MODULAR INVARIANT THEORY OF GRADED LIE ALGEBRAS

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Let G be a connected reductive algebraic group with Lie algebra $\mathfrak{g}$, defined over an algebraically closed field k of positive characteristic $p > 0$. The group G acts on $\mathfrak{g}$ via the adjoint action. Two distinct problems are considered in this thesis.

To begin with, assume that $\mathfrak{g}$ is a Lie algebra of exceptional type. We present a complete classification of the G-conjugacy classes of balanced toral elements of $\mathfrak{g}$. As a result, we also obtain the classification of conjugacy classes of balanced inner torsion automorphisms of $\mathfrak{g}$ of order p when $\operatorname{char} k = 0$.

Secondly, we look at the invariant theory of periodically graded Lie algebras. Assume the group G satisfies the standard hypothesis, and let the Lie algebra $\mathfrak{g}$ admit an $\mathbb{F}_p$-grading $\mathfrak{g} = \oplus_{i \in \mathbb{F}_p} \mathfrak{g}(i)$ given by the adjoint action of a toral element $h \in \mathfrak{g}$, in other words $\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = ix\}$ for each $i \in \mathbb{F}_p$. Denote by $G(0) = C_G(h)^\circ$ the connected centralizer of h in G; the group $G(0)$ acts on each subspace $\mathfrak{g}(i)$. We prove that for $i \neq 0$ the ring of invariant functions $k[\mathfrak{g}(i)]^{G(0)} \subseteq k[\mathfrak{g}(i)]$ is a polynomial ring generated by s homogeneous polynomials, where sp is the dimension of a p-cyclic subspace. We show moreover that the quotient morphism $\mathfrak{g}(i) \to \mathfrak{g}(i)/\!\!/G(0)$ induced by the inclusion $k[\mathfrak{g}(i)]^{G(0)} \hookrightarrow k[\mathfrak{g}(i)]$ is a flat morphism, that $k[\mathfrak{g}(i)]$ is a free $k[\mathfrak{g}(i)]^{G(0)}$-module and that there exists a Kostant-Weierstrass section for the action of $G(0)$ on $\mathfrak{g}(i)$.

Declaration

No portion of the work referred to in the dissertation has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

The present work has the purpose of outlining our approach to two distinct problems arising in the theory of modular algebraic groups and Lie algebras. We shall assume that the reader is familiar with the general theory on these topics, we refer in particular to expositions such as [Bo], [Hu1] and [Sp1]. This introduction provides a summary of the strategies applied and the results obtained, along with some literature review. Yet, it is by no means self-contained or exhaustive: more rigorous definitions, terminology and arguments will be given in subsequent chapters.

1.0.1

Let G be a reductive algebraic group, $\mathfrak{g}$ its Lie algebra, and assume they are defined over an algebraically closed field k of positive characteristic $p > 0$. We shall be interested in the case where $\mathfrak{g}$ admits a *periodic grading* given by a toral element, as described hereafter.

For all $x \in \mathfrak{g}$ the derivation $(\operatorname{ad} x)^p$ is an inner endomorphism of $\mathfrak{g}$ that coincides with $\operatorname{ad} x^{[p]}$ for a certain $x^{[p]} \in \mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is canonically endowed with a G-equivariant p-semilinear p-th power map (see [Bo, I.3], for example):

$$x \in \mathfrak{g} \longmapsto x^{[p]} \in \mathfrak{g}.$$

Definition 1.0.1. An element $h \in \mathfrak{g}$ is said to be *toral* if it satisfies $h^{[p]} = h$.

A toral element is semisimple and acts on $\mathfrak{g}$ with eigenvalues belonging to the finite field $\mathbb{F}_p \subseteq k$, since the minimal polynomial of $\operatorname{ad} h$ divides the polynomial $X^p - X$.

We will call $\mathfrak{g}(i) = \{x \in \mathfrak{g} \,|\, [h, x] = ix\}$ the eigenspace relative to the eigenvalue $i \in \mathbb{F}_p$. A toral element h is called *balanced* if the eigenspaces $\mathfrak{g}(i)$ corresponding to nonzero eigenvalues are all of the same dimension. If in addition their dimension is divisible by a certain $d \in \mathbb{N}$, the element is called *d-balanced* (see Definition 3.0.1 for explicit notation). We stress that for a toral element h to be balanced we require that $\dim \mathfrak{g}(i) \neq 0$ for all $i \in \mathbb{F}_p^\times$

The first problem we consider, treated in Chapter 3, is the classification of G-conjugacy classes of balanced toral elements when the Lie algebra $\mathfrak{g}$ is assumed to be of exceptional type.

If h is a toral element of $\mathfrak{g}$, the vector space decomposition $\mathfrak{g} = \bigoplus_{i \in \mathbb{F}_p} \mathfrak{g}(i)$ is in fact a Lie algebra grading. The subspace $\mathfrak{g}(0)$ is a subalgebra, which moreover is the Lie algebra of the identity component of the centralizer $C_G(h)$ of h in G; we will call $G(0) = C_G(h)^\circ$ this connected closed subgroup. It acts on each graded subspace $\mathfrak{g}(i)$, thus inducing an action of $G(0)$ on the ring of polynomial functions, without loss of generality say on $k[\mathfrak{g}(1)]$. Chapters 4 and 5 are concerned with finding a description of the ring of invariant polynomials $k[\mathfrak{g}(1)]^{G(0)} \subseteq k[\mathfrak{g}(1)]$ for this action.

Due to these two problems being unrelated in nature, we saw fit to divide this introduction into two parts dealing specifically with each of them.

1.1 Classification of balanced toral elements

The question of classifying orbits of balanced toral elements stems from work of Premet ([Pr2]), where he gives a complete account of p-balanced toral elements of exceptional Lie algebras. Here p is the characteristic of the field k and it is moreover a good prime for the root system of $\mathfrak{g}$. Premet observes that there is a natural way of associating a nilpotent orbit (more specifically, a Richardson orbit) to a p-balanced toral element h. Indeed, this can be done by inducing from the 0-orbit on the centralizer of h in $\mathfrak{g}$, a Levi subalgebra of $\mathfrak{g}$. This orbit enjoys some particular properties, notably its dimension is divisible by $p(p-1)$ and its elements belong to the restricted nullcone

$\mathcal{N}_p(\mathfrak{g})$. Hence, one can resort to case-by-case computations involving Richardson orbits of this type, since they are not too many.

In Chapter 3 we extend this theory, classifying all balanced toral elements of a simple Lie algebra $\mathfrak{g}$ of exceptional type in positive characteristic p. We allow d to be any nonnegative integer and we drop the assumption of p being a good prime for the root system of $\mathfrak{g}$, hence we also look at bad characteristics. In characteristic 2 and 3 every toral element is automatically balanced, therefore the only nontrivial bad prime to consider is $p = 5$ for the root system of type E_8; this is examined after settling the good characteristic case. Some of the features used in [Pr2] can be adapted to this more general setting. For instance, one can perform the same construction of a Richardson orbit from a conjugacy class of balanced toral elements (at least in good characteristic), but under our assumptions the dimension of the orbit has to be divisible only by $p-1$ (see Section 3.1.1).

The description of conjugacy classes of balanced elements is given in terms of Kac coordinates. These are a sequence of nonnegative integers introduced by Kac ([Ka, Chapter 8]) in order to classify finite order (or *torsion*) automorphisms of a simple Lie algebra in characteristic 0. This draws a connection between toral elements in characteristic $p > 0$ and torsion automorphisms of order p of a simple Lie algebra defined over $\mathbb{C}$, so that results valid for the former admit natural analogues holding for the latter.

Here $G_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}$ will denote an algebraic group and its Lie algebra defined over $\mathbb{C}$. Moreover, let $\mathfrak{g}_{\mathbb{C}}$ be simple. The identity component of its automorphism group $(Aut\,\mathfrak{g}_{\mathbb{C}})^\circ = G_{\mathbb{C}}$ is an algebraic group of adjoint type with $Lie\,G_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$, while the other irreducible components are cosets of the form $\varphi G_{\mathbb{C}}$, where φ is a graph automorphism of the Dynkin diagram. Torsion automorphisms of $\mathfrak{g}_{\mathbb{C}}$ of order m are in bijection with $\mathbb{Z}_m$-gradings on $\mathfrak{g}_{\mathbb{C}}$. Moreover, a finite order automorphism is inner (that is, it belongs to $G_{\mathbb{C}}$) if and only if a maximal toral subalgebra of $\mathfrak{g}_{\mathbb{C}}$ is pointwise fixed.

Our case of interest is the grading of $\mathfrak{g}$ induced by the adjoint action of a toral element h. This is a $\mathbb{Z}_p$-grading in characteristic p, for which any Cartan subalgebra containing h is included in the 0-degree component. Although automorphisms of order m are in bijection with $\mathbb{Z}_m$-gradings on $\mathfrak{g}$ only when m and p are relatively prime, at least in the inner case Kac coordinates can still be used over fields of positive characteristic to parameterize embeddings of the affine group scheme $\boldsymbol{\mu}_p$ in G ([Se]). It turns out that, in characteristic $p > 0$, Kac coordinates relative to such embeddings of $\boldsymbol{\mu}_p$ parameterize conjugacy classes of toral elements of $\mathfrak{g}$.

The algorithm we devised for classifying G-classes of balanced toral elements is described in 3.1.5. Each balanced toral element yields a Richardson orbit of dimension divisible by $p-1$; since there are only finitely many of these orbits, it is possible to check them one by one. Roughly speaking, we seek tuples of Kac coordinates that provide a balanced element. In many instances one can rule out straightaway certain choices of Kac coordinates using combinatorial arguments. Unfortunately this approach is unsuccessful or too involved when both the dimension of the nilpotent orbit and the characteristic of k are large. In order to tackle all the possible cases, we implemented the algorithm in the coding language C ([C]).

As remarked, manual computations are doable and effective in almost all cases. Before realising the trickiest cases could be dealt with by using a computer, we had proved that for exceptional Lie algebras there is a natural injection from the set of connected subgraphs of the extended Dynkin diagram to the set of positive roots Φ^+. A proof of this can be found in 3.5, along with examples of how we used it.

1.1.1 Statement of results

It emerges from the Tables in 3.2 that sometimes there exist more than one conjugacy class of balanced toral elements in char p having isomorphic (and G-conjugate) centralizers. Moreover, if h is balanced, rh is balanced for any $r \in \mathbb{F}_p^\times$. One may ask whether rh is still conjugate to h. Combining the results in Section 3.2 and the arguments in Section 3.3 we prove the following:

Theorem 1.1.1. *Let G be a connected reductive algebraic group of exceptional type over a field k of characteristic $p > 0$ and let $\mathfrak{g} = Lie\, G$.*

(a) *The Kac coordinates of G-orbits of balanced toral elements are listed in Tables 3.1 to 3.5.*

(*b*) *Let* $h, h' \in \mathfrak{g}$ *be balanced toral elements with isomorphic centralizers. There exists* $r \in \mathbb{F}_p^\times$ *such that* rh *is conjugate to* h' *under an automorphism of* $\mathfrak{g}$.

An equivalence relation that extends G-conjugacy can be defined on balanced toral elements by saying that h and h' are in relation if there exists $r \in \mathbb{F}_p^\times$ such that rh is conjugate to h' under an element of $Aut(\mathfrak{g})$. Theorem 1.1.1 states that in characteristic p, once the type of centralizer is fixed, there exists at most one equivalence class for this relation.

These results have an analogous interpretation in the setting of torsion automorphisms of order p of a complex simple Lie algebra $\mathfrak{g}_\mathbb{C}$ of exceptional type.

Let $\sigma \in Aut(\mathfrak{g})$ be an element of order $p > 0$, so that the Lie algebra decomposes as $\mathfrak{g} = \bigoplus_{i \in \mathbb{F}_p} \mathfrak{g}(\sigma, i)$, where $\mathfrak{g}(\sigma, i) = \{x \in \mathfrak{g} \mid \sigma(x) = \xi^i x\}$, for $\xi \in \mathbb{C}$ a fixed primitive p-th root of 1.

Definition 1.1.2. The automorphism σ is said to be balanced if $\dim \mathfrak{g}(\sigma, i) = \dim \mathfrak{g}(\sigma, 1)$ for all $i \in \mathbb{F}_p^\times$.

The analogue of Theorem 1.1.1 holds:

Theorem 1.1.3. *Let* $\mathfrak{g}_\mathbb{C}$ *be a simple Lie algebra of exceptional type over* $\mathbb{C}$, $G_\mathbb{C}$ *a simple linear algebraic group with Lie algebra* $\mathfrak{g}_\mathbb{C}$ *and* $Aut(\mathfrak{g}_\mathbb{C})$ *the automorphism group of* $\mathfrak{g}_\mathbb{C}$.

(*a*) *The Kac coordinates of* $G_\mathbb{C}$*-orbits of balanced automorphisms of prime order* $p > 0$ *are listed in Tables 3.1 to 3.5.*

(*b*) *Let* $\sigma, \sigma' \in \mathfrak{g}$ *be balanced torsion automorphisms with* $G_\mathbb{C}$*-conjugate fixed points subalgebras. There exists* $r \in \mathbb{F}_p^\times$ *such that* σ^r *is conjugate to* σ' *under an element of* $Aut(\mathfrak{g}_\mathbb{C})$.

1.2 Modular invariants of graded Lie algebras

We will now assume that G is a connected reductive linear algebraic group. The adjoint representation of G on $\mathfrak{g}$ yields an action of G on the algebra of polynomial functions $k[\mathfrak{g}]$ via:

$$(g \cdot f)(x) = f(g^{-1} \cdot x),$$

for any $g \in G, f \in k[\mathfrak{g}], x \in \mathfrak{g}$. A classical result of Chevalley (see for example [Jan1, 7.13]) states that the subring $k[\mathfrak{g}]^G \subseteq k[\mathfrak{g}]$ of functions invariant under this action is a polynomial ring, generated by $\operatorname{rank}\mathfrak{g}$ algebraically independent homogeneous generators. This was first proved in characteristic 0 but it also holds for groups satisfying the standard hypothesis (these will be explicitly stated in Section 2.1.2).

The strategy to obtain this result can be summarized as follows. A G-orbit on $\mathfrak{g}$ is closed if and only if it is the orbit of a semisimple element, and each such orbit intersects nontrivially a fixed Cartan subspace $\mathfrak{c} \subseteq \mathfrak{g}$. Two elements of the Cartan subspace are G-conjugate iff they are conjugate under the action of the Weyl group W of $\mathfrak{g}$ on $\mathfrak{c}$. This leads to the existence of a ring isomorphism:

$$k[\mathfrak{g}]^G \simeq k[\mathfrak{c}]^W.$$

The original problem is then reduced to studying the invariants of a *finite* group acting on a finite dimensional vector space. Moreover, the Chevalley-Shephard-Todd Theorem (we include the full statement as Theorem 5.3.1) states that the ring of invariants for the action of a finite group generated by reflections is polynomial. Since this hypothesis holds for the Weyl group, $k[\mathfrak{g}]^G$ is indeed a polynomial ring.

This fact can be seen from a more geometric perspective. In general, if a reductive group acts on an algebraic variety X (see 2.4.1 for all the relevant assumptions), the subring $k[X]^G \subseteq k[X]$ is finitely generated. Hence, it is the coordinate ring of an algebraic variety $X/\!\!/G$, which comes with a surjective morphism $X \twoheadrightarrow X/\!\!/G$ that is constant on G-orbits. Chevalley's result states that the variety $\mathfrak{g}/\!\!/G$ is isomorphic to an affine space of dimension $\operatorname{rank}\mathfrak{g} = \dim\mathfrak{c}$, and there exists an isomorphism of varieties $\mathfrak{g}/\!\!/G \simeq \mathfrak{c}/\!\!/W$.

Periodically graded Lie algebras in the literature. An extension of this problem consists in studying the ring of invariant functions of a *periodically graded* Lie algebra $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}_m} \mathfrak{g}(i)$, for m a nonnegative integer. The 0-degree component $\mathfrak{g}(0)$ is a Lie subalgebra, which is the Lie algebra of a connected closed subgroup $G(0) \subseteq G$. This latter acts on each graded component, so that one can wonder, without loss of generality, what the $G(0)$-orbit structure on $\mathfrak{g}(1)$ and the ring $k[\mathfrak{g}(1)]^{G(0)}$ look like.

Here G is assumed only to be reductive, not necessarily connected. In characteristic 0, the problem for periodic gradings was first tackled by Kostant and Rallis ([KR]) in the special case of a symmetric space, i.e. a $\mathbb{Z}_2$-graded Lie algebra where the grading is induced by an involution. Afterwards, Vinberg ([Vi]) gave a complete account for periodically graded Lie algebras in characteristic 0.

This was partially extended to positive characteristic by work of Levy ([Le1]). More specifically, he studied orbits and invariants of a periodically graded Lie algebra $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{g}(i)$ in the case where the integer m is coprime with the characteristic of the field and the group G satisfies the standard hypothesis.

Notice that for all the cases described so far, the grading can be induced by a finite order automorphism $\theta \in G$ of $\mathfrak{g}$, in the sense that if θ has order m and a primitive m-th root of unity $\xi \in k$ is fixed, then

$$\mathfrak{g}(i) = \{x \in \mathfrak{g} \,|\, \theta(x) = \xi^i x\}.$$

The subgroup $G(0)$ consists of the fixed points for the action of θ on G by conjugation.

All the cases above present several common features. To begin with, every nilpotent element of $\mathfrak{g}(1)$ turns out to be $G(0)$-*unstable*, that is, the closure of its $G(0)$-orbit contains 0. Again, the only closed $G(0)$-orbits on $\mathfrak{g}(1)$ are the orbits of semisimple elements. As a consequence, it is possible to define an object which plays the role of a Cartan subalgebra in this setting. This is called a *Cartan subspace* $\mathfrak{c}$, and is defined as a maximal abelian subspace of $\mathfrak{g}(1)$ consisting of semisimple elements. Any two Cartan subspaces are $G(0)$-conjugate, and two elements of $\mathfrak{c}$ are $G(0)$-conjugate iff they are conjugate under the action of the *little Weyl group* $W_{\mathfrak{c}}$, defined as the quotient $N_{G(0)}(\mathfrak{c})/C_{G(0)}(\mathfrak{c})$, where $N_{G(0)}(\mathfrak{c})$ and $C_{G(0)}(\mathfrak{c})$ are respectively the normalizer and the centralizer of $\mathfrak{c}$ in $G(0)$. This gives an isomorphism of rings:

$$k[\mathfrak{g}(1)]^{G(0)} \simeq k[\mathfrak{c}]^{W_{\mathfrak{c}}}.$$

It follows that the ring $k[\mathfrak{g}(1)]^{G(0)}$ is isomorphic to a polynomial ring in $\dim \mathfrak{c}$ indeterminates. This is a consequence of the fact that in all cases the little Weyl group is a finite group generated by *pseudoreflections*, finite order linear automorphisms of $\mathfrak{c}$ that fix pointwise a codimension 1 subspace. The fact that the ring of invariants for the action of such a group admits a free set of homogeneous generators is, once again, an application of the Chevalley-Shephard-Todd Theorem.

1.2.1 Contents of Chapters 4 to 6

The present work is aimed at addressing this problem in a yet untreated case: that of a $\mathbb{Z}_p$-graded Lie algebra when the field k is of characteristic $p > 0$. We assume that the group G satisfies the standard hypothesis.

Some difficulties arise in this setting, making the strategy and the proofs slightly different from the cases described above.

To begin with, the grading is not induced by an automorphism of the Lie algebra, as in characteristic $p > 0$ an automorphism of order p has 1 as its only eigenvalue. Still, it has been observed ([Se]) that such a grading is induced by the adjoint representation on $\mathfrak{g}$ of a toral element h. Indeed, we already remarked that all the eigenvalues of this endomorphism belong to $\mathbb{Z}_p$. Of course this is true provided the subalgebra $\mathfrak{g}(0)$ contains a maximal toral subalgebra of $\mathfrak{g}$, so we will stick to this assumption throughout. The subgroup $G(0)$ is the identity component of the centralizer of h in G.

Even in this case a nilpotent element of $\mathfrak{g}(1)$ is $G(0)$-unstable. Unlike the characteristic 0 case, where this can be proved by exploiting the existence of $\mathfrak{sl}_2$-triples, we have to resort to the Kempf-Rousseau theory and a generalization of the Bala-Carter theory by Premet ([Pr1]); this is similar to Levy's approach, and it is described in 4.1.2. Remarkably, finiteness of nilpotent $G(0)$-orbits of $\mathfrak{g}(1)$, a feature peculiar to the classical cases, holds for these gradings as well. More generally, each fibre of the quotient morphisms $\mathfrak{g}(1) \twoheadrightarrow \mathfrak{g}(1)/\!\!/G(0)$ consists of finitely many $G(0)$-orbits.

Another issue is the fact that the subspace $\mathfrak{g}(1)$ does not contain any semisimple elements, hence the usual definition of a Cartan subspace does not provide a suitable candidate for parameterizing closed orbits. The correct analogue of this object is what we call a *p-cyclic subspace*, denoted again by $\mathfrak{c}$ and defined in Section 4.3. Any two p-cyclic subspaces are $G(0)$-conjugate, and every element whose $G(0)$-orbit is closed (we called these elements $G(0)$-*semisimple*) belongs to a p-cyclic subspace.

