on $|\Im(\mathcal{L})|$ on the domain $|\xi| \le 2|\lambda|$ when we note that

$$|\Im(\mathcal{L})| = |\Im(\mathcal{L}) - \Im(\tilde{\mathcal{L}}) + \Im(\tilde{\mathcal{L}})| \le |\Im(\tilde{\mathcal{L}})| + |\Im(\phi_\lambda(\omega_2/2) - \phi_\lambda(\omega_1/2))| \le 2016 + 2\pi \le 2023.$$

9. Proof of Proposition 5

We first note that if $(\lambda, \xi) \in \mathcal{F} \times_\lambda X_\lambda$, then

$$\Im(i\pi z/\omega_1 - i\pi) \le 64\pi.$$

This follows on writing $z = b_1\omega_1 + b_2\omega_2$, observing that the imaginary part of $i\omega_2/\omega_1$ is bounded in modulus by $1/2$ and applying Proposition 4.

To prove Proposition 5, we consider two cases. First, if $|\lambda/\xi| > \frac{1}{2}$, we have shown that $|\Im(\mathcal{L})| \le 2023$ at the end of the previous section. On the other hand, if $|\lambda/\xi| \le \frac{1}{2}$, use (40), and then apply Lemma 24,

Lemma 21, Lemma 23 and Lemma 24 together with the observation above to find that

$$|\Im(\mathcal{L})| \leq 2409.$$

10. Appendix

Here we establish some limits on how far our results can be pushed. To begin, we show that no analogue of our results for $\wp, \zeta$ and φ holds for σ. Here it will be convenient to use some terminology from logic.

We assume that $\lambda \in \mathcal{F}$ and recall the singular expansion of ω_2 (28) and the relations (10),(11). We set $\omega = \omega_1 + \omega_2, \eta = \eta_1 + \eta_2$. We first note that as

$$\omega_2 + \omega_1 = i(\omega_1/\pi)\log(\lambda) + \omega_1 + u$$

we have

$$\omega_2' + \omega_1' = i(\omega_1'/\pi)\log(\lambda) + i\omega_1/(\lambda\pi) + \omega_1' + u'.$$

Using the expansion $\omega_1 = \pi + O(\lambda)$ we get

$$2\lambda(1-\lambda)(\omega_2' + \omega_1') = 2i + O(\lambda\log(\lambda))$$

and also using $u = i4\log 2 + O(\lambda)$ we have

$$\omega_2 + \omega_1 = i\log(\lambda) + \pi + i4\log 2 + O(\lambda\log(\lambda)).$$

With the help of these expressions we compute

$$\omega\eta = -\frac{1}{3}\log(\lambda)^2 + \frac{2}{3}(i\pi - 4\log 2)\log(\lambda) - 2\log(\lambda) + O(1).$$

If we express $\log(\lambda)^2 = \log(|\lambda|) + 2i\arg(\lambda)\log(|\lambda|) - \arg(\lambda)^2$ we can further compute

$$\Im(\omega\eta) = \frac{2}{3}(\pi - \arg(\lambda))\log|\lambda| + O(1).$$

It follows from this that as $\lambda \in \mathcal{F}$ approaches 0, the imaginary part of $\omega\eta$ tends to infinity, for we have $|\arg(\lambda)| \leq \pi/2$ for such λ.

Let σ_λ be the Weierstrass sigma function associated to the lattice spanned by ω_1, ω_2. This function is odd and satisfies

$$\sigma_\lambda(z + \omega) = -\sigma_\lambda \exp(\eta(z + \frac{\omega}{2}))$$

[L, Theorem 1, page 241]. Using these two properties we deduce that

$$\frac{\sigma_\lambda((\frac{1}{2} + r)\omega)}{\sigma_\lambda((\frac{1}{2} - r)\omega)} = \exp(r\eta\omega).$$

For $r \in [0, \frac{1}{2})$ both $(\frac{1}{2} - r)\omega$ and $(\frac{1}{2} + r)\omega$ lie in the fundamental parallelogram spanned by ω_1, ω_2. Thus the function

$$\psi_\lambda(r) = \Im\left(\frac{\sigma_\lambda((\frac{1}{2} + r)\omega)}{\sigma_\lambda((\frac{1}{2} - r)\omega)}\right)$$

restricted to $[0, \frac{1}{2})$ is definable in the expansion of the real field by the restrictions of $\Re\sigma_\lambda, \Im\sigma_\lambda$ to this fundamental parallelogram. However $\psi_\lambda(r_0) = 0$ whenever $\theta(r_0) = \frac{1}{\pi}(\Im(r_0\eta\omega))$ is an integer. The function θ is real and continuous on $[0, \frac{1}{2})$ and its image contains an interval of length

$$\left| \frac{\Im(\eta\omega)}{2\pi} \right|$$

and so at least $|\frac{\Im(\eta\omega)}{2\pi}| - 1$ distinct integers. By our observations above, this latter expression tends to $+\infty$ as λ approaches 0 and so the number of zeroes of ψ_λ is unbounded as λ tends to 0. Using this we have the following.