A further difference is that not every element of a p-cyclic subspace has a closed $G(0)$-orbit, whereas in the cases mentioned above a Cartan subspace parameterizes closed orbits (or, more precisely, the variety $\mathfrak{c}/W_\mathfrak{c}$ does). Moreover, it is not true that any two elements of $\mathfrak{c}$ are $G(0)$-conjugate iff they are conjugate under the action of the little Weyl group $W_\mathfrak{c} = N_{G(0)}(\mathfrak{c})/C_{G(0)}(\mathfrak{c})$, but this does hold for $G(0)$-semisimple elements (Lemma 4.4.8). This fact, along with some results on separability of the quotient morphism $\mathfrak{g}(1) \to \mathfrak{g}(1)/\!\!/G(0)$ (section 5.1), is enough to prove the existence of an isomorphism of rings:

$$k[\mathfrak{g}(1)]^{G(0)} \simeq k[\mathfrak{c}]^{W_\mathfrak{c}}.$$

We stress that the little Weyl groups we obtain for this action are not finite groups. Still, they possess some nice properties that make it possible to understand the structure of the ring of invariants (for more details, see 5.2), that turns out to be a polynomial ring in $\dim \mathfrak{c}/p$ generators.

1.2.2 Statement of results

The main results can be summarised as follows (see Section 5.3.2). We assume that G is reductive and defined over an algebraically closed field k of characteristic $p > 0$, and G satisfies the standard hypothesis. Let $\mathfrak{g} = \sum_{i \in \mathbb{F}_p} \mathfrak{g}(i)$ be the $\mathbb{F}_p$-grading induced by a toral element $h \in \mathfrak{g}$ and set $G(0) = C_G(h)^\circ$.

Theorem 1.2.1. *Let G be a connected reductive algebraic group satisfying the standard hypothesis. Then the ring $k[\mathfrak{g}(1)]^{G(0)}$ is a polynomial ring.*

Chapter 6 comprises further results that crucially rely on polynomiality of the ring $k[\mathfrak{g}(1)]^{G(0)}$. Namely, in Section 6.1 we prove flatness of the quotient morphism $\mathfrak{g}(1) \twoheadrightarrow \mathfrak{g}(1)/\!\!/G(0)$, which in turn entails freeness of $k[\mathfrak{g}(1)]$ as a $k[\mathfrak{g}(1)]^{G(0)}$-module. Finally, Section 6.2 is devoted to proving the existence of a *Kostant-Weierstrass section* for the action of $G(0)$ on $\mathfrak{g}(1)$, that is a subvariety $\mathfrak{s} \subseteq \mathfrak{g}(1)$ such that the inclusion $\mathfrak{s} \hookrightarrow \mathfrak{g}(1)$ induces a ring isomorphism

$$k[\mathfrak{s}] \simeq k[\mathfrak{g}(1)]^{G(0)}.$$

More explicitly, we obtain (see Section 6.2.9):

Theorem 1.2.2. *If G is a connected reductive algebraic group satisfying the standard hypothesis, the action of $G(0)$ on $\mathfrak{g}(1)$ admits a KW-section.*

1.3 Notation

We assume that the reader possesses some familiarity with the general theory of algebraic groups and Lie algebras, standard references are [Bo], [Hu1] and [Sp1].

Unless otherwise stated, throughout this thesis we will be consistent with the notation hereafter.

1.3.1 The symbol G will denote a connected reductive linear algebraic group, and $\mathfrak{g}$ its Lie algebra. The field of definition is $k = \overline{k}$, algebraically closed and of characteristic $p > 0$. Whenever we want to refer to an algebraic group and its Lie algebra defined over $\mathbb{C}$ (but also other notions, e.g. orbits), we will simply add a subscript, as in $\mathfrak{g}_{\mathbb{C}}$ and $G_{\mathbb{C}}$.

We stress that all topological notions will pertain to the Zariski topology.

1.3.2 G acts on $\mathfrak{g}$ via the adjoint representation $Ad : G \to GL(\mathfrak{g})$. For $g \in G$ and $x \in \mathfrak{g}$, we will usually shorten the notation as $Ad\,g(x) = g \cdot x = gx$. The differential of Ad is $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. The G-orbit of x will be denoted by $G \cdot x$ or $\mathcal{O}_x$ interchangeably. We shall be using the following notations for centralizers of $x \in \mathfrak{g}$ in G and $\mathfrak{g}$ respectively:

$$\begin{aligned} G_x &= C_G(x) = \{ g \in G \mid g \cdot x = x \} \\ \mathfrak{g}_x &= C_{\mathfrak{g}}(x) = \{ y \in \mathfrak{g} \mid [y, x] = 0 \} \end{aligned}$$

Analogously, the normalizer of a subset $\mathfrak{s} \subseteq \mathfrak{g}$ will be denoted by:

$$N_G(\mathfrak{s}) = \{ g \in G \mid g \cdot \mathfrak{s} \subseteq \mathfrak{s} \}.$$

We will let $Z(H)$ stand for the centre of a group H, while $z(\mathfrak{a})$ will indicate the centre of a Lie algebra $\mathfrak{a}$.

1.3.3 Borel subgroups, maximal tori, parabolic subgroups and Levi subgroups will be typically denoted by B, T, P and L respectively, with $\mathfrak{b}, \mathfrak{t}, \mathfrak{p}$ and $\mathfrak{l}$ their corresponding Lie algebras. The symbol Φ will indicate the root system of G and $\mathfrak{g}$.

The choice of a Borel subgroup B and a maximal torus $T \subseteq B$ corresponds to fixing a system of positive roots $\Phi^+ \subseteq \Phi$ and hence a base of simple roots $\Delta \subseteq \Phi$. If

$\mathfrak{g}$ is simple, we will use Bourbaki's numbering for a system of simple roots. As usual, $W = N_G(T)/C_G(T)$ will stand for the Weyl group of G.

For a root $\alpha \in \Phi$, we will let $\mathfrak{g}_\alpha \subseteq \mathfrak{g}$ be the corresponding root subspace. The element e_α will be a fixed vector spanning the subspace $\mathfrak{g}_\alpha$. We recall that the rank of $\mathfrak{g}$ is the dimension of a Cartan subalgebra.

We will denote by $Y(G)$ the set of 1-parameter subgroups of G, that is, algebraic morphisms $k^\times \to G$. On the other hand, $X(G)$ will be the set of rational characters of G (morphisms $G \to k^\times$). These are both $\mathbb{Z}$-modules.

For the ring of integers modulo a certain $m \in \mathbb{N}$ we will use the notation $\mathbb{Z}_m$.

Chapter 2

Preliminaries

2.1 General setting

Let $\mathfrak{g}$ be the Lie algebra of a connected reductive linear algebraic group G, defined over an algebraically closed field k. The group G naturally acts on $\mathfrak{g}$ via the adjoint action.

As remarked in the previous chapter, we shall be mainly concerned with the *modular* case, thus we will assume (unless otherwise stated) that k is a field of characteristic $p > 0$. This entails that for all $x \in \mathfrak{g}$ the derivation $(\operatorname{ad} x)^p$ is inner and equals $\operatorname{ad} x^{[p]}$ for a certain $x^{[p]} \in \mathfrak{g}$. As a consequence, the Lie algebra $\mathfrak{g}$ is canonically endowed with a G-equivariant p-semilinear p-th *power map* (see [Bo, I.3], for example):

$$\begin{aligned} \mathfrak{g} &\longrightarrow \mathfrak{g} \\ x &\longmapsto x^{[p]}. \end{aligned} \tag{2.1}$$

In general, a Lie algebra admitting such a map is called *restricted*.

For more properties of the p-th power map we refer to [Jac, Chapter V] and [FS, Chapter 2]. Of particular relevance is formula (63) in [Jac, Chapter V], stating that for any two elements $a, b \in \mathfrak{g}$ the p-th power map satisfies:

$$(a+b)^{[p]} = a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a,b), \tag{2.2}$$

where $is_i(a,b)$ is the coefficient of λ^i in the expansion of $\operatorname{ad}(\lambda a + b)^{p-1}(a)$.

The following result, a direct consequence of Jacobson's formula (2.2), will be crucial for our arguments in Chapters 4 and 5. We state it in the form of a lemma:

Lemma 2.1.1. *The p-th power map (2.1) is a morphism of varieties given by homogeneous polynomials of degree p.*

Proof. Let $a, b \in \mathfrak{g}$ and $\mu, \lambda \in k$. Then $(\mu a + \lambda b)^{[p]} = \mu^p a^{[p]} + \lambda^p b^{[p]} + \sum_{i=1}^{p-1} s_i(\mu a, \lambda b)$ by applying (2.2) along with p-semilinearity of the p-th power map. By definition of $s_i(a, b)$, one has $s_i(\mu a, \lambda b) = \mu^{i+1}\lambda^{p-i-1}s_i(a, b)$. Fix a basis $\{w_1, \ldots, w_n\}$ of $\mathfrak{g}$, write a vector $v \in \mathfrak{g}$ with respect to such basis as $v = \eta_1 w_1 + \ldots + \eta_n w_n$ for some scalars $\eta_1, \ldots, \eta_n \in k$. By taking $a = \eta_1 w_1$ and $b = \eta_2 w_2 + \ldots + \eta_n w_n$, applying the identity above and iterating this procedure, one obtains $v^{[p]} = \sum_{i=1}^{n} p_i(\eta_1, \ldots, \eta_n)w_i$ for suitable homogeneous polynomials p_i of degree p. □

Corollary 2.1.2. *For $i \geq 0$, the p^i-th power map is a morphism of varieties defined by homogeneous polynomials of degree p^i.* □

2.1.0.1 An element $x \in \mathfrak{g}$ is *semisimple* if and only if $x \in \operatorname{span}_k\langle x^{[p]^i} | i > 0\rangle$; it is called *toral* if $x^{[p]} = x$ ([FS, 2.3]). As a result, a toral element is semisimple and acts on $\mathfrak{g}$ with eigenvalues belonging to the finite field $\mathbb{F}_p \subseteq k$, because the minimal polynomial of $\operatorname{ad} h$ divides the polynomial $X^p - X$. We will call $\mathfrak{g}(i) = \{x \in \mathfrak{g} \,|\, [h, x] = ix\}$ the eigenspace for $\operatorname{ad} h$ relative to the eigenvalue $i \in \mathbb{F}_p$.

A toral element h is called *balanced* if the eigenspaces $\mathfrak{g}(i)$ corresponding to nonzero eigenvalues are all of the same dimension; more explicitly $\dim \mathfrak{g}(i) = \dim \mathfrak{g}(1)$ for all $i \in \mathbb{F}_p^\times$ (notice that we set no assumptions on the dimension of the centralizer of h). If, in addition, their dimension is divisible by a certain $d \in \mathbb{N}$, the element is called *d-balanced.*

2.1.1 Moore determinant

Since k is algebraically closed, any toral subalgebra $\mathfrak{t}$ admits a basis consisting of toral elements, and it can be expressed as $\mathfrak{t} = \mathfrak{t}^{tor} \otimes_{\mathbb{F}_p} k$, where $\mathfrak{t}^{tor}$ is the set of toral elements of $\mathfrak{t}$ (see for example [FS, 2.3]). Even more is true: let $\{t_1, \ldots, t_n\}$ be a basis of $\mathfrak{t}$ consisting of toral elements, then for a generic n-tuple of scalars $(\lambda_1, \ldots, \lambda_n) \in k^n$, the subalgebra $\mathfrak{t}$ is spanned by p-th powers of the element $\sum_{i=1}^{n} \lambda_i t_i$. This depends on the fact that the *Moore matrix*

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \lambda_1^p & \lambda_2^p & \dots & \lambda_n^p \\ \vdots & \vdots & & \vdots \\ \lambda_1^{p^{n-1}} & \lambda_2^{p^{n-1}} & \dots & \lambda_n^{p^{n-1}} \end{pmatrix}$$

is nonsingular iff $\lambda_1, \dots, \lambda_n$ are $\mathbb{F}_p$-linearly independent (see [Go, 1.3], for example).

2.1.2 The standard hypothesis

It will often be necessary to assume that G satisfies the following conditions, known as the *standard hypothesis*:

(a) The characteristic p is a good prime for the root system Φ of G.

(b) The derived subgroup G' is simply connected.

(c) There exists a nondegenerate symmetric bilinear G-equivariant invariant form $\kappa : \mathfrak{g} \times \mathfrak{g} \to k$.

A prime p is good for Φ if it is good for all its irreducible components.

Assume Φ is irreducible and let $\Delta = \{\alpha_1, \dots, \alpha_l\}$ be a base of simple roots for Φ. If $\widetilde{\alpha}_0 = \sum_{i=1}^{l} b_i \alpha_i$ is the highest root of Φ, we say that p is a *good prime* for G if it does not divide any of the integers b_i for $i = 1, \dots, l$. Thus the good primes for irreducible root systems are the following: any prime for type A; $p > 2$ for types B, C and D; $p > 3$ for E_6, E_7, F_4, G_2; $p > 5$ for E_8.

In view of our discussion in Chapter 3, we remark that when G is of exceptional type the standard hypothesis are satisfied if and only if p is a good prime for G, provided G is simply connected. It will not be restrictive to stick to this assumption as we will be only concerned with the adjoint representation of G on $\mathfrak{g}$.

Finally, if p is a good prime for G, it is a good prime for any Levi subgroup of G.

2.2 Induced nilpotent orbits and sheets

Parabolic and Levi subgroups. Every parabolic subgroup of G admits a *Levi decomposition*, in other words it is the semidirect product $P = L \ltimes U$, where U is the unipotent radical of P and L is a Levi subgroup. Correspondingly, the Lie algebra decomposes as $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$, with $\mathfrak{n}$ the nilradical of $\mathfrak{p}$.

Every parabolic subgroup is G-conjugate to a unique *standard parabolic subgroup* P_I; this latter is a subgroup determined by the choice of a subset $I \subseteq \Delta$ (see [Hu1, Ch. 29, 30]). Analogously, every Levi subgroup is conjugate to a standard Levi subgroup L_I for a certain $I \subseteq \Delta$.

2.2.1 Generalities on nilpotent orbits.

As $\mathfrak{g}$ is an algebraic Lie algebra, an element $x \in \mathfrak{g}$ admits a Jordan-Chevalley decomposition (see [Hu1, Ch. 15], for example): it can be written in a unique way as a sum $x = x_s + x_n$ where $x_s \in \mathfrak{g}$ is semisimple, $x_n \in \mathfrak{g}$ is nilpotent and $[x_s, x_n] = 0$. Moreover, for $y \in \mathfrak{g}$ one has $[x, y] = 0$ iff $[x_s, y] = [x_n, y] = 0$.

Nilpotent orbits, that is G-orbits of nilpotent elements, will be a recurring theme throughout. [Jan1] contains an extensive treatment on this topic, from which we source most of the generalities on orbits we recall hereafter.

2.2.1.1 For $x \in \mathfrak{g}$ (not necessarily nilpotent), the orbit $G \cdot x$ is a smooth and locally closed subset of $\mathfrak{g}$ which is open in its closure $\overline{G \cdot x}$, a G-stable subset of $\mathfrak{g}$. The set $\overline{G \cdot x} \setminus G \cdot x$ is G-stable as well and consists of orbits of dimension strictly less than $\dim G \cdot x = \dim G - \dim C_G(x)$. The orbit $G \cdot x$ is closed iff x is semisimple, and $x_s \in \overline{G \cdot x}$ for any $x \in \mathfrak{g}$.

The inclusion $Lie\, G_x \subseteq \mathfrak{g}_x$ always holds, however it can be strict (this is always the case, for example, for a simple Lie algebra not of type A if the characteristic of the field is a bad prime for the root system and x is a principal nilpotent element, see [Sp2]). For reductive groups satisfying the standard hypothesis, the equality $Lie\, G_x = \mathfrak{g}_x$ always holds, and for the tangent space at an orbit one has the identity $T_x(\mathcal{O}_x) = [\mathfrak{g}, x]$.

2.2.1.2 The number of nilpotent G-orbits of $\mathfrak{g}$ is always finite; this is true also for reductive algebraic groups in bad characteristic. Nilpotent orbits are conical subsets: if $x \in \mathfrak{g}$ is nilpotent and $a \in k^\times$, then $ax \in G \cdot x$.

The set of nilpotent elements of $\mathfrak{g}$ is the *nilpotent cone* $\mathcal{N}(\mathfrak{g})$. This is a Zariski closed subset of $\mathfrak{g}$. Indeed, recall the subring $k[\mathfrak{g}]^G \subseteq k[\mathfrak{g}]$ mentioned in 1.2 (we shall discuss it in more depth in 2.4). It is a polynomial ring generated by $l = \operatorname{rank} \mathfrak{g}$ algebraically independent homogeneous polynomials $f_1, \ldots, f_l$, and $\mathcal{N}(\mathfrak{g})$ is the zero locus of the ideal of $k[\mathfrak{g}]$ generated by these polynomials.

The set $\mathcal{N}(\mathfrak{g})$ is irreducible and of dimension $2 \dim \mathfrak{n}$. It contains a (unique) dense open orbit $\mathcal{O}_{reg}$, the *regular* (or *principal*) *orbit*.

2.2.2 Induced nilpotent orbits

The results in this section follow from [LS]. Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ that admits a Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$. Let $\mathcal{O}_\mathfrak{l}$ be a nilpotent L-orbit of $\mathfrak{l}$. The subset $\mathcal{O}_\mathfrak{l} + \mathfrak{n}$ consists entirely of nilpotent elements of $\mathfrak{g}$. Owing to finiteness of nilpotent G-orbits remarked above, there exists a unique nilpotent G-orbit $\mathcal{O}_\mathfrak{g}$ in $\mathfrak{g}$ meeting $\mathcal{O}_\mathfrak{l} + \mathfrak{n}$ in an open dense subset.

We say that the orbit $\mathcal{O}_\mathfrak{g}$ is *induced* by $\mathcal{O}_\mathfrak{l}$ in the sense of *Lusztig-Spaltenstein*. If $\mathcal{O}_\mathfrak{g}$ is induced by the 0-orbit $\mathcal{O}_\mathfrak{l} = \{0\}$ of a certain Levi subalgebra $\mathfrak{l}$, then $\mathcal{O}_\mathfrak{g}$ is called a *Richardson orbit*.

To give a couple of examples, assume $\mathfrak{g}$ is simple. Then the principal nilpotent orbit $\mathcal{O}$ is a Richardson orbit as it is induced from the 0-orbit on any Cartan subalgebra of $\mathfrak{g}$. The variety $\mathcal{N}(\mathfrak{g})$ always contains a unique *subregular orbit*, that is an orbit $\mathcal{O}'$ with $\dim \mathcal{O}' = \dim \mathcal{O} - 2$. The subregular orbit is Richardson as well as it is induced by the 0-orbit of any Levi subalgebra whose root system is of type A_1.

The orbit induced by $\mathcal{O}_\mathfrak{l}$ depends only on the Levi subalgebra $\mathfrak{l}$, not on the choice of a parabolic subalgebra containing it; for this reason it can (and will) be denoted $\operatorname{Ind}_\mathfrak{l}^\mathfrak{g}(\mathcal{O}_\mathfrak{l})$.

Any induced orbit verifies the dimensional equality $\operatorname{codim}_\mathfrak{l}(\mathcal{O}_\mathfrak{l}) = \operatorname{codim}_\mathfrak{g}(\operatorname{Ind}_\mathfrak{l}^\mathfrak{g}(\mathcal{O}_\mathfrak{l}))$. Moreover, induction is transitive in the sense that $\operatorname{Ind}_{\mathfrak{l}_2}^\mathfrak{g}(\operatorname{Ind}_{\mathfrak{l}_1}^{\mathfrak{l}_2}(\mathcal{O}_{\mathfrak{l}_1})) = \operatorname{Ind}_{\mathfrak{l}_1}^\mathfrak{g}(\mathcal{O}_{\mathfrak{l}_1})$, where $\mathfrak{l}_1 \subseteq \mathfrak{l}_2$ are two Levi subalgebras of $\mathfrak{g}$ and $\mathcal{O}_{\mathfrak{l}_1}$ is a nilpotent orbit of $\mathfrak{l}_1$.

There exist nilpotent orbits that can be induced in more than one way; an orbit which is not induced is called *rigid*.

2.2.3 Sheets

The notion of sheet will be crucial for our discussion on balanced toral elements. Here we will assume that G is connected. Let $m \in \mathbb{Z}$ be fixed, and let $\mathfrak{g}_{(m)}$ be the set of elements in $\mathfrak{g}$ whose centralizer in G has dimension m. The irreducible components of the $\mathfrak{g}_{(m)}$ are called the *sheets* of $\mathfrak{g}$. In other words, the sheets of $\mathfrak{g}$ are the irreducible components of each of the varieties consisting of the union of all G-orbits of the same dimension.

A few brief remarks are needed here. First of all, as G satisfies the standard hypothesis, the equality $Lie\, G_x = \mathfrak{g}_x$ holds for all $x \in \mathfrak{g}$. It follows that the definition of $\mathfrak{g}_{(m)}$ given is equivalent to requiring that $x \in \mathfrak{g}_{(m)}$ iff $\dim \mathfrak{g}_x = m$. By observing that $\dim \mathfrak{g}_x = m$ amounts to $\dim[\mathfrak{g}, x] = \dim \mathfrak{g} - m$, it is immediate to observe that the sheets of $\mathfrak{g}$ are locally closed subsets. In fact, the set of $x \in \mathfrak{g}$ for which $\dim[\mathfrak{g}, x] < n$ for $n \in \mathbb{Z}_{\geq 0}$ is a closed subset of $\mathfrak{g}$. The group G being connected, each sheet is a G-invariant set.

2.2.3.1 Decomposition classes. A simple Lie algebra admits only a *finite number* of sheets. Moreover, sheets can be parameterized by G-conjugacy classes of pairs $(\mathfrak{l}, (\mathrm{Ad}L)e)$, where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ and $(\mathrm{Ad}L)e$ (or, more briefly, Le) is a *rigid* nilpotent orbit of $\mathfrak{l}$ for $e \in \mathfrak{l}$. A nice exposition on this parameterization for the characteristic 0 case is contained in [TY, Ch. 39]. The same results for groups satisfying the standard hypothesis follow from [PrSt], we briefly summarize how the correspondence works as it will be essential for building the theory in Chapter 3.

Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$, let $z(\mathfrak{l})_{reg}$ be the set of all elements x in the centre of $\mathfrak{l}$ such that $\mathfrak{l} = C_{\mathfrak{g}}(x)$ and let a nilpotent element $e_0 \in \mathfrak{l}$ be fixed. The *decomposition class* of $\mathfrak{g}$ associated with the pair $(\mathfrak{l}, e_0)$ is defined as:

$$\mathcal{D}(\mathfrak{l}, e_0) = G \cdot (e_0 + z(\mathfrak{l})_{reg}).$$

The number of decomposition classes of $\mathfrak{g}$ is finite. This is again due to finiteness of nilpotent orbits of a reductive Lie algebra (the Levi subalgebra $\mathfrak{l}$ in this case) and the fact that the number of G-conjugacy classes of Levi subalgebras of $\mathfrak{g}$ is finite as well, due to each of them being conjugate to a standard Levi subalgebra.

The subset $e_0 + z(\mathfrak{l})_{reg}$ is irreducible. Thanks to connectedness of G, a decomposition class is an irreducible subset since it is the image of the irreducible variety $G \times (e_0 + z(\mathfrak{l})_{reg})$ under the morphism:

$$\begin{array}{rcl} G \times (e_0 + z(\mathfrak{l})_{reg}) & \longrightarrow & \mathfrak{g} \\ (g, e_0 + x) & \longmapsto & g \cdot (e_0 + x). \end{array}$$

If $x \in \mathcal{D}(\mathfrak{l}, e_0)$, then $x = g \cdot (e_0 + y) = g \cdot z$, where $y \in z(\mathfrak{l})_{reg}$ and $z = e_0 + y$. In particular, $\dim \mathfrak{g}_x = \dim \mathfrak{g}_z$. The Jordan-Chevalley decomposition of z is exactly $z = e_0 + y$, then $\mathfrak{g}_z = \mathfrak{g}_y \cap \mathfrak{g}_{e_0}$ and $\mathfrak{g}_y = \mathfrak{l}$ by assumption. In particular $\dim \mathfrak{g}_x = \dim \mathfrak{l}_{e_0}$, an integer uniquely determined by the decomposition class. As a consequence, every decomposition class is contained in a sheet of $\mathfrak{g}$.

This entails that there exist only finitely many sheets, and every sheet contains a unique open decomposition class.

Nonetheless, *every sheet contains a unique nilpotent orbit*. Again, this holds for groups satisfying the standard hypothesis and it has been proven in [PrSt, Proposition 2.5]. More generally, the closure of every decomposition class $\mathcal{D}(\mathfrak{l}, e_0)$ contains a unique *open* nilpotent orbit $\mathcal{O}(e)$, that intersects densely with $Le_0 + \mathfrak{n}$ ([PrSt, Theorem 2.3 and Proposition 2.5]). This means in particular that $\mathcal{O}(e)$ is *the orbit obtained from* Le_0 *by Lusztig-Spaltenstein induction*: if $\mathcal{D}(\mathfrak{l}, e_0)$ is the open decomposition class of $\mathcal{S}$, the unique nilpotent orbit in the sheet $\mathcal{S}$ is $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(Le_0)$.

2.2.4 Sheet diagrams and Richardson orbits

2.2.4.1 Weighted Dynkin diagrams. It is well-known that over $\mathbb{C}$ (and in characteristic $p \gg 0$), a nilpotent orbit $\mathcal{O}_{\mathbb{C}}$ is uniquely determined by its *weighted Dynkin diagram*, a way of labelling the Dynkin diagram described hereafter. An element $e \in \mathcal{O}_{\mathbb{C}}$ is included in an $\mathfrak{sl}_2$-triple (h, e, f) of $\mathfrak{g}_{\mathbb{C}}$, that is a triple of elements h, e, f verifying $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. The $\mathfrak{sl}_2$-triple is uniquely determined up to $G_{\mathbb{C}}$-conjugacy by the $G_{\mathbb{C}}$-orbit of h (and that of e).

There exist a maximal toral subalgebra $\mathfrak{t}_{\mathbb{C}}$ containing h and a system of simple roots Δ such that $\alpha(h) \geq 0$ for all $\alpha \in \Delta$. There are not many possibilities for the value assumed by $\alpha(h)$ for $\alpha \in \Delta$: it can only be equal to $0, 1$ or 2 (see [Ca, Section 5.6], for example).

Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$; we label the node corresponding to α_i in the Dynkin diagram of Φ with the integer $\alpha_i(h)$. This is the *weighted Dynkin diagram* of the nilpotent orbit $\mathcal{O}_{\mathbb{C}}$.

2.2.4.2 In the case where $\operatorname{char} k$ is a good prime for the root system of $\mathfrak{g}$, nilpotent orbits are still parameterized by the same weighted Dynkin diagrams. Yet, they must be interpreted in a different way, since the theory of $\mathfrak{sl}_2$-triples fails when $\operatorname{char} k$ is not sufficiently big (see [Pr1, Section 2], for example).

Let $e \in \mathcal{N}(\mathfrak{g})$; there exists a one-parameter subgroup λ optimal for e in the sense of the Kempf-Rousseau theory (this will be introduced in Section 2.5), so that $e \in \mathfrak{g}(\lambda, 2)$, where $\mathfrak{g}(\lambda, i) = \{x \in \mathfrak{g} \mid (Ad\,\lambda(t))x = t^i x\}$.

Replacing e and λ by conjugates if necessary, there exists a system of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ such that $\forall\ x \in \mathfrak{g}_{\alpha_i}$ one has $(Ad\,\lambda(t))x = t^{r_i}x$ with $r_i \in \{0, 1, 2\}$, for $i = 1, \ldots, l$. The Dynkin diagram labelled with the integers $r_1, \ldots, r_l$, let us call it D, coincides with one of the weighted Dynkin diagrams relative to an orbit over $\mathbb{C}$. Moreover, any weighted Dynkin diagram over $\mathbb{C}$ can be obtained in this way, and the dimension of orbits corresponding to the same weighted Dynkin diagram is independent of the characteristic.

2.2.4.3 Sheet diagrams. As remarked in 2.2.3.1, the sheets of $\mathfrak{g}$ are parameterized by G-orbits of pairs $(\mathfrak{l}, e_0)$, where $(\mathrm{Ad}L)e_0$ is rigid in $\mathfrak{l}$. [dGE] contains a list of rigid nilpotent orbits of exceptional Lie algebras in characteristic 0 and their Dynkin diagrams (Tables 1 to 5). Such information can be exploited in our setting as well: for $\mathfrak{g}$ of exceptional type, rigid orbits in good characteristic admit the same weighted Dynkin diagram as their counterparts over $\mathbb{C}$ ([PrSt, Theorem 3.8]).

Let $\mathcal{S}$ be the sheet associated to the G-orbit of pairs $(\mathfrak{l}, e_0)$. Up to conjugacy, we may assume that $\mathfrak{l}$ is a standard Levi subalgebra of a standard parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_\Pi$ corresponding to a subset $\Pi \subseteq \Delta$. The *sheet diagram* ([dGE]) is a labelling of the Dynkin diagram obtained by applying the procedure hereafter. Nodes corresponding to roots in $\Delta \setminus \Pi$ are labelled 2. As Π is a system of simple roots for the root system of $\mathfrak{l}$, the nodes corresponding to roots in Π are labelled just as in the weighted Dynkin diagram of the L-orbit of e_0.

A nilpotent orbit is Richardson if and only if the sheet diagram does not contain any node labelled 1 ([dGE, 5.1]).

Two standard Levi subalgebras can be G-conjugate, and therefore the sheet diagram of a Richardson orbit is determined up to conjugation of the standard Levi subalgebra corresponding to the choice of simple roots with label 0. Nevertheless, a certain orbit can have more than one sheet diagram in the case it belongs to more than one sheet.

2.2.4.4 In order to achieve the classification of balanced toral elements, we will look in detail at Richardson orbits of exceptional Lie algebras, and in particular at the structure of the Levi subalgebra from which they are induced. This latter describes the centralizer of the balanced toral element we are looking for (this will be explained in Section 3.1.2), which can be recovered from the sheet diagram of the orbit. We will make use of [dGE, Tables 6-10], which provide information on the dimension and the sheet diagram of every induced orbit. Notice that data provided *loc. cit.* can be used more generally in good characteristic thanks to [PrSt, Theorem 1.4].

2.3 Kac coodinates and finite order automorphisms of simple Lie algebras

2.3.1 Inner torsion automorphisms.

Until further notice, we will be working with a simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and the algebraic group of adjoint type $G_{\mathbb{C}} = Aut(\mathfrak{g}_{\mathbb{C}})^{\circ}$. This section follows closely [Re].

Let us fix a Borel subgroup $B_{\mathbb{C}} \subseteq G_{\mathbb{C}}$ and a maximal torus $T_{\mathbb{C}} \subseteq B_{\mathbb{C}}$; this amounts to the choice of a system of positive roots Φ^+ for the root system Φ of $G_{\mathbb{C}}$, and hence a subset of simple roots $\Delta = \{\alpha_1, \dots, \alpha_l\}$. The system Δ is a basis of the weight lattice $X = X(T_{\mathbb{C}})$, that admits a natural pairing $\langle\, , \rangle : X \times Y \to \mathbb{Z}$, where $Y = Y(T_{\mathbb{C}})$ is the lattice of cocharacters of $T_{\mathbb{C}}$ (see Section 1.3.3). Let $\{\check{\omega}_1, \dots, \check{\omega}_l\}$ be the $\mathbb{Z}$-basis of Y consisting of fundamental coweights dual to Δ, explicitly $\langle \alpha_i, \check{\omega}_j \rangle = \delta_{ij}$ for all i, j.

2.3.1.1 The full automorphism group $Aut(\mathfrak{g}_{\mathbb{C}})$ consists of cosets $\sigma G_{\mathbb{C}}$, where σ is a graph automorphism of the Dynkin diagram ([St1]). An automorphism of $\mathfrak{g}_{\mathbb{C}}$ belonging to $G_{\mathbb{C}}$ is called *inner*. An inner torsion automorphism is semisimple, so we can assume, up to $G_{\mathbb{C}}$-conjugacy, that it belongs to $T_{\mathbb{C}}$. If it is of order $m \in \mathbb{Z}_{>0}$, it is uniquely determined by its action on root subspaces $(\mathfrak{g}_{\mathbb{C}})_{\alpha_i}$, that is, multiplication by an m-th root of unity. If we denote $V = \mathbb{R} \otimes Y$, torsion elements are the image of $\mathbb{Q} \otimes Y \subseteq V$ under the homomorphism

$$\exp : V \longrightarrow T_{\mathbb{C}}$$

determined by

$$\alpha(\exp(x)) = e^{2\pi i \langle \alpha, x \rangle}$$

for all $\alpha \in \Phi$, where the pairing $\langle \, , \rangle$ is extended to V by $\mathbb{R}$-bilinearity. Since Y is the kernel of the map above, the homomorphism exp yields an isomorphism between V/Y and the subtorus $\mathrm{Im}(\exp) \subseteq T_{\mathbb{C}}$.

The Weyl group W acts on V as a group generated by reflections about simple roots; every semisimple element of $G_{\mathbb{C}}$ is conjugate to an element of $T_{\mathbb{C}}$, and two elements of $T_{\mathbb{C}}$ are $G_{\mathbb{C}}$-conjugate if and only if they are W-conjugate ([MFK, Lemma 2.8]). Therefore, $G_{\mathbb{C}}$-conjugacy classes of semisimple elements of $G_{\mathbb{C}}$ are in bijection with W-conjugacy classes on $T_{\mathbb{C}}$, so that two elements $x, y \in \mathbb{Q} \otimes Y$ give conjugate torsion elements $\exp(x)$ and $\exp(y)$ if and only if x, y are conjugate in V under the *extended affine Weyl group*

$$\widetilde{W} = W \ltimes Y,$$

where the action of Y on V is given by translations.

2.3.1.2 The hyperplane arrangement on V consisting of hyperplanes

$$L_{\alpha,n} = \{x \in V \mid \langle \alpha, x \rangle = n\} \qquad \text{for } \alpha \in \Phi, n \in \mathbb{Z}$$

has been extensively studied; we refer in particular to [Bou, VI.2]. The group of isometries generated by reflections about the hyperplanes $L_{\alpha,n}$ naturally arises in this setting, and it is isomorphic to the *affine Weyl group* $\widetilde{W}^{\circ} = W \ltimes \mathbb{Z}\check{\Phi}$, where $\mathbb{Z}\check{\Phi} \subseteq Y$ is the lattice of coroots of $T_{\mathbb{C}}$. The space V can be partitioned into *alcoves*; the closure of any alcove is a fundamental domain for the action of $\widetilde{W}^{\circ}$, which is a Coxeter group generated by reflections about the $l + 1$ *walls* of any alcove.

Let $\widetilde{\alpha}_0 = \sum_{i=1}^{l} b_i\alpha_i$ be the highest root in Φ^+, $\alpha_0 = 1 - \widetilde{\alpha}_0$ (as an affine linear function on V), and $b_0 = 1$, so that $\sum_{i=0}^{l} b_i\alpha_i = 1$. The simplex:

$$C = \{x \in V \mid \langle \alpha_i, x\rangle > 0 \quad \forall\, 0 \leq i \leq l\,\}$$

is the alcove determined by Δ.

For sake of notation, let $\check{\omega}_0 = 0$. The closure of C can be written adding an extra coordinate x_0 as

$$\overline{C} = \left\{ \sum_{i=0}^{l} x_i\check{\omega}_i \;\middle|\; x_i \geq 0 \,\text{and}\, \sum_{i=0}^{l} b_i x_i = 1 \right\}.$$

Notice that $\overline{C}$ is the convex hull of its vertices defined as $v_i = b_i^{-1}\check{\omega}_i$ for $0 \leq i \leq l$.

The extended affine Weyl group $\widetilde{W}$ acts transitively on alcoves since the affine Weyl group $\widetilde{W}^\circ$ does. Yet, unlike this latter, $\widetilde{W}$ need not act simply transitivity on alcoves, so there exists a possibly nontrivial stabilizer $\Omega = \{\rho \in \widetilde{W} \,|\, \rho \cdot C = C\}$, which gives the extended affine Weyl group a semidirect product structure as

$$\widetilde{W} = \Omega \ltimes \widetilde{W}^\circ.$$

Hence for $x, y \in \overline{C}$ the elements $\exp(x)$ and $\exp(y)$ are $G_{\mathbb{C}}$-conjugate if and only if x, y are Ω-conjugate.

2.3.2 Action of Ω.

Looking at the definition of the extended and the unextended affine Weyl group one sees that $\Omega \simeq Y/\mathbb{Z}\check{\Phi}$, the fundamental group of $G_{\mathbb{C}}$. We shall be interested with exceptional types, so that Ω is the trivial group for types E_8, F_4 and G_2, while for types E_6 and E_7 it is isomorphic to a cyclic group of order 3 and 2 respectively.

The action of Ω on the vertices of the alcove C can be visualized via symmetries of the extended Dynkin diagram, as described in [Bou, VI.2.3]. Elements of Ω are in bijection with *minuscule* coweights, namely those coweights $\check{\omega}_i$ for which $b_i = 1$.

In type E_6 the minuscule coweights are $\check{\omega}_1$ and $\check{\omega}_6$ in Bourbaki's notation, and the group Ω is isomorphic to the group of rotations of the extended Dynkin diagram.

In type E_7 there is only one minuscule coweight, $\check{\omega}_7$, and the generator of Ω is the unique symmetry of the extended Dynkin diagram.

2.3.2.1 Kac coordinates. Kac coordinates parameterize elements of $\overline{C}$ that correspond to finite order automorphisms. Let $x \in \overline{C}$ and suppose that $s = \exp(x)$ is a torsion element of order $m \in \mathbb{Z}_{>0}$, that is, $\exp(mx) = 1$. Then $mx \in Y$, so there exist nonnegative integers $a_1, \ldots, a_l$ (with $\gcd(m, a_1, \ldots, a_l) = 1$) such that

$$x = \frac{1}{m} \sum_{i=1}^{l} a_i \check{\omega}_i.$$

Since $x \in \overline{C}$,

$$0 \leq \langle \alpha_0, x \rangle = 1 - \frac{1}{m} \sum_{i=1}^{l} b_i a_i.$$

If we define the nonnegative integers $b_0 = 1$ and $a_0 = m - \sum_{i=1}^{l} b_i a_i$, the $(l+1)$-tuple $(a_0, \ldots, a_l)$ satisfies

$$\sum_{i=0}^{l} b_i a_i = m. \tag{2.3}$$

We call the integers $(a_0, \ldots, a_l)$ the *Kac coordinates* of $s \in G_{\mathbb{C}}$.

Fix a primitive m-th root of unity ζ; Kac coordinates uniquely determine a torsion element s since the action of s on every root subspace $(\mathfrak{g}_{\mathbb{C}})_\alpha$ for a root $\alpha = \sum_{i=1}^{l} c_i \alpha_i \in \Phi$ is given by multiplication by $\zeta^{\sum_{i=1}^{l} c_i a_i}$.

Two elements $s = \exp(x)$ and $s' = \exp(x')$ are $G_{\mathbb{C}}$-conjugate if and only if their Kac coordinates $(a_0, \ldots, a_l)$ and $(a'_0, \ldots, a'_l)$ can be obtained from one another under the action of Ω as permutation of the indices $\{0, \ldots, l\}$ corresponding to its action on the vertices of the alcove C.

2.3.3 Kac coordinates for toral elements

From now on, $\mathfrak{g}$ and G will be considered again as defined over an algebraically closed field k of positive characteristic p.

Unfortunately the theory depicted so far does not always apply to this more general setting as in the characteristic $p > 0$ case Kac coordinates can only be used to classify torsion automorphisms of order m with $\gcd(p, m) = 1$. Indeed, an automorphism of $\mathfrak{g}$ of order m divisible by p does not endow the Lie algebra with a $\mathbb{Z}/m\mathbb{Z}$-grading.

Serre ([Se]) makes up for this issue by resorting to the theory of affine group schemes. Here we include a brief digression on this topic since the language of group schemes will be repeatedly used.

2.3.3.1 Affine group schemes. All the group schemes we shall encounter will be affine, for a more complete treatment we refer to [Wa].

Definition 2.3.1. An *affine group scheme* over a field K is a representable functor from K-algebras to groups.

K-algebras are intended to be commutative with 1. Let F be a functor as in Definition 2.3.1, and say it is represented by a K-algebra A. Then for every K-algebra R there is a natural correspondence between the group $F(R)$ and $Hom_K(A, R)$. The existence of a group structure on the functor is equivalent to requiring that A is a Hopf algebra over K, and this correspondence between affine group schemes and Hopf algebras over K is a bijection ([Wa, 1.4]). Moreover, homomorphism of affine group schemes (natural transformations which are group homomorphisms) correspond to Hopf algebra homomorphisms. In other words, if G and H are affine group schemes represented by A and B respectively and $G \to H$ is a homomorphism of group schemes (that is, $G(R) \to H(R)$ is a homomorphism for every K-algebra R), then this comes from a unique Hopf algebra map $B \to A$.

Example 2.3.2. Only in this part shall the symbol Δ stand for Hopf algebra comultiplication. The affine group schemes we will encounter more often are $\mathbf{G}_m$ and $\boldsymbol{\mu}_n$, which are respectively the multiplicative group and the group of integers modulo $n \in \mathbb{N}$. They are represented by the algebras $K[X, X^{-1}]$ and $K[X]/(X^n - 1)$ respectively. The comultiplication Δ for the Hopf algebra structure is given in both cases by $\Delta(X) = X \otimes X$.

Notice that an algebraic torus T is a finite product of copies of $\boldsymbol{G}_m$.