Proposition 26. *Let L be the language of the real ordered field together with two binary functions f and g. Then there is a formula $\theta(x)$ with the following property. For every positive integer n there exists $\varepsilon > 0$ such that if $\lambda \in \mathcal{F}$ with $|\lambda| < \varepsilon$ then, upon interpreting f and g as the real and imaginary parts of σ_λ, the set defined by θ has at least n connected components.*

To prove this from the above, we simply let $\theta(x)$ be a formula which expresses $\psi_\lambda(x) = 0$ in the structures mentioned in the proposition.

From the proposition, it follows immediately that there is no analogue of Peterzil and Starchenko's well-known result on the two-variable $\wp$-function [PS] for σ as a function of two variables. Similarly, it follows easily from the proposition (and Khovanskii's theorem) that if $B(\lambda)$ is a bound on the entries of the format of a representation of $\sigma|_{\mathfrak{F}_\Omega}$ as a piecewise subpfaffian set, then $B(\lambda)$ is unbounded as λ varies in $\mathcal{F}$. So there is no analogue for σ of our results for $\wp, \zeta$ and φ.

Finally, we discuss the choice of the fundamental domain $\mathfrak{F}_\Omega$. We have chosen ω_1 and ω_2 such that ω_2/ω_1 lies in the standard fundamental domain in the upper half plane. Surprisingly (to us at least), this choice is important. In fact, if we change the fundamental domain of Ω, the format of the corresponding definition of $\wp$ might go up. To see this, let

$$\mathfrak{F}_\Omega^{(a,b,c,d)} = \{r_1(a\omega_1 + b\omega_2) + r_2(c\omega_1 + d\omega_2); 0 \le r_1, r_2 < 1, r_1^2 + r_2^2 \ne 0\}.$$

for some integers a, b, c, d with $ad - bc = 1$. Let $B'(a, b, c, d)$ be a bound on the entries of the format of a representation of $\wp|_{\mathfrak{F}_\Omega^{(a,b,c,d)}}$ as a piecewise subpfaffian set.

Proposition 27. *The number B' tends to infinity as $\max\{|a|, |b|, |c|, |d|\}$ tends to infinity.*

Proof. Suppose not. Then we can find an infinite sequence of distinct tuples for which B' is smaller than a fixed constant. Take a

tuple (a, b, c, d) of that sequence and pick the entry that has modulus $n = \max\{|a|, |b|, |c|, |d|\}$. Say it is a. By our assumption there is a representations of the curves $C = \{\wp(r\omega_2) : r \in (0, 1)\}$ and $C_n = \{\wp(r(a\omega_1 + b\omega_2)) : r \in (0, 1)\}$ as piecewise subpfaffian sets whose formats are bounded independently of n. And then there is a similar representation of $C \cap C_n$. So by Khovanskii's theorem the number of connected components of this set is again bounded independently of n. However, this set contains at least $(n-1)/2$ isolated points. If one of the other entries has modulus n we can make an analogous construction. Thus n is bounded along that sequence. This is a contradiction. $\square$

Motivated by a question that Corvaja and Zannier asked us, we now show how the results in this paper lead to an effective bound on the Betti map of a section of E_λ restricted to a small triangle in $\mathbb{C}$ with a vertex 0.

In order to keep the discussion brief we restrict our attention to triangles contained in $\mathcal{F}$ but this could be extended without difficulty.

Let $Q \in \mathbb{C}[X, Y] \neq 0$ and choose Δ an open triangle in the set $\mathcal{F}$ with 0 as one of its vertices such that there is an analytic solution ξ to

$$Q(\xi, \lambda) = 0$$

on Δ and each such solution satisfies $\xi(\lambda) \neq 0, 1, \lambda$. Now we fix such an analytic solution ξ. If ξ is non-constant we pick $\lambda_0 \in \Delta$ such that $\xi(\lambda) \in X_\lambda$ for λ in a neighbourhood of λ_0 and define the Betti coordinates for $\xi(\lambda)$ as usual. If ξ is constant we also define the Betti coordinates with z defined as the continuation of z from the north if ξ lies on the boundary of X_λ. We continue the Betti-coordinates from this neighbourhood analytically to Δ.

Proposition 28. *The Betti map of ξ on Δ is bounded effectively by a constant that depends only on the degree of Q.*

Proof. Let $\lambda_1 \in \Delta$ and $l : [0, 1] \to \Delta$ be a straight line joining λ_0 and λ_1. The set

$$\{t \in [0, 1]; Q(\xi, l(t)) = 0 \text{ and } \xi \in (-\infty, 0] \cup [1, \infty) \cup \{\text{image of } l\}\}$$

has a finite number of connected components $N(\lambda_1)$ and this number is bounded solely and effectively by the degree of Q.