More generally, let M be any abelian group, and give $K[M]$ a Hopf algebra structure by setting comultiplication $\Delta(m) = m \otimes m$, counit $\varepsilon(m) = 1$ and antipode $S(m) = m^{-1}$ for every $m \in M$. The group schemes corresponding to such algebras $K[M]$ are called *diagonalizable.* For example, $\boldsymbol{\mu}_n$ is a diagonalizable group scheme with $M = \mathbb{Z}_n$; the same holds for an algebraic torus T where $M = X(T)$, the character group of T. Assume G and H are diagonalizable group schemes that correspond to algebras $K[M_G]$ and $K[M_H]$ respectively, where M_G is the *group of characters* of G, namely the group of homorphisms $G \to \mathbf{G}_m$ (analogously M_H is the group of characters of H). Then homomorphisms $G \to H$ are in bijective correspondence with group homomorphisms $M_H \to M_G$ ([Wa, Th. 2.2]).

2.3.3.2 Going back to inner torsion automorphisms, Serre ([Se]) remarks that already in the characteristic 0 case they correspond to embedding of group schemes $\mathrm{Hom}_k(\boldsymbol{\mu}_n, G)$, in a sense that we will now make more precise. Assume that for a maximal torus $T \subseteq G$ there is an embedding $\boldsymbol{\mu}_n \to T$. Thanks to the discussion at the end of 2.3.3.1, this map is uniquely determined by a group homomorphism $X(T) \to \mathbb{Z}_n$. Typically we will apply these results in the case where G is of adjoint type, so that the character group $X(T)$ is a lattice, a $\mathbb{Z}$-basis of which can be taken to be a base of simple roots Δ for the root system. Therefore an embedding as above is uniquely determined by a map $\Delta \to \mathbb{Z}_n$, which in turn translates into a $\mathbb{Z}_n$-grading on the Lie algebra $\mathfrak{g}$.

This makes it possible to generalize the theory of Kac coordinates to the case of characteristic $p > 0$, with $p|n$, in the case in which the 0-degree subspace of $\mathfrak{g}$ contains a maximal toral subalgebra (*inner case*). In fact, under this assumption Kac coordinates parameterize classes of embeddings of the group scheme $\boldsymbol{\mu}_n$, or equivalently $\mathbb{Z}_n$-gradings on $\mathfrak{g}$, and these correspond to elements $x \in \mathbb{Q} \otimes Y$ (here we have to replace the map exp with what Serre ([Se]) calls a *parameterization of roots of unity*, that is, a homomorphism $\mathbb{Q} \to k$ of kernel $\mathbb{Z}$).

The difference is that in this context an embedding of group schemes need not give an automorphism of order n, but when $n = p$ it can be interpreted as a toral element $h \in \mathfrak{g}$. Using arguments identical to those applied before, G-conjugacy classes on $\mathfrak{g}$ correspond to $\widetilde{W}$-orbits on $\mathbb{Q} \otimes Y$. A representative h of a class of toral elements is uniquely defined by the Kac coordinates $(a_0, a_1, \ldots, a_l)$ of a point $x \in \mathbb{Q} \otimes Y \cap \overline{C}$, so that $\alpha_i(h) = a_i$ for all $i = 1, \ldots, l$, and $a_0 = n - \sum_{i=1}^{l} b_i a_i$.

2.4 Elements of Geometric Invariant Theory

2.4.1 The categorical quotient.

Suppose H is a reductive algebraic group acting algebraically on an affine variety X, both defined over a field $k = \bar{k}$ (for this subsection only we set no restrictions on the characteristic). This yields a linear action of H on the coordinate ring $k[X]$ given by

$$(g \cdot f)(x) = f(g^{-1} \cdot x),$$

for any $g \in H, f \in k[X], x \in X$. The fixed points of this action form a subring $k[X]^H \subseteq k[X]$. The ring $k[X]^H$ is finitely generated (see [MFK, Chapter 1]; in positive characteristic this has been proved in [Ha]).

There exists a *categorical quotient*, i.e. a variety $X/\!\!/H$ which comes with a surjective morphism:

$$\pi_X : X \to X/\!\!/H,$$

that is constant on H-orbits. These satisfy the following universal property: for any affine variety Y and any morphism $\varphi : X \to Y$ constant on H-orbits, φ factors through a unique morphism $\widetilde{\varphi} : X/\!\!/H \to Y$.

More explicitly, $X/\!\!/H = \operatorname{Specmax} k[X]^H$ and the map π_X is the morphism corresponding to the inclusion $k[X]^H \subseteq k[X]$. Every fibre of π_X contains a unique closed H-orbit. If $\mathcal{O}_y$ is the closed orbit in $\pi_X^{-1}(y)$, then for $x \in X$ one has $\pi_X(x) = y$ iff $\mathcal{O}_y \subseteq \overline{H \cdot x}$. Furthermore, if X is irreducible (resp. normal) $X/\!\!/H$ is irreducible (resp. normal) as well (see [BR], [MFK] or [Br] for more details).

In the sequel we will almost exclusively deal with rational actions of G on vector spaces.

2.4.2 The adjoint quotient map.

Since the characteristic of the field k is a good prime, it is well-known that the ring $k[\mathfrak{g}]^G$ is a polynomial ring generated by $l = \operatorname{rank} \mathfrak{g}$ algebraically independent homogeneous functions (see for example [Jan1, 7.13]):

$$k[\mathfrak{g}]^G \simeq k[f_1, \ldots, f_l]. \tag{2.4}$$

This yields a map

$$\begin{aligned} F : \mathfrak{g} &\longrightarrow \mathfrak{g}/\!\!/G \simeq \mathbb{A}^l \\ x &\longmapsto (f_1(x), \ldots, f_l(x)), \end{aligned}$$

independent of the choice of generators. Here are some properties of the map F:

(i) F is surjective and all the fibers are irreducible and of dimension $\dim \mathfrak{g} - l$.

(ii) For $\xi \in \mathbb{A}^l$, let $P_\xi = \{x \in \mathfrak{g} \mid F(x) = \xi\}$ be the corresponding fibre of F. Each P_ξ consists of finitely many G-orbits. It contains a unique closed orbit, whose elements are semisimple, and a unique orbit of maximal dimension.

(iii) $F^{-1}(0) = \mathcal{N}(\mathfrak{g})$, the nilpotent cone of $\mathfrak{g}$.

2.4.2.1 Chevalley Restriction Theorem. Let $\mathfrak{c} \subseteq \mathfrak{g}$ be a Cartan subalgebra. For any element $x \in \mathfrak{c}$ we have $(G \cdot x) \cap \mathfrak{c} = W \cdot x$ (see [Jan1, 7.12], for example). Then the inclusion $\mathfrak{c} \subseteq \mathfrak{g}$ induces a morphism:

$$k[\mathfrak{g}]^G \longrightarrow k[\mathfrak{c}]^W. \tag{2.5}$$

This map is in fact an *isomorphism* when the characteristic of k is 0 or $p > 0$ with some mild assumptions (see [Jan1, Proposition 7.12] or [SS, II, 3.17']). In particular, it holds for groups satisfying the standard hypothesis.

Notice that this entails the existence of an isomorphism of varieties $\mathfrak{c}/W \simeq \mathfrak{g}/\!\!/G$.

2.5 The Kempf-Rousseau theory

Retain notation from Section 1.3.3. Recall once again that there is a natural pairing $\langle\,,\rangle$ between $X(G)$ and $Y(G)$: for $\chi \in X(G)$ and $\lambda \in Y(G)$, $\langle \chi, \lambda\rangle$ is the integer occurring in the composite $\chi(\lambda(t)) = t^{\langle \chi, \lambda\rangle}$.

For a maximal torus $T \subseteq G$ the sets $Y(T)$ and $X(T)$ are free $\mathbb{Z}$-modules and the restriction of the pairing is nondegenerate and invariant under the action of the Weyl group W. Thanks to the well-known isomorphism of varieties $Y(T)/W \simeq Y(G)/G$ ([MFK, Lemma 2.8]), one sees that $Y(G) = \cup_{g\in G} Y(g^{-1}Tg)$.

There is a canonical choice of a parabolic subgroup $P(\lambda)$ associated to a 1-parameter subgroup $\lambda \in Y(T)$: it is defined to be the subgroup generated by the root subgroups U_α, $\alpha \in \Phi$, with $\langle \alpha, \lambda\rangle \geq 0$. $P(\lambda)$ contains a Levi subgroup $L(\lambda)$, the connected closed subgroup of elements commuting with λ. We let $U(\lambda)$ stand for the unipotent radical of $P(\lambda)$.

It is possible to introduce a notion of *length* on $Y(G)$. One starts by defining a W-invariant integer-valued inner product $(\ ,\ )$ on $Y(T)$; the square length of a 1-parameter subgroup λ of T will be $\parallel \lambda \parallel^2 = (\lambda, \lambda)$. For a general 1-parameter subgroup $\mu \in Y(G)$ the notion can be extended by taking an element $g \in G$ such that $g \cdot \mu \cdot g^{-1} \in Y(T)$ and setting $\parallel \mu \parallel = \parallel g \cdot \mu \cdot g{-}1 \parallel$. This is well defined.

Assume G acts on a vector space V and the point 0 is fixed by the action of G. We say that a vector $v \in V$ is *G-unstable* if 0 belongs to the orbit closure $\overline{G \cdot v}$. The set of unstable elements of V will be denoted $\mathcal{N}(V)$ and it is the vanishing locus of a set of generators for the ring $k[V]^G$. Observe that such notation is consistent with our earlier remarks since for $V = \mathfrak{g}$ with the adjoint G-action, $\mathcal{N}(\mathfrak{g})$ is the nilpotent cone of $\mathfrak{g}$.

The *Hilbert-Mumford criterion* (see [MFK, Chapter 2]) states that $v \in V$ is G-unstable if and only if there exists a 1-parameter subgroup λ of G such that v is unstable for the induced $\mathbf{G}_m$-action on V. The vector space decomposes under the action of a 1-parameter subgroup λ into a direct sum $V = \bigoplus_{i \in \mathbb{Z}} V(\lambda, i)$, where $V(\lambda, i) = \{v \in V \mid \lambda(t) \cdot v = t^i v, \quad \forall t \in k^\times\}$. Each $v \in V$ decomposes uniquely as $v = \sum_{i \in \mathbb{Z}} v_i$ according to this grading. We define:

$$m(\lambda, v) := min\{\, i \in \mathbb{Z} \mid v_i \neq 0 \,\}.$$

According to the Hilbert-Mumford criterion, a vector $0 \neq v \in V$ is G-unstable if and only if there exists $\lambda \in Y(G)$ such that $m(\lambda, v) > 0$. The choice of such a 1-parameter subgroup is by no means unique; yet, once a length on $Y(G)$ has been fixed, we say that $0 \neq \lambda \in Y(G)$ is *optimal* for a nonzero unstable vector $v \in V$ if for all $\mu \in Y(G)$

$$\frac{m(\lambda, v)}{\parallel \lambda \parallel} \geq \frac{m(\mu, v)}{\parallel \mu \parallel}.$$

If λ is optimal, all its nonnegative scalar multiples are. We say that λ optimal is *primitive* if there does not exist $\mu \in Y(G)$ such that $\lambda = n\mu$ with $n \in \mathbb{N}$, $n \geq 2$.

The following theorem summarizes the main results of the *Kempf-Rousseau theory* (see [Ke]):

Theorem 2.5.1. *Let $0 \neq v \in \mathcal{N}(V)$ and let Λ_v be the set of optimal primitive elements for v in $Y(G)$. The following are true:*

(i) Λ_v *is nonempty and for all* $\lambda, \mu \in \Lambda_v$ *one has* $P(\lambda) = P(\mu)$*; therefore there exists a well-defined parabolic subgroup* $P(v) \subseteq G$ *such that* $P(v) = P(\lambda)$ *for all* $\lambda \in \Lambda_v$.

(ii) Λ_v *is a principal homogeneous space under the action of the unipotent radical* $U(v)$ *of* $P(v)$*; otherwise stated,* $\Lambda_v = \{g\lambda g^{-1} | g \in U(v)\}$ *for any* $\lambda \in \Lambda_v$.

(iii) Any maximal torus of $P(v)$ *contains a unique element of* Λ_v.

(iv) For any $g \in G$, $\Lambda_{g \cdot v} = g\Lambda_v g^{-1}$ *and* $gP(v)g^{-1} = P(g \cdot v)$.

(v) $P(v)$ *contains the stabilizer of* v *in* G.

2.6 The cone associated with a variety

An algebraic geometric notion that we shall use in a crucial way in 6.1 is that of the *cone associated to an affine variety* X. The treatment here follows closely [PrSk, 5.1].

Consider the affine space $\mathbb{A}^n$ over the field k, for $n \in \mathbb{N}$. The algebra of regular functions of this space is a polynomial ring in n variables $k[\mathbb{A}^n] = k[x_1, \ldots, x_n]$ and it is endowed with the grading $k[\mathbb{A}^n] = \bigoplus_{j\in\mathbb{N}} k[\mathbb{A}^n]_j$ given by polynomial degree. Starting from any ideal $I \subseteq k[x_1, \ldots, x_n]$ one can build the homogeneous ideal grI whose components of degree $i \in \mathbb{N}$ are given by:

$$gr_i I = \{\ f \in k[x_1, \ldots, x_n]_i \mid \exists\, g \in I \text{ such that } g - f \in \bigoplus_{j<i} k[x_1, \ldots, x_n]_j\ \}.$$

For any subset $X \subseteq \mathbb{A}^n$, let $I(X)$ be the ideal of regular functions vanishing on X. The closed conical subset:

$$\mathbb{K}X = \{\ v \in V \mid f(v) = 0 \text{ for all } f \in grI(X)\ \}$$

is the *cone associated with* X. It verifies $\dim \mathbb{K}X = \dim \overline{X}$, and if $0 \notin \overline{X}$ then $\overline{k^\times X} = k^\times X \cup \mathbb{K}X$.

2.7 Structure of classical Lie algebras

For our discussion in Section 5.2 it will be necessary to set notation and introduce some generalities on the structure of Lie algebras of classical type. Algebraic groups and Lie algebras of these types can be realized as subgroups and subalgebras of the space of $m \times m$ matrices, for suitable $m \in \mathbb{Z}_{\geq 0}$ (which should be clear for each type). Such sets of matrices have a natural representation on the vector space k^m. Once a basis of k^m has been fixed, a maximal torus of G and a maximal toral subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ are given by diagonal matrices in this basis. Throughout this section we will let γ_i stand for the function that associates to a diagonal $m \times m$ matrix its i-th eigenvalue, while $e_{i,j}$ will denote the square $m \times m$ matrix whose only nonzero entry is in row i and column j and it is equal to 1.

2.7.1 Type A

We briefly give the standard realization of a Lie algebra of type A. Fixed $n \in \mathbb{N}$, we can assume $\mathfrak{g} = \mathfrak{sl}_{n+1}$, while $G = SL(n+1)$ is the simply connected group with Lie algebra $\mathfrak{g}$ (it will suffice to look at this case as we are only interested in the action of G on $\mathfrak{g}$). We will usually shorten the notation as $\mathfrak{gl}_n = \mathfrak{gl}_n(k)$ or $\mathfrak{sl}_n = \mathfrak{sl}_n(k)$ whenever the field of definition is clear.

Both G and $\mathfrak{g}$ naturally act on the vector space k^{n+1}. Diagonal matrices form a Cartan subalgebra $\mathfrak{h}$; this is isomorphic to k^n under the map:

$$(a_1, \dots, a_n) \in k^n \longmapsto \operatorname{diag}(a_1, \dots, a_n, a_{n+1}) \tag{2.6}$$

using $a_{n+1} = -a_1 - \dots - a_n$.

2.7.1.1 The root system consists of roots $\gamma_i - \gamma_j$ for $1 \leq i \neq j \leq n$, with a system of positive roots being $\{\gamma_i - \gamma_j \,|0 \leq i < j \leq n\}$. Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be the corresponding base of simple roots for the root system. We can regard each of them as $\alpha_i = \gamma_i - \gamma_{i+1}$ for $i = 1, \dots, n$ as functionals on $\mathfrak{h}$. The root subspace $\mathfrak{g}_{\alpha_i}$ is the line $ke_{i,i+1}$ for $i = 1, \dots n$.

The Weyl group in type A is isomorphic to S_{n+1}, the symmetric group on $n+1$ elements. This acts on the roots by permuting the subscripts of the functionals γ_i.

2.7.2 Other classical types: generalities

From now on we will assume that G is one of $SO(n)$ or $Sp(2n)$. $SO(n)$ is the subgroup of $SL(n)$ leaving invariant a symmetric bilinear form $B(\cdot, \cdot)$ defined on k^n. Analogously,

$Sp(2n) \subseteq SL(2n)$ consists of matrices fixing a skew symmetric bilinear form $B'(\cdot,\cdot)$ of k^{2n}. It is worth stressing that since the characteristic of k is not 2 (this is a bad characteristic for types B, C and D), we can exploit the bijection between symmetric bilinear forms and quadratic forms.

The description we give is obtained by using the same means as in [FH, Part III]. Yet, the choice we make is the same as in [Le1] as it will be notationally convenient later. More specifically, for $d \geq 0$ let J_d be the following $d \times d$ matrix:

$$J_d = \begin{pmatrix} 0 & \dots & 0 & 1 \\ \vdots & & \iddots & 0 \\ 0 & \iddots & & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}.$$

Then we assume that in the standard basis for k^n the matrix associated with B is J_n, whereas in the standard basis for k^{2n} the matrix of B' is

$$\bar{J}_n = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}.$$

Under these assumptions $SO(n)$ consists of matrices $M \in SL(n)$ verifying

$$M^T J_n M = J_n \tag{2.7}$$

and its Lie algebra $\mathfrak{so}(n)$ is given by $n \times n$ matrices X satisfying

$$X^T J_n + J_n X = 0. \tag{2.8}$$

On the other hand, the group $Sp(2n)$ consists of matrices $M \in SL(2n)$ that satisfy

$$M^T \bar{J}_n M = \bar{J}_n \tag{2.9}$$

and the Lie algebra $\mathfrak{sp}(2n)$ is the subalgebra of matrices X belonging to $\mathfrak{sl}(2n)$ with the property

$$X^T \bar{J}_n + \bar{J}_n X = 0. \tag{2.10}$$

2.7.3 The Lie algebra $\mathfrak{sp}(2n)$ (Type C)

Assume $X \in \mathfrak{sp}(2n)$, so that it verifies $X^T \bar{J}_n + \bar{J}_n X = 0$. Writing:

$$X = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

where A, B, C, D are $n \times n$ matrices, condition (2.10) amounts to requiring that A is the negative of D transposed about its antidiagonal, while C and B are invariant under transposition about their antidiagonal.

Diagonal matrices in this realization form a Cartan subalgebra $\mathfrak{h}$; this has dimension n and can be identified to k^n via

$$(a_1, \dots, a_n) \in k^n \mapsto \mathrm{diag}(a_1, \dots, a_n, -a_n, \dots, -a_1). \tag{2.11}$$

Analogously, diagonal matrices in $Sp(2n)$ form a maximal torus of dimension n isomorphic to the torus $(t_1, \dots, t_n) \in (k^\times)^n$ via associating to each such n-tuple the matrix $\mathrm{diag}(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1})$.

2.7.3.1 The root system consists of roots $\gamma_i - \gamma_j$ $(0 < i \neq j \leq n)$, $\gamma_i + \gamma_j$ (for $i < j \leq n$) and roots of the form $2\gamma_i$ $(i = 1, \dots, n)$. A base of simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\}$ corresponds to the following functionals on $\mathfrak{h}$:

$$\alpha_i = \gamma_i - \gamma_{i+1} \text{ for } 1 \leq i \leq n-1, \ \alpha_n = 2\gamma_n.$$

The corresponding root subspaces are spanned by $e_{i,i+1} - e_{2n-i,2n-i+1}$ for $i = 1, \dots n-1$ and $e_{n,n+1}$ respectively.

The Weyl group is the semidirect product $S_n \rtimes \mathbb{Z}_2^n$, where S_n acts by permuting subscripts of the functionals $\gamma_1, \dots, \gamma_n$ and $\mathbb{Z}_2^n$ is the direct product of sign changes $\gamma_i \mapsto -\gamma_i$.

2.7.4 The Lie algebra $\mathfrak{so}(2n)$ (Type D)

Let $X \in \mathfrak{so}(2n)$, this means $X^T J_{2n} + J_{2n} X = 0$. As in the previous case, let us express X as:

$$X = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

where A, B, C, D are $n \times n$ matrices. Then the condition (2.8) is verified if and only if A is the negative of D transposed about its antidiagonal, while both C and B are skew-symmetric with respect to tranposition about their antidiagonal.

Diagonal matrices are a Cartan subalgebra $\mathfrak{h}$ of dimension n isomorphic to k^n via:

$$(a_1, \dots, a_n) \in k^n \mapsto \mathrm{diag}(a_1, \dots, a_n, -a_n, \dots, -a_1). \tag{2.12}$$

Again, diagonal matrices in $SO(2n)$ form a maximal torus of dimension n isomorphic to the torus $(t_1, \dots, t_n) \in (k^\times)^n$ under the identification with the matrix $\mathrm{diag}(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1})$.

2.7.4.1 The root system consists of roots $\gamma_i - \gamma_j$ $(0 < i \neq j \leq n)$, $\gamma_i + \gamma_j$ (for $0 < i < j \leq n$). A base of simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\}$ corresponds to the following functionals on $\mathfrak{h}$:

$$\alpha_i = \gamma_i - \gamma_{i+1} \text{ for } i \leq n-1, \text{ and } \alpha_n = \gamma_{n-1} + \gamma_n.$$

The corresponding root subspaces are the lines spanned by $e_{i,i+1} - e_{2n-i,2n-i+1}$ with $i = 1, \dots, n-1$ and $e_{n-1,n+1} - e_{n,n+2}$ respectively.

The Weyl group is isomorphic to the semidirect product $S_n \ltimes \mathbb{Z}_2^{n-1}$. This is the subgroup of the Weyl group of type BC where each transformation involves only an even number of sign changes.

2.7.5 The Lie algebra $\mathfrak{so}(2n+1)$ (Type B)

Now we will assume $X \in \mathfrak{so}(2n+1)$, so that it satisfies $X^T J_{2n+1} + J_{2n+1} X = 0$. It is convenient to write X as:

$$X = \left(\begin{array}{ccc|c|ccc} & & & e_1 & & & \\ & A & & \vdots & & B & \\ & & & e_n & & & \\ \hline g_1 & \dots & g_n & I & h_1 & \dots & h_n \\ \hline & & & f_1 & & & \\ & C & & \vdots & & D & \\ & & & f_n & & & \end{array}\right).$$

As before A, B, C, D are $n \times n$ matrices, while $E = (e_1, \ldots, e_n)$ and $F = (f_1, \ldots, f_n)$ are $n \times 1$ column vectors, $G = (g_1, \ldots, g_n)$ and $H = (h_1, \ldots, h_n)$ are $1 \times n$ row vectors and $I \in k$ is a scalar. It is easy to carry on computations upon writing:

$$J_{2n+1} = \left(\begin{array}{c|c|c} 0 & 0 & J_n \\ \hline 0 & 1 & 0 \\ \hline J_n & 0 & 0 \end{array} \right).$$

Then the condition (2.8) leads to the following:

- A is the negative of D transposed about its antidiagonal;
- both C and B are skew-symmetric with respect to tranposition about their antidiagonal;
- $(g_1, \ldots, g_n) = -(f_n, \ldots, f_1)$ and $(h_1, \ldots, h_n) = -(e_n, \ldots, e_1)$;
- $I = 0$.

As for a Cartan subalgebra, as usual $\mathfrak{h}$ is given by diagonal matrices in the standard basis of k^{2n+1} and it is of dimension n. It can be identified to k^n via

$$(a_1, \ldots, a_n) \in k^n \mapsto \mathrm{diag}(a_1, \ldots, a_n, 0, -a_n, \ldots, -a_1). \tag{2.13}$$

Diagonal matrices in $SO(2n+1)$ form a maximal torus of dimension n isomorphic to the torus $(t_1, \ldots, t_n) \in (k^\times)^n$ via $\mathrm{diag}(t_1, \ldots, t_n, 1, t_n^{-1}, \ldots, t_1^{-1})$.

2.7.5.1 The root system consists of the roots $\gamma_i - \gamma_j$ $(0 < i \neq j \leq n)$, $\gamma_i + \gamma_j$ (for $0 < i < j \leq n$) and γ_i (for $i = 1, \ldots, n$). A base of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ is given by the following functionals on $\mathfrak{h}$:

$$\alpha_i = \gamma_i - \gamma_{i+1} \text{ for } i \leq n-1, \text{ and } \alpha_n = \gamma_n.$$

The corresponding root subspaces are spanned by $e_{i,i+1} - e_{2n-i+1,2n-i+2}$ with $i = 1, \ldots n-1$ and $e_{n,n+1} - e_{n+1,n+2}$ respectively.

The Weyl group is the same as for type C, that is $S_n \ltimes \mathbb{Z}_2^n$, with the same interpretation in terms of the action on the γ_i's.

Remark 2.7.1. It will be useful to set some conventions on elements of the Weyl group for a Lie algebra of classical type; this shall be used in Section 5.2.

We will express an element of the symmetric group S_n as a product of cyclic permutations in the indices of the functionals γ_i. As for an element $\eta \in \mathbb{Z}^n$ representing a sign change, in order to shorten notation we will usually identify γ_i with its index i and write $\eta(i) = i$ (resp. $\eta(i) = -i$) if $\eta(\gamma_i) = \gamma_i$ (resp. $\eta(\gamma_i) = -\gamma_i$).

Remark 2.7.2. The exposition in [Jan1, 7.2 to 7.10] gives a very explicit account on the algebra $k[\mathfrak{g}]^G$ when G is one of the classical groups above (orthogonal, symplectic or special linear group). The results can be summarized as follows. Assume G has rank n; we will identify a Cartan subalgebra with k^n via (2.6), (2.11), (2.12) and (2.13). Using the Chevalley Restriction Theorem (Section 2.4.2.1), we can restrict to the subalgebra of diagonal matrices $\mathfrak{h} \subseteq \mathfrak{g}$. Then there exist ring isomorphisms $k[\mathfrak{g}]^G \simeq k[\mathfrak{h}]^W \simeq k[f_1, \ldots, f_n]$ where the functions $f_1, \ldots, f_n$ are algebraically independent.

We recall that given a set of variables $x_1, \ldots, x_n$, the j-th elementary symmetric polynomial in these variables (for $1 \leq j \leq n$) is:

$$\sigma_j = \sum_{i_1 < i_2 < \ldots < i_j} x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_j}.$$

The polynomial σ_j is homogeneous of degree j.

The generators $f_1, \ldots, f_n$ for $k[\mathfrak{c}]^W$ can be chosen as follows:

- $G = SL(n+1)$. For $i = 1, \ldots, n$ one sets $f_i = \sigma_{i+1}(a_1, \ldots, a_{n+1})$.
- $G = Sp(2n)$ or $G = SO(2n+1)$. For $i = 1, \ldots, n$ one sets $f_i = \sigma_i(a_1^2, \ldots, a_n^2)$.
- $G = SO(2n)$. For $i = 1, \ldots, n-1$ one sets $f_i = \sigma_i(a_1^2, \ldots, a_n^2)$ and $f_n = \sigma_n(a_1, \ldots, a_n)$ (a direct proof of this fact will be given in Section 5.3).

As a result, if G in one of SL, Sp or SO, if two diagonal matrices in $\mathfrak{g}$ have the same eigenvalues, then they are G-conjugate. This is a consequence of the definition of $f_1, \ldots f_n$ above combined with the fact that $\overline{G \cdot x} = G \cdot x$ iff $x \in \mathfrak{g}$ is semisimple.

Chapter 3

Classification of balanced toral elements

The results in this chapter appear in [A].

Retain notations from Section 2.1. Recall that a *toral* element $h = h^{[p]}$ of $\mathfrak{g}$ decomposes the Lie algebra as a direct sum $\mathfrak{g} = \oplus_{i\in\mathbb{F}_p}\mathfrak{g}(i)$, where $\mathfrak{g}(i)$ is the eigenspace of eigenvalue $i \in \mathbb{F}_p$ for $\operatorname{ad} h$.

Definition 3.0.1. A toral element h is said to be *balanced* if the dimension of $\mathfrak{g}(i)$ for $i \neq 0$ is independent of $i \in \mathbb{F}_p^\times$; more explicitly $\dim \mathfrak{g}(i) = \dim \mathfrak{g}(1)$ for all $i \in \mathbb{F}_p^\times$. If a positive integer d divides $\dim \mathfrak{g}(i)$, then h is called *d-balanced.*

3.1 Preliminary discussion and outline of method

We will now make use of the machinery from Chapter 2 to obtain the classification of G-conjugacy classes of balanced toral elements. In characteristic $p = 2$ and $p = 3$ all toral elements are balanced. This is obvious for $p = 2$; for $p = 3$ it is just as immediate by observing that $\mathfrak{g}_\alpha \subseteq \mathfrak{g}(i)$ if and only if $\mathfrak{g}_{-\alpha} \subseteq \mathfrak{g}(-i)$, for $\alpha \in \Phi$ and $i \in \mathbb{F}_3$. Apart from the root system of type E_8 in characteristic $p = 5$, this reduces the classification of balanced toral elements in characteristic 2 and 3 to listing all the possible G-classes of toral elements. Although this does not immediately seem to simplify the problem, it is worth remarking that in small characteristic there are very few options for Kac coordinates. This will be carried out in Section 3.1.6.

The strategy is based on case-by-case computations on Richardson orbits; 3.1.1 follows closely Premet ([Pr2]); the goal is to adapt some of his considerations to our setting. We will assume that *p is a good prime for the root system of* $\mathfrak{g}$. After working on this case, we will separately examine type E_8 in characteristic $p = 5$, since it requires some slight modifications to the strategy.

3.1.1 The Richardson orbit associated to a balanced toral element

There is a canonical way of associating a Richardson orbit to a G-conjugacy class of toral elements. Let h be a representative of one of these orbits; up to conjugacy it belongs to the Lie algebra $\mathfrak{t}$ of a fixed maximal torus $T \subseteq G$, and the centralizer $\mathfrak{l} = C_{\mathfrak{g}}(h)$ is a Levi subalgebra of $\mathfrak{g}$.

As the 0-orbit on $\mathfrak{l}$ is rigid, the nilpotent orbit $\mathcal{O}_0 = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\{0\})$ obtained by induction from $0 \in \mathfrak{l}$ is the unique nilpotent orbit contained in the sheet $\mathcal{S}$ whose open decomposition class is $\mathcal{D}(\mathfrak{l}, 0) = \mathrm{Ad}\, G(z(\mathfrak{l})_{reg})$.

If in addition h is balanced, there are some restrictions on the values $\dim \mathcal{O}_0$ can assume. Let us consider $e \in \mathcal{O}_0$ which lies in the same sheet as h. It satifies $\dim \mathfrak{g}_e = \dim G_e = \dim G_h = \dim \mathfrak{g}_h$. The following equalities hold:

$$\dim \mathcal{O}_0 = \dim \mathfrak{g} - \dim G_e =$$
$$= \dim \mathfrak{g} - \dim \mathfrak{g}_h = \sum_{i \in \mathbb{F}_p^\times} \dim \mathfrak{g}(i) = (p-1) \cdot \dim \mathfrak{g}(1).$$

Hence $\dim \mathcal{O}_0 \equiv 0 \mod (p-1)$, so that a balanced toral element h naturally corresponds to a Richardson orbit $\mathcal{O}_0$ of dimension divisible by $p - 1$.

3.1.2 Structure of the centralizer.

Let h be a balanced toral element, without loss of generality we can assume it corresponds to a choice of Kac coordinates $(a_0, a_1, \ldots, a_l)$ as in 2.3.3. Because of the limitations on its dimension, there are not too many options for the orbit $\mathcal{O}_0$ associated to it as in 3.1.1. We make use of [dGE, Tables 6-10]; the sheet diagram describes the structure of the centralizer $\mathfrak{l}$ of h. We need to consider all possible ways of embedding a centralizer of the relevant type in the Dynkin diagram, since Kac coordinates for a balanced element might be relative to a standard Levi subalgebra conjugated to the centralizer we can read off from the tables. For exceptional Lie algebras two isomorphic Levi subalgebras are always conjugated, except for three cases in type E_7. Still, what is chiefly needed is the dimension of the orbit $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(0)$, that depends only on the type of the centralizer $\mathfrak{l}$. Finally, simple roots belonging to a base for the standard Levi subalgebra $\mathfrak{l}$ can be chosen up to a permutation of Δ corresponding to an element of Ω.

For every simple Lie algebra of exceptional type, we produce the list:

Ω-orbits of standard Levi subalgebras $\bar{\mathfrak{l}}$	$\dim \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(0)$

3.1.3 Kac coordinates for h.

If h is a balanced element in char$k = p > 0$, $\dim \mathcal{O}_0$ is divisible by $p-1$ (3.1.1). All the potential centralizers of h are those for which $\dim \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(0)$ is divisible by $p-1$, and so they can be read off from the list as in 3.1.2. Any of these choices of centralizers tells exactly which Kac coordinates are equal to 0, while all the others must be strictly positive, let them be $a_0, a_{i_1}, \ldots, a_{i_r}$. Notice that $a_0 > 0$ because the characteristic is good for the root system. We inspect all possible strictly positive coordinates that satisfy the condition $a_0 = p - \sum_{j=1}^{r} b_{i_j} a_{i_j}$. Each one of the a_i is the eigenvalue of ad$h$ on $\mathfrak{g}_{\alpha_i}$, so by linearity one obtains the eigenvalue of ad$h$ on any root space $\mathfrak{g}_\alpha$, for $\alpha \in \Phi$. Finally, we check that $\dim \mathfrak{g}(i)$ for $i \in \mathbb{F}_p^\times$ is independent of $i > 0$.

3.1.4 Restrictions on the characteristic.

We repeat this procedure varying the characteristic $p > 0$. There is always an upper bound for a prime number p such that a balanced toral element in characteristic p can potentially exist, given the type of the Lie algebra. Indeed, such a value depends on the root system: *e.g.* $p \leq |\Phi| + 1$, but this can actually be refined. Observe that $|\Phi|$ equals the dimension of the regular nilpotent orbit for the Lie algebra $\mathfrak{g}$, and assume $\frac{|\Phi|}{2} + 1 < p < |\Phi| + 1$. If a balanced toral element h exists in characteristic p, all its eigenspaces must be of dimension 1. Moreover, the orbit $\mathcal{O}_0$ corresponding to h is not the regular nilpotent orbit. Thus h is not a regular semisimple element, so its centralizer contains a nonzero root space $\mathfrak{g}_\alpha$, and without loss of generality $\alpha \in \Delta$.

Otherwise stated, in the sheet diagram of $\mathcal{O}_0$ there is at least one node with label 0. We apply a classical result of Bourbaki ([Bou, VI.1.6 Corollaire 3 à la Proposition 19]) to show that for at least one $i \in \mathbb{F}_p^\times$ we have $\dim \mathfrak{g}(h, i) > 1$. There exist simple roots $\alpha, \beta \in \Delta$ such that $\alpha(h) = 0$, $\beta(h) > 0$ and $\alpha + \beta$ is a root; this is clear considering two adjacent nodes on the sheet diagram such that one is labelled 0 and the other has label 2. But then $\beta, \alpha + \beta \subseteq \mathfrak{g}(\beta(h))$, and so this subspace has dimension strictly greater than 1.

In any case, the subregular orbit is the only one whose sheet diagram contains exactly one node labelled 0, and its dimension is $|\Phi| - 2$. Using an argument similar to the one above, if neither $|\Phi|$ nor $|\Phi| - 2$ is of the form $2(q-1)$ for a prime q, then we can reduce even further the number of primes to check: it is not necessary to consider the range $\frac{|\Phi|}{3} + 1 < p < |\Phi| + 1$.

As a result, for every root system it is enough to check only the following primes:

- Type G_2, $5 \leq p \leq 7$ and $p = 13$.
- Type F_4, $5 \leq p \leq 23$.
- Type E_6, $5 \leq p \leq 37$ and $p = 73$.
- Type E_7, $5 \leq p \leq 43$ and $p = 127$.
- Type E_8, $7 \leq p \leq 79$ and $p = 241$.

Except for type F_4, $|\Phi| + 1$ is a prime number, hence it is necessary to check if there exist balanced toral elements in characteristic $|\Phi|+1$, that is, an element h whose adjoint action on the Lie algebra yields all nonzero eigenspaces of dimension 1. This represents the trickiest case for every root system, and it turns out that only in type G_2 there exist balanced elements in characteristic $|\Phi| + 1$.

3.1.5 The algorithm

The code we wrote is absolutely elementary and does not make use of any functions in the C++ library. It varies slightly for each distinct type as we have to input two sets of data: the root system and the sheet diagram of Richardson orbits. In order to obtain the root system we use some basic routines in GAP ([GAP]).

Call $n = \operatorname{rank} \mathfrak{g}$ and $q =$ number of Richardson orbits.

First of all, we initialize the integers $s_0 = 1, s_1, \ldots, s_n$ representing the coefficients of the highest roots.

The positive roots are given in C in the form of a $\frac{|\Phi|}{2} \times n$ matrix M whose rows correspond to coefficients of positive roots in Φ. For instance, when $\mathfrak{g}$ is of type G_2 the matrix `M` is:

`M[0][0]=1; M[0][1]=0;`	α_1
`M[1][0]=0; M[1][1]=1;`	α_2
`M[2][0]=1; M[2][1]=1;`	$\alpha_1 + \alpha_2$
`M[3][0]=2; M[3][1]=1;`	$2\alpha_1 + \alpha_2$
`M[4][0]=3; M[4][1]=1;`	$3\alpha_1 + \alpha_2$
`M[5][0]=3; M[5][1]=2;`	$3\alpha_1 + 2\alpha_2$.

The list as in 3.1.2 is inputted in the form of a $q \times (n+1)$ matrix O. Data for these orbits are sourced from the Tables in [dGE]. As for the first n entries in each row, simple roots generating the centralizer Levi subalgebra correspond to 0-entries in the matrix, whereas all the others are given the value 2 (this could have been any nonzero value, but we chose 2 in analogy with the sheet diagram). For example, in type F_4 a centralizer generated by roots α_1 and α_2 (in Bourbaki's notation) is associated to the i-th row of the matrix in this way: $O_{i,1} = 0, O_{i,2} = 0, O_{i,3} = 2, O_{i,4} = 2$.

On the other hand, the last column of the matrix O contains the dimension of the orbit considered. Carrying on with the same example, we have $O_{i,5} = 42$, as one can read off from [dGE, Table 10] (indeed, the orbit whose sheet diagram is 0022 has dimension 42).

3.1.5.1 The characteristic p varies according to our discussion in 3.1.4. We initialize the range of primes in the form of an array $P = (P_1, \ldots, P_r)$. Let $p = P_i$ be fixed. The program finds all the rows of the matrix O whose entry in the last column is divisible by $p - 1$. The other entries in the same row describe the structure of the centralizer. Kac coordinates are contained in an array a. In case $O_{i,j} = 0$ we set $a_j = 0$, while the remaining entries of a (including a_0) must be strictly positive.

The code examines all the tuples with $a_0 = p - \sum_{i=1}^{n} s_i a_i$ over all the suitable a_i's. For each of them an array v of dimension $|\Phi|$ has the task of counting the dimension of the different eigenspaces $\mathfrak{g}(i)$. This is easy to compute: if the root α corresponds to the l-th row of M, then the root space $\mathfrak{g}_\alpha$ has eigenvalue $j = \sum_{k=1}^{\operatorname{rank}\mathfrak{g}} M_{l,k}a_k$, and the component v_j is increased.

Finally, the program returns tuples of Kac coordinates for which $v_i = v_1$ for all $i \in \mathbb{F}_p^\times$. Some of the outputs coincide up to the action of the group Ω; since any two such tuples define the same G-class of balanced toral elements, only one representative is listed in the Tables in Section 3.2.

Here is the pseudocode for the algorithm described.

Algorithm 1: Kac coordinates of balanced toral elements

```
Initialize with n, q, r, s_0, ..., s_n, M, P, O
Define a = 0, v = 0
Define p, i, j, m dummy indices
for p ← P_1 to P_r do
    for m ← 1 to q do
        if O_{n+1,m} ≡ 0 mod p − 1 then
            Set v = 0 array
            Set a = 0 array
            for i ← 1 to n do
                if O_{i,m} ≠ 0 then
                    Set a_i = 1
            while s_1 a_1 + ... s_n a_n < p do
                Set a_0 = p − (s_1 a_1 + ... s_n a_n)
                for i ← 1 to |Φ| do
                    Set j = M_{i,1} a_1 + ... + M_{i,n} a_n
                    Increase v_j by 1
                    Increase v_{p−j} by 1
                if v_1 = v_2 = ... = v_{p−1} then
                    Return characteristic p
```

3.1.6 The bad characteristic case

Let us now focus on type E_8 in characteristic 5. Unlike the good characteristic case, here the centralizer of a semisimple element need not be a Levi subalgebra, more generally it is a *pseudo-Levi subalgebra* (see, for example, [So, 2.1]). Kac coordinates can still be used to parameterize G-classes of toral elements, yet we have to allow $a_0 = 0$. As the extended Dynkin diagram in type E_8 does not admit any symmetry, our list as in 3.1.2 simply contains all the potential centralizers of a toral element. For $p = 5$, we check only the possible centralizers, not the dimension of the orbit (nilpotent orbits may be different from good characteristic, and the description of sheets in 2.2.3.1 does not apply). However, we stress that this does not hinder the efficiency of the algorithm, mainly because there are very few centralizers that can work for a toral element as in 2.3.3, as Kac coordinates are nonnegative integers satisfying $a_0 = 5 - 2a_1 - 3a_2 - 4a_3 - 6a_4 - 5a_5 - 4a_6 - 3a_7 - 2a_8$.

The Tables in the next section contain all conjugacy classes of toral elements in characteristic 2 and 3 as well; elements in these classes are automatically balanced. We simply list Kac coordinates satisfying (2.3) (for $m = 2$ and 3) up to the action of Ω, since they are not too many.