The map $L = (l, \xi \circ l)$ defines a path in the space $S = \Delta \times \mathbb{C} \setminus ((\Delta \times \{0\}) \cup (\Delta \times \{1\}) \cup \{(\lambda, \lambda); \lambda \in \Delta\})$ and we can choose a path from $(\lambda_1, \xi(\lambda_1))$ to $p_0 = (\lambda_0, \xi(\lambda_0))$ lying entirely in the fibred product $\Delta \times_\lambda X_\lambda$. We can compose those two paths to get a loop $\gamma \in \pi_1(p_0, S)$.

The fundamental group $F = \pi_1(p_0, S)$ is generated by the three loops γ_1, γ_2 and γ_3 around $\Delta \times \{0\}, \Delta \times \{1\}$ and $\{(\lambda, \lambda); \lambda \in \Delta\}$ respectively. These are chosen such that, say, the compositum with ξ of the first two are small loops around 0 and 1 respectively while the compositum of the third with $\xi - \lambda$ is a small loop around 0. There is a group homomorphism $\rho : F \to S_2 \ltimes \mathbb{Z}^2$ where the group law on $S_2 \ltimes \mathbb{Z}^2$ is defined by $(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2, x_1 y_1 + y_2)$ (where we identified S_2 with $\{\pm 1\}$). From (7) and (8) we can deduce that it is given by

$$\rho(\gamma_1) = (-1, (0, 1)), \rho(\gamma_2) = (-1, (1, 0)), \rho(\gamma_3) = (-1, (1, 1))$$

and the action of F on the Betti-coordinates (given by analytic continuation) can be expressed by $(a, b) : (b_1, b_2) \to a(b_1, b_2) + b$.

Now by an elementary geometric argument the word-length of γ as a word in $\gamma_1, \gamma_2, \gamma_3$ can be bounded from above by $N(\lambda_1)$ which as remarked above is bounded independently of λ_1. It follows that if $\rho(\gamma) = (x, y)$ then y is bounded independently of λ_1. Since by Proposition 4 the Betti-coordinates on X_λ are bounded effectively so are the Betti coordinates of ξ on Δ. $\square$

We note that although there is some choice involved in Δ, the bound obtained is independent of the choices made. To obtain a statement about a general open triangle with vertex 0, contained in $\mathbb{C} \setminus \{0, 1\}$ we cut it into several simple regions. This construction can also be carried out in an effective manner.

Finally, again in connection with correspondence with Corvaja and Zannier, we discuss the definability of Betti maps (of sections of E_λ), viewed now as functions of λ. To this end we fix $U \subseteq \mathbb{C} \setminus \{0, 1\}$, an open set, definable (by which we shall always mean definable in $\mathbb{R}_{an,exp}$). We suppose that U is simply connected. For instance U could be a sector of the unit circle. On U we take some choice of period maps ω_1 and ω_2 (we need not make the particular choice made elsewhere in the paper, but we do number them such that the quotient below takes values in the upper half plane). We now write λ for the usual λ-function on the upper half plane, and so we will write t for the variable in U. The quotient ω_2/ω_1 is a branch of the inverse of λ. By a theorem of Peterzil and Starchenko [PS], λ is definable on its usual fundamental domain and on the image of this domain under finitely many elements in $\Gamma(2)$ (the elements needed will depend on U and on the choices of the periods). As the inverse of a definable function, the quotient above is definable. It follows that the derivative of this quotient is also definable (see, for example, Chapter 7 of van den Dries's book [vdD]). Computing, we find that

$$\left(\frac{\omega_2}{\omega_1}\right)'(t) = \frac{c}{t(1-t)\omega_1(t)^2},$$

for some absolute $c \neq 0$. So ω_2 is too.

To get the definability of the elliptic logarithms, we use the definability, also due to Peterzil and Starchenko [PS], of the map $\wp$, as a function of both z and τ, on the domain $\{(\tau, z) : \tau \in \mathcal{S} \text{ and } z \in \mathfrak{F}_{\Omega_\tau}\}$. Here $\mathcal{S}$ is the usual fundamental domain in the upper half plane, and Ω_τ is the lattice generated by 1 and τ. This definability clearly extends to the domain with $\mathcal{S}$ replaced by the union of fundamental domains for the λ function that we used above.

Suppose that ξ is an algebraic function of t, and that we have fixed a definite well-defined branch on U. We now write z for some branch on U of the elliptic logarthim determined by ξ. This satisfies

$$\wp(\lambda^{-1}(t), z(t)) = \xi(t) - \frac{1}{3}(t + 1),$$

with the inverse of λ that we gave above. From this follows the definability of z on U. And once we have the periods and the logarithm we get the definability of the Betti maps on U, defined as usual by (5) and (6) but with the periods and logarithm as above. It then follows from general facts on definability, which can again be found for instance in Chapter 7 of van den Dries's book [vdD], that the differential of the Betti map (considered in more general setting in [CMZ2]) is also definable.

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