3.2 The Tables

We implemented the algorithm discussed in 3.1.5 using the programming language C. The Tables hereafter represent all the G-conjugacy classes of balanced toral elements and are sorted by type of the root system and characteristic (denoted p). Every G-conjugacy class is described by a choice of Kac coordinates. Let h be an element in a fixed class, and denote by Φ_0 the root system of the centralizer of h. The second last column contains the type of Φ_0; notice that in a few cases h is regular semisimple. Finally, the value in last column is the dimension $d = \dim \mathfrak{g}(i)$ of each h-eigenspace for $i \neq 0$.

Type G_2

p	Kac coord.			Type	d
	a_0	a_1	a_2	of Φ_0	
2	0	1	0	$\widetilde{A}_1 A_1$	8
3	0	0	1	A_2	3
3	1	1	0	$\widetilde{A}_1$	5
7	2	1	1	regular	2
13	6	1	2	regular	1
13	1	2	3	regular	1

Table 3.1

Type F_4

p	Kac coordinates					Type	d
	a_0	a_1	a_2	a_3	a_4	of Φ_0	
2	0	1	0	0	0	$A_1 C_3$	28
2	0	0	0	0	1	B_4	16
3	1	1	0	0	0	C_3	15
3	0	0	1	0	0	$\widetilde{A}_2 A_2$	18
3	1	0	0	0	1	B_3	15
5	1	1	0	0	1	B_2	10
7	1	0	0	1	1	A_2	7
7	2	1	1	0	0	$\widetilde{A}_2$	7
13	2	1	1	1	1	regular	4
17	4	2	1	1	1	regular	3
17	1	1	2	1	2	regular	3

Table 3.2

Type E_6

p	Kac coordinates							Type	d
	a_0	a_1	a_2	a_3	a_4	a_5	a_6	of Φ_0	
2	1	1	0	0	0	0	0	D_5	32
2	0	0	0	1	0	0	0	A_1A_5	40
3	2	1	0	0	0	0	0	D_5	16
3	1	2	0	0	0	0	0	D_5	16
3	1	1	0	0	0	0	1	D_4	24
3	1	0	1	0	0	0	0	A_5	21
3	1	0	0	1	0	0	0	A_1A_4	25
3	1	0	0	0	0	1	0	A_1A_4	25
3	0	0	0	0	1	0	0	A_2^3	27
5	1	1	1	0	0	0	1	A_3	15
7	2	4	0	0	0	0	1	D_4	8
7	1	4	0	0	0	0	2	D_4	8
7	2	0	1	0	1	0	0	A_2^2	10
7	1	1	1	0	0	1	1	A_2	11
11	2	1	1	0	1	1	1	A_1	7
11	1	1	1	0	1	1	2	A_1	7
13	2	1	1	1	1	1	1	regular	6
19	4	1	2	2	1	1	1	regular	4
19	1	4	2	2	1	1	1	regular	4
37	9	5	4	3	2	1	1	regular	2
37	5	9	3	4	2	1	1	regular	2
37	8	9	4	1	1	2	3	regular	2
37	9	8	1	4	1	2	3	regular	2

Table 3.3

Type E_7

p	Kac coordinates								Type	d
	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	of Φ_0	
2	1	0	0	0	0	0	0	1	E_6	54
2	0	1	0	0	0	0	0	0	A_1D_6	64
2	0	0	1	0	0	0	0	0	A_7	70
3	2	0	0	0	0	0	0	1	E_6	27
3	1	0	0	0	0	0	1	0	A_1D_5	42
3	1	0	1	0	0	0	0	0	A_6	42
3	1	1	0	0	0	0	0	0	D_6	33
3	0	0	0	1	0	0	0	0	A_2A_5	45
5	1	1	0	0	0	0	1	0	D_4A_1	25
7	2	1	0	1	0	0	0	0	A_5	16
7	2	1	0	0	0	0	1	1	D_4	17
7	1	0	0	0	1	0	1	0	$A_1^3A_2$	19
11	1	1	1	0	0	1	1	1	A_2	12
13	2	1	0	1	1	0	1	0	A_1^3	10
19	2	1	1	1	1	1	1	1	regular	7

Table 3.4

Type E_8

p	Kac coordinates									Type	d
	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	of Φ_0	
2	0	1	0	0	0	0	0	0	0	D_8	128
2	0	0	0	0	0	0	0	0	1	A_1E_7	112
3	1	1	0	0	0	0	0	0	0	D_7	78
3	0	0	1	0	0	0	0	0	0	A_8	84
3	0	0	0	0	0	0	0	1	0	A_2E_6	81
3	1	0	0	0	0	0	0	0	1	E_7	57

5	1	1	0	0	0	0	0	0	1	D_6	45
5	1	0	0	1	0	0	0	0	0	A_1A_6	49
5	0	0	1	0	0	0	0	0	1	A_1A_6	49
5	0	0	0	0	0	1	0	0	0	A_4^2	50
7	2	0	0	0	0	0	0	1	1	E_6	28
7	1	1	0	0	0	0	1	0	0	D_4A_2	35
11	1	1	1	0	0	0	0	1	1	A_4	22
13	2	1	0	0	0	0	1	1	1	D_4	18
13	1	1	1	0	0	1	0	0	1	A_2^2	19
19	1	1	1	1	0	0	1	1	1	A_2	13
31	2	1	1	1	1	1	1	1	1	regular	8
41	5	3	2	1	1	1	1	1	1	regular	6
41	1	2	1	1	1	1	1	2	4	regular	6
61	7	5	1	3	1	1	1	2	4	regular	4
61	3	2	2	1	1	1	4	1	7	regular	4

Table 3.5

3.3 Conjugacy of balanced toral elements up to scalar multiples

Let us stick to the assumption that char k is a good prime for the root system of $\mathfrak{g}$. The invariant subalgebra $k[\mathfrak{g}]^G$ is a polynomial ring generated by $l = \operatorname{rank} G$ algebraically independent homogeneous polynomials $\{f_1, \ldots, f_l\}$, and the zero locus of the ideal generated by the f_i's is the nullcone $\mathcal{N}(\mathfrak{g})$ (see, for example, [Jan1]).

Even though generators are not uniquely determined, the set $\{d_1, \ldots, d_l\}$ of their degrees is (in characteristic 0 this is a result of Chevalley, for good characteristic $p > 0$ see [De]). Remarkably, the degrees in good characteristic are the same as in characteristic 0 ([De]); they can be found, for example, in [Hu2]. There is always an invariant of degree 2, due to the existence of a G-invariant symmetric bilinear form κ on $\mathfrak{g}$.

3.3.1 Scalar multiples of balanced toral elements.

Throughout this section we will write $h \sim h'$ to indicate that two elements h and h' are in the same G-orbit. Let h be the usual representative of a class of balanced toral elements as in Section 3.2; for any $r \in \mathbb{F}_p^\times$ the element rh is still balanced. A natural question to ask is whether rh is in the same conjugacy class of h. Once the characteristic $p > 0$ is fixed, if h represents the unique conjugacy class with a certain type of centralizer, of course $rh \sim h$ for any $r \in \mathbb{F}_p^\times$. For any fixed p there are at most two conjugacy classes with isomorphic centralizer, except for type E_6 in characteristic 37, where there are four classes. The cases with exactly two conjugacy classes with the same type of centralizer are:

(i) type G_2, characteristic 13;

(ii) type F_4, characteristic 17;

(iii) type E_6, characteristic 3, centralizer of type D_5;

(iv) type E_6, characteristic 3, centralizer of type A_1A_4;

(v) type E_6, characteristic 7;

(vi) type E_6, characteristic 11;

(vii) type E_6, characteristic 19;

(viii) type E_8, characteristic 5;

(ix) type E_8, characteristic 41;

(x) type E_8, characteristic 61.

Suppose we are in the good characteristic case. Since h is semisimple, at least one of the invariant polynomials does not vanish on h; assume for instance that f_i has such property. Notice that the invariant of degree 2 always annihilates a balanced toral element h. This is because, up to replacing the invariant form κ with a scalar multiple, one can write:

$$\kappa(h,h) = \mathrm{Tr}(\mathrm{ad}h)^2 = \sum_{i \in \mathbb{F}_p^\times} \dim \mathfrak{g}(i) i^2 = \dim \mathfrak{g}(1) \sum_{i \in \mathbb{F}_p^\times} i^2 \equiv 0 \bmod p.$$

Indeed, $\sum_{i=0}^{p-1} i^2 = (p-1)p(2p-1)/6$, which is congruent to 0 modulo p for $p \geq 5$. This is because for such primes $p-1$ is always divisible by 2, and $2p-1$ is divisible by 3 whenever $p-1$ is not.

3.3.1.1 The elements of $\mathbb{F}_p^\times$ are all the $r \in k$ satisfying $r^{p-1} = 1$. Let us assume that $rh \sim h$ for all $r \in \mathbb{F}_p^\times$, and suppose $f_i(h) \neq 0$. By invariance $f_i(h) = f_i(rh) = r^{d_i} f_i(h)$, and so $r^{d_i} = 1$ for all $r \in \mathbb{F}_p^\times$. The number of solutions of the polynomial $x^{d_i} = 1$ over the field $\mathbb{F}_p$ is at most d_i. Therefore, if $p - 1 > \max_{i=1,\dots,l} d_i$, there exists a multiple rh of h which is not conjugate to h. When there are two conjugacy classes with representatives h and h', this implies that at least one multiple of h is conjugate to h' and viceversa.

Since maxima for the degrees of the f_i's in type G_2, F_4, E_6 and E_8 are respectively $6, 12, 12$ and 30, this argument suffices to settle the problem in all cases with two conjugacy classes except for $(iii), (iv), (v), (vi)$ and $(viii)$.

A closer look shows that case (vi) can actually be tackled in the same way. Indeed, consider the root system of type E_6 in characteristic 11. There are two classes of balanced toral elements with representatives h and h', whose centralizer is in both cases of type A_1. Assume rh is conjugate to h for all $r \in \mathbb{F}_{11}^\times$. The degrees of invariants in type E_6 are $2, 5, 6, 8, 9, 12$. The argument above shows that $f_1(h) = f_2(h) = f_3(h) = f_4(h) = f_5(h) = 0$. But $f_6(h) \neq 0$ implies $r^{12} = 1$ for all $r \in \mathbb{F}_{11}^\times$, and so $r = \pm 1$.

3.3.1.2 Type E_6 in characteristic 37**.** Only in this case are there four conjugacy classes with isomorphic centralizers; let their representatives be h_1, h_2, h_3, h_4. Since the maximum degree for a generator of the invariant subalgebra is $12 < 36$, none of the h_i's has the property that all its multiples are conjugate to h_i itself.

Consider the set $X_i = \{r \in \mathbb{F}_{37}^\times | rh_1 \sim h_i\}$. Clearly $X_1 \neq \emptyset$ and $\mathbb{F}_{37}^\times = \coprod_{i=1}^4 X_i$. Assume $X_j \neq \emptyset$ for $j \neq 1$. For $s_j \in X_j$ we have the inclusion $s_j X_1 \subseteq X_j$. This means that there exists $g \in G$ such that $s_j h_1 = (\mathrm{Ad} g) h_j$, and so $(\mathrm{Ad} g) s_j^{-1} h_j = h_1$. In particular, for any $t \in X_j$, $s_j^{-1} t h_1 \sim s_j^{-1} h_j \sim h_1$. In other words, $s_j^{-1} X_j \subseteq X_1$.

The two inclusions imply that $|X_1| = |X_j|$ for every $X_j \neq \emptyset$. Then $\mathbb{F}_{37}^\times$ is the disjoint union of at least two sets of the same cardinality. Notice that we cannot have $|X_1| = 12$. In that case we could assume without loss of generality $h_1 \sim h_2 \sim h_3$. But that would lead to $rh_4 \sim h_4$ for all $r \in \mathbb{F}_{37}^\times$.

Finally, it is possible to exclude the case in which exactly two of the X_i's are nonempty. Each of them should have cardinality 18, but the highest degree of an invariant generator in type E_6 is 12, and so $|X_1| \leq 12$. Then for every choice of h_i and h_j, there exists $r \in \mathbb{F}_{37}^\times$ such that $rh_i \sim h_j$.

3.3.2 Remaining cases

The arguments in 3.3.1 allow us to conclude in almost all instances that whenever there is more than one conjugacy class of balanced toral elements with isomorphic centralizer, two representatives of distinct classes are conjugate up to a scalar multiple. Something similar happens for (iii), (v) and $(viii)$ with elements in distinct classes being conjugate up to multiplication by a suitable $r \in \mathbb{F}_p^\times$. However, for type E_6 in characteristic 3 with centralizer of type A_1A_4 (case (iv)) it turns out that two representatives are conjugate under an element of the full automorphisms group $Aut(\mathfrak{g})$, not of its identity component (see Section 3.5.5).

We are going to prove these facts by finding explicit Kac coordinates for suitable scalar multiples of each element.

3.3.2.1 Type E_6 in characteristic 7. Retain the notation from Section 2.3. Recall that on the vector space V the symmetry with respect to the hyperplane $L_{\alpha,n} = \{x \in V | \langle \alpha, x \rangle = n\}$ is given by $S_{\alpha,n}(x) = x + (n - \langle \alpha, x \rangle)\alpha^\vee$, where $\alpha^\vee$ is the coroot corresponding to α.

The representative h of one of the two classes (let h' be a representative of the other) has Kac coordinates $(2, 4, 0, 0, 0, 0, 1)$, and therefore it corresponds to the point $\frac{4}{7}\check{\omega}_1 + \frac{1}{7}\check{\omega}_6$. The balanced element $3h$ corresponds to $\frac{12}{7}\check{\omega}_1 + \frac{3}{7}\check{\omega}_6$; we will conjugate it via the extended affine Weyl group to a point in the fundamental alcove. Translating by $-\check{\omega}_1$ we obtain the point $\frac{5}{7}\check{\omega}_1 + \frac{3}{7}\check{\omega}_6$. Let $\alpha = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$. We apply the reflection about the hyperplane $L_{\alpha,1}$, observing that the expression of the coroot $\alpha^\vee$ in terms of the fundamental coweights is $\alpha^\vee = \check{\omega}_1 - \check{\omega}_2 + \check{\omega}_6$. Then $S_{\alpha,1}(\frac{5}{7}\check{\omega}_1 + \frac{3}{7}\check{\omega}_6) = \frac{5}{7}\check{\omega}_1 + \frac{3}{7}\check{\omega}_6 - \frac{1}{7}(\check{\omega}_1 - \check{\omega}_2 + \check{\omega}_6) = \frac{4}{7}\check{\omega}_1 + \frac{1}{7}\check{\omega}_2 + \frac{2}{7}\check{\omega}_6$.

As for the highest root, $\widetilde{\alpha}_0^\vee = \check{\omega}_2$, and so $S_{\widetilde{\alpha}_0,1}(\frac{4}{7}\check{\omega}_1 + \frac{1}{7}\check{\omega}_2 + \frac{2}{7}\check{\omega}_6) = \frac{4}{7}\check{\omega}_1 + \frac{2}{7}\check{\omega}_6$, which is a point in the fundamental alcove corresponding to the 7-tuple with Kac coordinates

$(1,4,0,0,0,0,2)$, so that $3h \sim h'$.

3.3.2.2 Type E_8 in characteristic 5. The bad characteristic case can be treated just as the previous one. There are two conjugacy classes, with representatives h and h', with isomorphic centralizers of type A_1A_6. If h has Kac coordinates $(1,0,0,1,0,0,0,0,0)$, it corresponds to the point $\frac{1}{5}\check{\omega}_3$, and the element $2h$ to the point $\frac{2}{5}\check{\omega}_3$. We apply to this element, in the order, reflections about the following hyperplanes:

- $L_{\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7,1}$;
- $L_{2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\alpha_8,1}$;
- $L_{2\alpha_1+2\alpha_2+4\alpha_3+5\alpha_4+4\alpha_5+3\alpha_6+2\alpha_7+\alpha_8,1}$;
- $L_{\widetilde{\alpha}_0-\alpha_8,1}$.

The point in the fundamental alcove thus obtained is $\frac{1}{5}\check{\omega}_2 + \frac{1}{5}\check{\omega}_8$, so that $2h \sim h'$.

3.3.2.3 Type E_6 in characteristic 3. As for the two conjugacy classes with centralizer of type D_5, if h is the element with Kac coordinates $(2,1,0,0,0,0,0)$, then the other is $2h$. The last case is centralizer of type A_1A_4; the points in the fundamental alcove corresponding to the two elements are $\frac{1}{3}\check{\omega}_3$ and $\frac{1}{3}\check{\omega}_5$. The reflection about the hyperplane $L_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5,1}$ sends $\frac{2}{3}\check{\omega}_3$ to $\frac{1}{3}\check{\omega}_3$, so that $2h$ is conjugate to h for elements in both these classes. This is the only case in which G-conjugacy does not hold for scalar multiples. Still, representatives of distinct classes are conjugated under the full automorphism group $Aut(\mathfrak{g})$ (see Section 3.5.5).

3.4 Summary of results

Combining the arguments presented so far in Chapter 3, we can explicitly state the main results of this chapter, that were mentioned also in Section 1.1:

Theorem 3.4.1. *Let G be a connected algebraic group of exceptional type over a field k of characteristic $p > 0$ and let $\mathfrak{g} = Lie\, G$.*

(a) *The Kac coordinates of G-orbits of balanced toral elements are listed in Tables 3.1 to 3.5.*

(*b*) *Let* $h, h' \in \mathfrak{g}$ *be balanced toral elements with isomorphic centralizers. There exists* $r \in \mathbb{F}_p^\times$ *such that* rh *is conjugate to* h' *under an automorphism of* $\mathfrak{g}$.

Again, as explicitly stated in Section 1.1, the corresponding results hold for torsion automorphisms of order p of a complex simple Lie algebra $\mathfrak{g}_\mathbb{C}$ of exceptional type:

Theorem 3.4.2. *Let* $\mathfrak{g}_\mathbb{C}$ *a simple Lie algebra of exceptional type over* $\mathbb{C}$, $G_\mathbb{C}$ *a simple linear algebraic group with Lie algebra* $\mathfrak{g}_\mathbb{C}$ *and* $Aut(\mathfrak{g}_\mathbb{C})$ *the automorphism group of* $\mathfrak{g}_\mathbb{C}$.

(*a*) *The Kac coordinates of* $G_\mathbb{C}$*-orbits of balanced automorphisms of prime order* $p > 0$ *are listed in Tables 3.1 to 3.5.*

(*b*) *Let* $\sigma, \sigma' \in \mathfrak{g}$ *be balanced torsion automorphisms with* $G_\mathbb{C}$*-conjugate fixed points subalgebras. There exists* $r \in \mathbb{F}_p^\times$ *such that* σ^r *is conjugate to* σ' *under an element of* $Aut(\mathfrak{g}_\mathbb{C})$.

3.5 Alternative computational methods

For the majority of primes p, a classification by hand of balanced toral elements can be obtained. Indeed, we were able to find Kac coordinates of balanced elements or disprove their existence in all cases apart from the following:

- Φ of type E_6, characteristic 73, regular nilpotent orbit;
- Φ of type E_7, characteristic 43 and 127, regular nilpotent orbit;
- Φ of type E_8, characteristic 79, orbit with label $E_8(a_3)$ of dimension 234;
- Φ of type E_8, characteristic $41, 61$ and 241, regular nilpotent orbit.

We chose to resort to computational methods in order to examine these cases as well; most of them are indeed characterized by the existence of balanced toral elements. Of course, the results obtained with the code coincide with our computations in all the other cases.

In this section we want to give an idea of how one can manually obtain the classification in good positive characteristic. For some orbits this approach requires lengthy computations. Nonetheless, while a computer checks all the possible $(l+1)$-tuples of Kac coordinates satisfying $a_0 = p - \sum_{i=0}^{l} b_i a_i$ and compatible with the restriction on the centralizer given by the relevant nilpotent orbit, a manual approach allows us to exclude straightaway most orbits or options for Kac coordinates.

3.5.1 In good positive characteristic there exists a G-equivariant isomorphism of varieties between the nilpotent cone $\mathcal{N}(\mathfrak{g})$ and the unipotent variety $\mathcal{U}$ of G, provided the root system is not of type A_n ([McN]). As in [Pr2], the Richardson orbit associated to a toral element (Section 3.1.1) is included in the restricted nullcone $\mathcal{N}_p(\mathfrak{g})$. One can then rely on the tables contained in [La] to figure out the order of elements of a Richardson orbit, immediately ruling out those whose elements do not satisfy $x^{[p]} = 0$.

3.5.2 Once a Richardson orbit is fixed, it may happen that there is only one option for both the centralizer of a toral element h and Kac coordinates verifying $a_0 = p - \sum_{i=0}^{l} b_i a_i$ (up to the action of Ω). Whenever this happens, there is no need to directly compute the dimension of eigenspaces $\mathfrak{g}(i)$ since the element h is automatically balanced ([Pr2], part (3) in the proof of Proposition 2.7).

As an example, consider the root system of type E_8 in characteristic $p = 19$. The Richardson orbit with label $E_8(a_3)$ has dimension 234, which is divisible by 18; it can be induced in two ways from the 0-orbit of a Levi subalgebra, and one of them is for centralizer of type A_2. There exists only one way of choosing this centralizer such that Kac coordinates satisfy $a_0 = p - \sum_{i=0}^{l} b_i a_i$, namely when $a_4 = a_5 = 0$. Only the 9-tuple $(1, 1, 1, 1, 0, 0, 1, 1, 1)$ is compatible with such condition, and therefore the corresponding toral element must be balanced.

3.5.3 Fix the Kac coordinates of a toral element h; whenever it is not immediately clear if h is balanced or not, we directly compute the dimension of the eigenspaces. In general basic arguments of graph theory are enough to exclude some cases.

As remarked before, [Bou, VI.1.6 Corollaire 3 à la Proposition 19] is an exceedingly useful tool in dealing with computations on the dimension of eigenspaces. For the same purpose, we prove the following:

Proposition 3.5.1. *Let $\mathfrak{g}$ be a simple Lie algebra of exceptional type with root system*

Φ, and let Ψ be the set of connected subgraphs of the extended Dynkin diagram of $\mathfrak{g}$. There exists a natural injection:

$$\phi : \Psi \longrightarrow \Phi^+.$$

Proof. We call i the node corresponding to the simple root α_i for $i = 1, \ldots, l$, while 0 indicates the node relative to $-\widetilde{\alpha}_0$. Let $J \subseteq \{0, 1, \ldots, l\}$ be a choice of nodes for a connected subgraph of the Dynkin diagram, the root $\phi(J)$ is assigned according to the following:

$$\begin{cases} \phi(J) = \sum_{i \in J} \alpha_i & \text{if } 0 \notin J \\ \phi(J) = \widetilde{\alpha}_0 - \sum_{i \in J \setminus \{0\}} \alpha_i & \text{if } 0 \in J. \end{cases}$$

In both cases $\phi(J) \in \Phi^+$; we only need to prove that for $\mathfrak{g}$ of exceptional type ϕ is injective. Clearly the function ϕ is injective when restricted to each of the subsets $\Psi' = \{J \in \Psi \,|\, 0 \notin J\}$ and $\Psi \setminus \Psi'$. For Lie algebras of type F_4, G_2, E_7 and E_8, injectivity of Ψ is due to:

$$\mathrm{ht}(\phi(J')) \leq l < h - l - 1 \leq \mathrm{ht}(\phi(J)),$$

whenever $J' \in \Psi', J \in \Psi \setminus \Psi'$, and where h is the Coxeter number.
For type E_6 it is enough to note that, retaining assumptions on J and J', if $\phi(J) = \sum_{i=1}^{l} n_i \alpha_i$ and $\phi(J') = \sum_{i=1}^{l} n'_i \alpha_i$, then $n'_4 < 2 \leq n_4$. □

Corollary 3.5.2. *In type E_6, there exists a bijective correspondence between Φ^+ and the set Ψ of connected subgraphs of the extended Dynkin diagram.*

Proof. In type E_6 the sets Φ^+ and Ψ have the same cardinality. □

It is possible to use Proposition 3.5.1 and Corollary 3.5.2 when trying to compute the dimension of eigenspaces $\mathfrak{g}(i)$ for a toral element h. Consider the extended Dynkin diagram of a Lie algebra of exceptional type, labelled with the Kac coordinates of h; we will always use Bourbaki's numbering when referring to simple roots. The proof of 3.5.1 implies that:

- $\mathfrak{g}_{\phi(J)} \subseteq \mathfrak{g}(\sum_{i \in J} a_i)$ for $J \in \Psi'$;

- $\mathfrak{g}_{-\phi(J)} \subseteq \mathfrak{g}(\sum_{i \in J} a_i)$ for $J \in \Psi \setminus \Psi'$.

For $A, B \subseteq \{0, 1, \ldots, l\}$ with $A \cap B = \emptyset$ we define the set:

$$_A\Psi_B = \{J \in \Psi \mid A \subseteq J \text{ and } J \cap B = \emptyset\}\,.$$

When it cannot be a source of confusion, we will drop the set notation, for example we will write $_1\Psi_4$ instead of $_{\{1\}}\Psi_{\{4\}}$.

Let us assume $C \subseteq \{1, \ldots, l\}$ is the subset of nodes of the Dynkin diagram corresponding to the simple roots in the centraliser of h. If $A \cup B = \{0, \ldots, l\} - C$ and the intersection $A \cap B = \emptyset$, consider $J \in {}_A\Psi_B$. By Proposition 3.5.1, if $0 \in B$ we have $\mathfrak{g}_{\phi(J)} \subseteq \mathfrak{g}(\sum_{i\in A} a_i)$, while if $0 \in A$ then $\mathfrak{g}_{-\phi(J)} \subseteq \mathfrak{g}(\sum_{i\in A} a_i)$. In any case, $\dim \mathfrak{g}(\sum_{i\in A} a_i) \geq |{}_A\Psi_B|$.

3.5.4 As an example of how this can be applied, consider the root system of type E_6, and suppose we are looking for balanced toral elements in characteristic $p = 7$. The Richardson orbit with label $2A_2$ has dimension 48, which is divisible by $p - 1$. It can be induced only by the 0-orbit on a Levi subalgebra whose root system is of type D_4. There is a unique way of embedding a centraliser of type D_4 in the Dynkin diagram of E_6; therefore, if a balanced toral element h exists for this orbit, its Kac coordinates satisfy $a_2 = a_3 = a_4 = a_5 = 0$, while a_0, a_1, a_6 are all strictly positive and verify $a_0 = 7 - a_1 - a_6$. Moreover, eigenspaces for h corresponding to nonzero eigenvalues must have dimension 8.

Since $|{}_0\Psi_{1,6}| = |{}_1\Psi_{0,6}| = |{}_6\Psi_{0,1}| = 6$, necessarily a_0, a_1, a_6 are three *distinct* strictly positive integers. Up to symmetries of the extended Dynkin diagram, the only option (see Remark 3.5.5 below) is

$$a_1 = 1 \qquad a_6 = 2 \qquad a_0 = 4.$$

Direct computation shows that this is indeed a d-balanced toral element for every positive integer d dividing 8.

3.5.5 When checking that an element h corresponding to a choice of Kac coordinates is balanced, one can actually consider its coordinates up to any symmetry of the extended Dynkin diagram, and not only up to the action of an element of Ω. The bigger group $Aut(\mathfrak{g})$ is the union of cosets σG where σ is a diagram automorphism of the Dynkin diagram, and the fact that Kac coordinates coincide up to a symmetry not in G only means that the corresponding toral elements are conjugated by some element in σG. This, despite giving a possibly different G-conjugacy class, does not change the dimension of eigenspaces.

For exceptional Lie algebras, only in type E_6 is the full automorphisms group $Aut(\mathfrak{g})$ not connected, in this case it consists of two distinct connected components.

Chapter 4

Modular invariants: structure of orbits and p-cyclic subspaces

4.1 Preliminaries

Let $h \in \mathfrak{g}$ be a toral element. The endomorphism $\operatorname{ad} h$ endows $\mathfrak{g}$ with a $\mathbb{Z}_p$-grading:

$$\mathfrak{g} = \sum_{i \in \mathbb{Z}_p} \mathfrak{g}(i),$$

where $\mathfrak{g}(i) = \{ x \in \mathfrak{g} \mid [h, x] = ix \}$.

We denote by $G(0) = C_G(h)^\circ$ the identity component of the centralizer of h in G. This is a connected reductive subgroup of G with Lie algebra $\mathfrak{g}(0)$.

For $g \in G(0)$ and $x \in \mathfrak{g}(i)$, one has $[h, g\cdot x] = gg^{-1}[h, g\cdot x] = g[g^{-1}\cdot h, x] = g[h, x] = i\, g \cdot x$, so that $G(0)$ acts on each graded subspace $\mathfrak{g}(i)$.

Our purpose is to study the ring of invariants of this action; without loss of generality we can restrict our attention to $k[\mathfrak{g}(1)]^{G(0)}$. This is because considering the action of $G(0)$ on $\mathfrak{g}(i)$ for $i \neq 0$ is equivalent to replacing h with $h' = i^{-1} \cdot h$ and looking at the component of degree 1 for the grading given by the toral element h'; notice moreover that $C_G(h)^\circ = C_G(h')^\circ = G(0)$. Unless explicitly stated, we will assume from now on that G satisfies the *standard hypothesis* (see 2.1.2).

4.1.1 Properties of the adjoint quotient map

We will now make use of the essentials introduced in Section 2.4. The ring $k[\mathfrak{g}(1)]^{G(0)}$ is finitely generated, we can therefore write $k[\mathfrak{g}(1)]^{G(0)} = k[\psi_1, \ldots, \psi_m]$ for suitable homogeneous polynomials $\psi_1, \ldots, \psi_m \in k[\mathfrak{g}(1)]$, that need not be algebraically independent. Consider the adjoint quotient map

$$\Psi : \mathfrak{g}(1) \to \mathfrak{g}(1)/\!\!/G(0) \tag{4.1}$$

given by the embedding $k[\mathfrak{g}(1)]^{G(0)} \subseteq k[\mathfrak{g}(1)]$. Since $G(0)$ is reductive, each fibre contains a unique closed orbit ([MFK, Corollary 1.2]).

We want to prove that the analogue of property (ii) in Section 2.4.2 holds for Ψ. Notice that for $x \in \mathfrak{g}(1)$ one has the equality:

$$T_x(G{\cdot}x) \cap \mathfrak{g}(1) = T_x(G(0) \cdot x) = [\mathfrak{g}(0), x], \tag{4.2}$$

where $T_x(G{\cdot}x)$ is the tangent space at x to the orbit $G{\cdot}x$. Indeed, $[\mathfrak{g}(0), x] \subseteq T_x(G(0){\cdot}x)$ as $\mathfrak{g}(0)$ is the Lie algebra of $G(0)$. Furthermore $T_x(G \cdot x) = [\mathfrak{g}, x]$ since G satisfies the standard hypothesis. But then we can apply the properties of the grading to obtain $T_x(G \cdot x) \cap \mathfrak{g}(1) = [\mathfrak{g}(0), x]$. As a consequence

$$T_x(G(0) \cdot x) \subseteq T_x(G \cdot x) \cap \mathfrak{g}(1) = [\mathfrak{g}(0), x], \tag{4.3}$$

but then $[\mathfrak{g}(0), x] = T_x(G(0) \cdot x)$ and every inclusion in (4.3) is an equality.

Thanks to this fact, we can make use of an argument of Richardson. We refer to the version in [Jan1], whose statement we include for completeness.

Lemma 4.1.1 ([Jan1], Lemma 2.4)**.** *Let G be an algebraic group and let H be a closed subgroup. Let M be a G-variety and let N be a closed and H-stable subvariety of M. Suppose that for all $x \in N$ the following holds:*

$$T_x(G \cdot x) \cap T_x(N) \subseteq [\mathfrak{h}, x],$$

where $\mathfrak{h} = LieH$. Then the intersection with N of each G-orbit in M is a union of finitely many H-orbits. Moreover, $T_x(H \cdot x) = [\mathfrak{h}, x]$ for all $x \in N$.

Lemma 4.1.2. *Each fibre of Ψ in (4.1) consists of finitely many $G(0)$-orbits.*

Proof. In view of (4.2), we can apply Lemma 4.1.1 by replacing G, H, M, N with $G, G(0), \mathfrak{g}, \mathfrak{g}(1)$ respectively. Thus the intersection $G \cdot x \cap \mathfrak{g}(1)$ of any G-orbit in $\mathfrak{g}$ with $\mathfrak{g}(1)$ consists of finitely many $G(0)$-orbits.

Notice that the restriction of any $f \in k[\mathfrak{g}]^G$ to $\mathfrak{g}(1)$ is an element of the ring $k[\mathfrak{g}(1)]^{G(0)}$. In particular, the restriction of each generator of $k[\mathfrak{g}]^G = k[f_1, \dots, f_l]$ can be written as $f_i|_{\mathfrak{g}(1)} = p_i(\psi_1, \dots, \psi_m)$ for a certain polynomial $p_i \in k[X_1, \dots, X_m]$ in m indeterminates. Hence the value $f_i(x)$ for $x \in \mathfrak{g}(1)$ is uniquely determined by the values $\psi_1(x), \dots, \psi_m(x)$.

Let $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{A}^m$, and let $\overline{P}_\alpha = \Psi^{-1}(\alpha)$ be the corresponding fibre. Call $\xi_i = p_i(\alpha_1, \dots, \alpha_m)$ for all $i = 1, \dots, l$. Then $\overline{P}_\alpha \subseteq P_\xi$, with P_ξ the fibre of F over the point $(\xi_1, \dots, \xi_l)$, where F is the adjoint quotient map introduced in Section 2.4.2. The set P_ξ is the union of finitely many G-orbits $G \cdot x_1, \dots, G \cdot x_q$ for some $x_1, \dots, x_q \in \mathfrak{g}$ (property (ii) in Section 2.4.2). Therefore we have the inclusion:

$$\overline{P}_\alpha \subseteq \bigcup_{i=1}^{q} (G \cdot x_i \cap \mathfrak{g}(1)).$$

As each $G \cdot x_i \cap \mathfrak{g}(1)$ is the union of finitely many $G(0)$-orbits, by Lemma 4.1.1 the same holds for $\overline{P}_\alpha$. □

4.1.2 Nilpotent orbits

Let $e \in \mathfrak{g}(1) \cap \mathcal{N}(\mathfrak{g})$ be a nilpotent element. If an element $e \in \mathfrak{g}(1)$ is $G(0)$-unstable it is automatically nilpotent since it is also G-unstable, hence $\mathcal{N}(\mathfrak{g}(1)) \subseteq \mathcal{N}(\mathfrak{g}) \cap \mathfrak{g}(1)$. The next lemma shows that under our assumptions (chiefly, the standard hypothesis) even the opposite inclusion holds, so that $\mathcal{N}(\mathfrak{g}(1)) = \mathcal{N}(\mathfrak{g}) \cap \mathfrak{g}(1)$. This was verified in Vinberg's paper ([Vi]), but his approach uses $\mathfrak{sl}_2$-triples, which are not very useful in positive characteristic. We will in fact resort to the Kempf-Rousseau theory (Section 2.5) and results in [Pr1].

Lemma 4.1.3. *Let $e \in \mathcal{N}(\mathfrak{g}) \cap \mathfrak{g}(1)$. Then e is $G(0)$-unstable.*

Proof. The element e being nilpotent, it is G-unstable by the Hilbert-Mumford criterion, therefore there exists $\lambda \in Y(G)$ such that $e \in \sum_{j>0} \mathfrak{g}(\lambda, j)$ (see Section 2.5). Our goal is to find $\eta \in Y(G(0))$ with the same properties. We retain notation from Section 2.5.

Without loss of generality we can choose λ to be optimal and satisfying the following properties (see [Pr1]):

- $e \in \mathfrak{g}(\lambda, 2)$;

- $C_{\mathfrak{g}}(e) \subseteq \sum_{j\geq 0} \mathfrak{g}(\lambda, j) = Lie\, P(\lambda)$.

As G satisfies the standard hypothesis, the subspaces $\mathfrak{g}(\lambda, i)$ and $\mathfrak{g}(\lambda, -i)$ are in duality via the invariant form κ. Consider the subspace $[\mathfrak{g}(\lambda, 0), e] \subseteq \mathfrak{g}(\lambda, 2)$. The κ-orthogonal complement to $[\mathfrak{g}(\lambda, 0), e]$ in $\mathfrak{g}(\lambda, -2)$ coincides with $\mathfrak{g}(\lambda, -2)_e = \mathfrak{g}(\lambda, -2) \cap C_{\mathfrak{g}}(e)$. Indeed, assume that for a certain $y \in \mathfrak{g}(\lambda, -2)$ we have $\kappa([x, e], y) = \kappa(x, [e, y]) = 0$ for all $x \in \mathfrak{g}(\lambda, 0)$; by nondegeneracy of the invariant form it must be $[e, y] = 0$.

On the other hand $\mathfrak{g}(\lambda, -2)_e = \{0\}$, hence $[\mathfrak{g}(\lambda, 0), e] = \mathfrak{g}(\lambda, 2)$. As a consequence, there exists $h_e \in \mathfrak{g}(\lambda, 0)$ satifying $[h_e, e] = e$. Recalling that $[h, e] = e$, one has $h - h_e \in C_{\mathfrak{g}}(e) \subseteq LieP(\lambda)$. But as $h_e \in LieP(\lambda)$, also $h \in LieP(\lambda)$.

We have that $h \in LieP(\lambda) = LieP(e)$. As h is semisimple, it belongs to the Lie algebra of a maximal torus $S \subseteq P(\lambda)$, so that $S \subseteq G(0)$. Nonetheless, there exists a primitive cocharacter $\lambda_S(k^\times) \subseteq S$ that is optimal for e (Theorem 2.5.1(iii)). But then $\lambda_S \subseteq G(0)$, and the lemma follows. □

Corollary 4.1.4. *The space* $\mathfrak{g}(1)$ *contains only finitely many nilpotent* $G(0)$*-orbits.*

Proof. The statement follows by combining Lemma 4.1.2 and Lemma 4.1.3 in view of $\Psi^{-1}(0) = \mathcal{N}(\mathfrak{g}(1)) = \mathcal{N}(\mathfrak{g}) \cap \mathfrak{g}(1)$. □

Corollary 4.1.5. *If* $\mathfrak{g}(1) \subseteq \mathcal{N}(\mathfrak{g})$*, then* $k[\mathfrak{g}(1)]^{G(0)} \simeq k$*.*

Proof. By Lemma 4.1.3 the only closed orbit in $\mathfrak{g}(1)$ is $\{0\}$. But any fibre of Ψ contains a unique closed orbit, so that the k-algebra $k[\mathfrak{g}(1)]^{G(0)}$ must be trivial. □

4.2 Regular elements

Fix $x \in \mathfrak{g}(1)$. First of all, observe that $x^{[p]^i} \in \mathfrak{g}(0)$ for $i > 0$. Indeed, $[x, h] = -x$ and therefore $[x^{[p]}, h] = (\mathrm{ad}\, x)^{p-2}[x, [x, h]] = 0$.

Furthermore, for a big enough integer $i \gg 0$ the element $x^{[p]^i}$ is semisimple; this fact is a consequence of Jacobson's formula (2.2) and one can take $N = \dim \mathfrak{g}$, for example. Notice that there exists $N \gg 0$ such that $x^{[p]^N}$ is semisimple for every $x \in \mathfrak{g}(1)$. The elements $x^{[p]^i}$ and $x^{[p]^j}$ commute for any $i, j \geq 0$, so we can define a toral subalgebra $\mathfrak{t}(x) \subseteq \mathfrak{g}(0)$ spanned by iterated p-th powers of x:

$$\mathfrak{t}(x) := \mathrm{span}_k \langle x^{[p]^i}, i \geq N \rangle.$$

This definition does not depend on the integer N chosen. Indeed, assume $x^{[p]^n}$ for $n \geq 0$ is semisimple, and call $\mathfrak{t}_n(x) = \mathrm{span}_k\langle x^{[p]^i}, i \geq n\rangle$; it is enough to prove the equality $\mathfrak{t}_n(x) = \mathfrak{t}_{n+1}(x)$. The element $x^{[p]^n}$ is semisimple, hence it belongs to a toral subalgebra $\mathfrak{t}$. Every toral subalgebra is spanned over k by the toral elements it contains as $\mathfrak{t} = \mathfrak{t}^{tor} \otimes_{\mathbb{F}_p} k$, where $\mathfrak{t}^{tor}$ is the set of toral elements of $\mathfrak{t}$ (see for example [FS, 2.3]). Assume $\{h_1, \ldots, h_l\}$ is a basis of $\mathfrak{t}$ consisting of toral elements, and write $x^{[p]^n} = \sum_{i=1}^{l} a_i h_i$ for $a_1, \ldots, a_l \in k$. As a consequence, $x^{[p]^{n+j}} = \sum_{i=1}^{l} a_i^{p^j} h_i$ for any $j \geq 0$. Then $\dim_k \mathfrak{t}_n(x) = \dim_{\mathbb{F}_p} \mathrm{span}_{\mathbb{F}_p}\langle a_1, \ldots, a_l\rangle$ thanks to the discussion in Section 2.1.1. Say $a_{i_1}, \ldots, a_{i_j}$ are a $\mathbb{F}_p$-basis of $\mathrm{span}_{\mathbb{F}_p}\langle a_1, \ldots, a_l\rangle$. We have that $a_{i_1}, \ldots, a_{i_j}$ are $\mathbb{F}_p$-linearly independent if and only if $a_{i_1}^p, \ldots, a_{i_j}^p$ are; this is because the field k is a separable extension of the perfect field $\mathbb{F}_p$. As a result, $\dim \mathfrak{t}_n(x) = \dim \mathfrak{t}_{n+1}(x)$, and since $\mathfrak{t}_{n+1}(x) \subseteq \mathfrak{t}_n(x)$, the equality $\mathfrak{t}_n(x) = \mathfrak{t}_{n+1}(x)$ must hold.

4.2.0.1 The centralizer $\mathfrak{l}(x) = C_{\mathfrak{g}}(\mathfrak{t}(x))$ is a Levi subalgebra of $\mathfrak{g}$. We now follow [PrSt, Section 2], for some insight on the structure of Levi subalgebras for groups satisfying the standard hypothesis. The Lie algebra $\mathfrak{l}(x)$ admits a decomposition as a sum of restricted ideals:

$$\mathfrak{l}(x) = \tilde{\mathfrak{l}}_1(x) \oplus \tilde{\mathfrak{l}}_2(x) \oplus \mathfrak{z}. \tag{4.4}$$

Here $\tilde{\mathfrak{l}}_1(x) \simeq \bigoplus_{i=1}^{\tilde{d}_x} \mathfrak{gl}_{r_i p}(k)$ is a sum of $\tilde{d}_x \in \mathbb{Z}$ ideals of type $A_{r_i p-1}$ for suitable nonnegative integers r_i. On the other hand, the subalgebra $\tilde{\mathfrak{l}}_2(x)$ is a direct sum of simple ideals not of type A_{rp-1} and $\mathfrak{z}$ is central in $\mathfrak{l}(x)$.

4.2.0.2 A short digression on the choice of the ideals isomorphic to $\mathfrak{gl}_{r_i p}(k)$ in $\tilde{\mathfrak{l}}_1(x)$ is needed, since the construction we perform differs slightly from that in [PrSt, Section 2]. This is necessary in order to ensure restrictedness of $\mathfrak{l}(x)$ under the p-th power map of $\mathfrak{g}$.

Let $L(x) \subseteq G$ denote the Levi subgroup with Lie algebra $\mathfrak{l}(x)$, and write $\mathfrak{l}(x) = \mathfrak{l}_1 \oplus \ldots \oplus \mathfrak{l}_m \oplus z(\mathfrak{l}(x))$ corresponding to the decomposition of $L(x)$ into its simple components; each summand in this direct sum is $L(x)$-invariant and restricted. The construction in [PrSt, Section 2] is as follows. If for a certain $1 \leq i \leq m$ the ideal $\mathfrak{l}_i$ is of type A_{r_ip-1} for an integer $r_i \geq 1$, then $\mathfrak{l}_i \simeq \mathfrak{sl}_{r_ip}(k)$ as restricted Lie algebras. There exists an element $h_0^i \in \mathfrak{l}(x)$ orthogonal to every $\mathfrak{l}_j$ with $j \neq i$ and such that h_0^i acts on $\mathfrak{l}_i \simeq \mathfrak{sl}_{r_ip}(k)$ as commutation by the matrix e_{11} (see the beginning of Section 2.7 for the notation). Then $k\,h_0^i \oplus \mathfrak{l}_i \simeq \mathfrak{gl}_{r_ip}(k)$ and this is the ideal contributing to the decomposition of $\tilde{\mathfrak{l}}_1(x)$.

We claim that, up to slightly modifying the construction of each of the elements h_0^i, the ideals isomorphic to $\mathfrak{gl}_{r_ip}(k)$ can be chosen to be $L(x)$-invariant and restricted under the p-th power map of $\mathfrak{g}$. Indeed, according to the construction in [PrSt, Section 2] described above, the element $(h_0^i)^{[p]} - h_0^i$ commutes with all the ideals $\mathfrak{l}_1, \dots, \mathfrak{l}_m$, hence it belongs to the toral subalgebra $z(\mathfrak{l}(x))$, the centre of $\mathfrak{l}(x)$. Thus the subalgebra $k\,h_0^i + z(\mathfrak{l}(x))$ is toral as well, and therefore spanned by toral elements. As a result, there exists a toral element $\overline{h_0^i} \in k\,h_0^i + z(\mathfrak{l}(x))$ such that the adjoint action of $\overline{h_0^i}$ on $\mathfrak{l}(x)$ coincides with that of h_0^i. Then $k\,\overline{h_0^i} + \mathfrak{sl}_{r_ip} \simeq \mathfrak{gl}_{r_ip}$, and this is the ideal we fix in the decomposition $\tilde{\mathfrak{l}}_1(x) \simeq \bigoplus_{i=1}^{\tilde{d}_x} \mathfrak{gl}_{r_ip}(k)$. Notice that the ideal $\mathfrak{gl}_{r_ip} = k\,\overline{h_0^i} + \mathfrak{sl}_{r_ip}(k)$ is $L(x)$-invariant since $\mathfrak{l}_i \simeq \mathfrak{sl}_{r_ip}$ is, the inclusion $L(x)h_0^i \subseteq k\,h_0^i + \mathfrak{l}_i$ holds by [PrSt, Section 2] and $L(x)$ acts trivially on the centre $z(\mathfrak{l}(x))$. Furthermore, the elements $\overline{h_0^i}$ obtained by each summand $\mathfrak{l}_i$ of type A_{r_ip-1} are all linearly independent because each of them acts trivially on the summands $\mathfrak{l}_j$ for $j \neq i$.

4.2.0.3 Both h and x commute with $\mathfrak{t}(x)$, so they belong to $\mathfrak{l}(x)$. Notice that this implies $\mathfrak{z} \subseteq \mathfrak{g}(0)$. Furthermore, $x = [h, x] \in [\mathfrak{l}(x), \mathfrak{l}(x)]$.

The Levi subalgebra $\mathfrak{l}(x)$ is restricted by the above, and so is its derived subalgebra $[\mathfrak{l}(x), \mathfrak{l}(x)]$. In fact $[\mathfrak{l}(x), \mathfrak{l}(x)] = [\tilde{\mathfrak{l}}_1(x), \tilde{\mathfrak{l}}_1(x)] \oplus [\tilde{\mathfrak{l}}_2(x), \tilde{\mathfrak{l}}_2(x)]$; more specifically $[\tilde{\mathfrak{l}}_1(x), \tilde{\mathfrak{l}}_1(x)] \simeq \bigoplus \mathfrak{sl}_{r_ip}$ and $[\tilde{\mathfrak{l}}_2(x), \tilde{\mathfrak{l}}_2(x)] = \tilde{\mathfrak{l}}_2(x)$, and all these ideals are restricted.

As a result, $x^{[p]^i} \in [\mathfrak{l}(x), \mathfrak{l}(x)]$ for all $i \geq 0$. But for i large enough these elements are also central in $\mathfrak{l}(x)$, thus $x^{[p]^i} \in [\mathfrak{l}(x), \mathfrak{l}(x)] \cap z(\mathfrak{l}(x))$ for $i \geq N$, where $z(\mathfrak{l}(x))$ is the centre of the Levi subalgebra. The intersection $\tilde{\mathfrak{l}}_2(x) \cap z(\mathfrak{l}(x)) = \{0\}$ since $\tilde{\mathfrak{l}}_2(x)$ consists of simple ideals. Thus $x^{[p]^i} \in z(\tilde{\mathfrak{l}}_1(x))$, a subalgebra spanned by the central elements $\mathbf{1}_{r_jp} \in \mathfrak{sl}_{r_jp}$, where $\mathbf{1}_{r_jp}$ stands for the $r_jp \times r_jp$ identity matrix. As a result, $\mathfrak{t}(x) \subseteq z(\tilde{\mathfrak{l}}_1(x))$ as well.

Let $\xi_j \mathbf{1}_{r_jp}$ be the component of $x^{[p]^i}$ (for a fixed $i \geq N$) on $\mathfrak{sl}_{r_jp}$, where $\xi_j \in k$.

Then $\mathfrak{l}(x)$ decomposes as:

$$\mathfrak{l}(x) = \mathfrak{l}_1(x) \oplus \mathfrak{l}_2(x). \tag{4.5}$$

Here $\mathfrak{l}_1(x) = \bigoplus_{i=1}^{d_x} \mathfrak{gl}_{r_i p}(k) \subseteq \tilde{\mathfrak{l}}_1(x)$ is the direct sum of the ideals of the form $\mathfrak{gl}_{r_j p}$ for which $\xi_j \neq 0$ and for a certain $d_x \leq \tilde{d}_x$. Roughly speaking, $\mathfrak{l}_1(x)$ is the sum of the ideals of the form $\mathfrak{gl}_{r_j p}$ on which $\mathfrak{t}(x)$ is supported. On the other hand, $\mathfrak{l}_2(x)$ is the sum of all ideals of $\mathfrak{l}(x)$ appearing in the decomposition (4.4) that are orthogonal to $\mathfrak{t}(x)$ with respect to the invariant form κ.

4.2.1 The Levi subalgebra $\mathfrak{l}(x)$ associated to a regular element

Recall that the p-th power map maps $\mathfrak{g}(1)$ to $\mathfrak{g}(0)$. In the discussion above we associated the toral subalgebra $\mathfrak{t}(x) \subseteq \mathfrak{g}(0)$ to an element $x \in \mathfrak{g}(1)$. Define the integer:

$$s := \max_{x \in \mathfrak{g}(1)} \dim \mathfrak{t}(x).$$

Definition 4.2.1. We say that $x \in \mathfrak{g}(1)$ is *regular* if $\dim \mathfrak{t}(x) = s$. We let $\mathfrak{g}(1)_{reg}$ denote the set of regular elements of $\mathfrak{g}(1)$.

Lemma 4.2.2. *The set $\mathfrak{g}(1)_{reg}$ is a $G(0)$-invariant Zariski open subset of $\mathfrak{g}(1)$.*

Proof. Let $\{w_1, \ldots, w_n\}$ be a basis of $\mathfrak{g}(0)$. Given $x \in \mathfrak{g}(1)$, the elements $x^{[p]^i}$ for $i \geq N$ are semisimple. Thanks to Corollary 2.1.2, for every $i = N, \ldots, N+s$ there exist homogeneous polynomials $\Psi_1^i, \ldots, \Psi_n^i$ of degree p^i such that

$$x^{[p]^i} = \sum_{j=1}^{n} \Psi_j^i(x) w_j.$$

Consider the $s \times n$ matrix:

$$\begin{pmatrix} \Psi_1^N & \cdots & \Psi_n^N \\ \vdots & & \vdots \\ \Psi_1^{N+s-1} & \cdots & \Psi_n^{N+s-1} \end{pmatrix}. \tag{4.6}$$

The set $\mathfrak{g}(1)_{reg}$ consists of elements $x \in \mathfrak{g}(1)$ for which this matrix has rank s. By definition of s, the set $\mathfrak{g}(1)_{reg}$ is nonempty. Thus $\mathfrak{g}(1)_{reg}$ is Zariski open as it is the complement of the vanishing set of all the $s \times s$ minors of the matrix (4.6). Nonetheless, $\mathfrak{g}(1)_{reg}$ is $G(0)$-invariant since the p-th power map is a $G(0)$-equivariant morphism of varieties. □

Recall the decomposition (4.5), with $\mathfrak{l}_1(x) \simeq \bigoplus_{i=1}^{d_x} \mathfrak{gl}_{r_i p}$.

Lemma 4.2.3. *If $x \in \mathfrak{g}(1)_{reg}$ then $r_i = 1$ for all $i = 1, \ldots, d_x$. In particular, $d_x = s$.*

Proof. Start by writing $h = h' + h''$ according to the decomposition $\mathfrak{l}(x) = \mathfrak{l}_1(x) + \mathfrak{l}_2(x)$. Analogously we can write $x = x' + x'' \in \mathfrak{l}_1(x) \oplus \mathfrak{l}_2(x)$. We can further decompose $h' = \sum_{i=1}^{l} h'_i$ and $x' = \sum_{i=1}^{l} x'_i$ according to the sum $\mathfrak{l}_1(x) = \bigoplus_{i=1}^{d_x} \mathfrak{gl}_{r_i p}$. Let us focus on the subalgebra $\mathfrak{gl}_{r_i p}$. Let us use the notation $H = h'_i$, $X = x'_i$ and call $\mathfrak{s} = \mathrm{span}_k\langle H, X\rangle$ the subalgebra of $\mathfrak{gl}_{r_i p}$ spanned by the elements H and X. The relations $[H, X] = X$ and $H^{[p]} = H$ hold, and $(\mathrm{ad} X)^p = 0$ in $\mathfrak{s}$. Consider the natural $\mathfrak{gl}_{r_i p}$-module $V = k^{r_i p}$; by restriction it is an $\mathfrak{s}$-module. As $X^{[p]^N}$ spans the centre of $\mathfrak{gl}_{r_i p}$, X acts invertibly on V.

Let $W \subseteq V$ be an irreducible $\mathfrak{s}$-submodule. By [Jan2, 1.5], we have $\dim W = p$ and there exists $w \in W$ such that $Xw = \lambda w$ for a certain $0 \neq \lambda \in k$. The vectors $w, Hw, \ldots, H^{p-1}w$ are linearly independent. The element H acts diagonally on W and its eigenspaces are spanned by the following vectors:

$$\begin{aligned} &(H^{p-1} - 1)w && \text{with eigenvalue } 0, \\ &\sum_{i=1}^{p-1} \alpha^{-i} H^i w && \text{with eigenvalue } \alpha \in \mathbb{F}_p^\times. \end{aligned}$$

In particular, H has p distinct eigenvalues on W, namely all the scalars in the prime subfield $\mathbb{F}_p$. Writing a composition series for V as an $\mathfrak{s}$-module we can obtain a decomposition $V = \oplus_{j=1}^{r_i} V_j$, where $\dim V_j = p$, each V_j is H-invariant and the spectrum of H on V_j is $\mathbb{F}_p$. As a consequence, H has eigenvalues $p-1, \ldots, 1, 0$ on V, each with multiplicity r_i. Each subalgebra $\mathfrak{gl}_p \simeq \mathfrak{gl}(V_j) \subseteq \mathfrak{gl}_{r_i p}$ is $(\mathrm{ad}\, H)$-invariant and decomposes under $\mathrm{ad}\, H$ as $\mathfrak{gl}(V_j) = \oplus_{\alpha \in \mathbb{F}_p} \mathfrak{gl}_j(\alpha)$, where $\mathfrak{gl}_j(\alpha) = \{M \in \mathfrak{gl}(V_j) \mid [H, M] = \alpha M\}$ and each of these eigenspaces has dimension p. Notice that $\mathfrak{gl}_j(0)$ is a maximal toral subalgebra of $\mathfrak{gl}(V_j)$. Let $\mathbf{1}_j$ stand for the identity matrix of $\mathfrak{gl}(V_j)$. Fix $x_j \in \mathfrak{gl}_j(1)$ such that $x_j^{[p]} = \mathbf{1}_j$. Choose an r_i-tuple of scalars $\lambda_1, \ldots, \lambda_{r_i}$ generic enough so that their Moore matrix (see Section 2.1.1) is invertible. Consider the element $\bar{x}_i = \sum_{j=1}^{r_i} \lambda_j x_j$. Then $\mathbf{1}_j \in \mathrm{span}\langle \bar{x}_i^{[p]^n} \mid n \geq 1\rangle$ for every $j = 1, \ldots, r_i$. Thus, for the element $\bar{x} = x - x'_i + \bar{x}_i$, we have $\dim \mathfrak{t}(\bar{x}) > \dim \mathfrak{t}(x)$ unless $r_i = 1$. $\square$

Corollary 4.2.4. *Let $x \in \mathfrak{g}(1)_{reg}$, then $\mathfrak{l}_2(x)(1) \subseteq \mathcal{N}(\mathfrak{g}(1))$.*

Proof. Let $y \in \mathfrak{l}_2(x)(1)$ be not nilpotent. As in the proof of Lemma 4.2.3, let x' be the component of x on $\mathfrak{l}_1(x)$. For $\lambda \in k$ generic enough the torus $\mathfrak{t}(x' + \lambda y)$ has dimension strictly greater than $\dim \mathfrak{t}(x)$, impossible. □

Remark 4.2.5. Corollary 4.2.4 states that, for $x \in g(1)_{reg}$, all the elements in the component of degree 1 of $\mathfrak{l}_2(x)$ are $G(0)$-unstable. As a consequence, if $\mathfrak{l}_1(x) = \{0\}$ for every $x \in \mathfrak{g}(1)_{reg}$, then $\mathfrak{g}(1) \subseteq \mathcal{N}(\mathfrak{g})$ and every $G(0)$-orbit is nilpotent. In this case $k[\mathfrak{g}(1)]^{G(0)} = k$ thanks to Corollary 4.1.5. This settles the question for most exceptional Lie algebras; the only cases that might present nontrivial rings of invariants are E_6 for $p = 5$, E_7 in characteristic 5 or 7 and E_8 for $p = 7$. Indeed, this depends on whether it is possible to embed a subgraph of type A_{p-1} into the Dynkin diagram of $\mathfrak{g}$.

Notation 4.2.6. Once and for all, we set some notation and standard facts we shall repeatedly utilise and we shall be consistent with, unless otherwise stated.

We will call $L(x)$ the Levi subgroup of G with Lie algebra $\mathfrak{l}(x)$. We can decompose $L(x)$ as the almost direct product of subgroups $L_1(x) \cdot L_2(x)$ corresponding to the Lie algebra decomposition (4.5), so that $Lie\, L_i(x) = \mathfrak{l}_i(x)$ for $i = 1, 2$. The toral element h belongs to $\mathfrak{l}(x)$, therefore it makes sense to consider the subgroup $L(x)(0) = C_{L(x)}(h)^\circ = (L(x) \cap G(0))^\circ$. This is a connected reductive subgroup that acts on each graded component $\mathfrak{l}(x)(i)$. Observe that $L_1(x)$ acts trivially on $\mathfrak{l}_2(x)$ and viceversa. If $x \in \mathfrak{g}(1)_{reg}$, combining Corollary 4.1.4 and Corollary 4.2.4 and observing that the subspace $\mathfrak{l}_2(x)(1)$ is irreducible, it follows that $\mathfrak{l}_2(x)(1)$ contains a dense open $L(x)(0)$-orbit $\mathcal{O}(x)$ (it is in fact a $L_2(x)(0)$-orbit).

For x regular, $\mathfrak{l}_1(x)$ is a Lie algebra of type $(A_{p-1})^s$, isomorphic to the direct sum of s copies of $\mathfrak{gl}_p$. By supposing that these copies are indexed over $i = 1, \dots, s$, we will call $\mathfrak{l}_1^i(x)$ the i-th copy; in order to make the notation friendlier and whenever the dependence on x is clear, we will also use the notation $\mathfrak{gl}_p^i = \mathfrak{l}_1^i(x)$. This decomposition of the subalgebra $\mathfrak{l}_1(x)$ implies that the group $L_1(x)$ is the almost direct product of subgroups $L_1^i(x)$ for $i = 1, \dots, s$, namely the connected closed subgroups of $L(x)$ with Lie algebra $\mathfrak{gl}_p^i$.

We will express $h = h' + h''$ and $x = x' + x''$ according to the decomposition $\mathfrak{l}(x) = \mathfrak{l}_1(x) \oplus \mathfrak{l}_2(x)$. Moreover, we will write $h' = \sum_{i=1}^s h_i'$ and $x' = \sum_{i=1}^s x_i'$ with respect to $\mathfrak{l}_1(x) = \bigoplus_{i=1}^s \mathfrak{gl}_p^i$.

Since h'_i has full spectrum on the natural module for $\mathfrak{gl}_p^i$ (proof of Lemma 4.2.3), h'_i is a regular semisimple element of $\mathfrak{gl}_p^i$, and its centralizer in $\mathfrak{gl}_p^i$ is a maximal toral subalgebra that we will denote by $\mathfrak{t}_i$. It is worth stressing that $x_i'^{[p]} \subseteq \mathfrak{t}_i$, and more generally $\mathfrak{t}(x) \subseteq \bigoplus_{i=1}^s \mathfrak{t}_i$.

As $h \in \mathfrak{l}(x)$, the Levi subalgebra $\mathfrak{l}(x)$ admits an $\mathbb{F}_p$-grading via $\operatorname{ad} h$, and so do all the subspaces $\mathfrak{gl}_p^i$. Such grading on $\mathfrak{gl}_p^i$ coincides with the $\mathbb{F}_p$-grading induced by h'_i.

Let us now look at a fixed component $\mathfrak{gl}_p^i = \mathfrak{l}_1^i(x)$. Using Bourbaki's numbering for a root system of type A, choose a base of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_{p-1}\}$ and let α_0 be the negative of the highest root for this system. Sticking to notation in Section 2.7.1, and assuming up to G-conjugacy that $h'_i = \operatorname{diag}(p-1, \ldots, 1, 0)$, we can give a description of the degree 1 subspace $\mathfrak{l}_1^i(x)(1) = \mathfrak{gl}_p^i(1) = \operatorname{span}\langle e_{\alpha_0}, e_{\alpha_1}, \ldots, e_{\alpha_{p-1}}\rangle$. It is not restrictive to assume that $(e_{\alpha_0} + e_{\alpha_1} + \ldots + e_{\alpha_p-1})^{[p]} = \mathbf{1}_i$, the identity matrix of $\mathfrak{gl}_p^i$.

In case we need to consider more than one component $\mathfrak{gl}_p^i$ at the same time, we will call $e^i_{\alpha_0}, e^i_{\alpha_1}, \ldots, e^i_{\alpha_{p-1}}$ the generators of root subspaces of $\mathfrak{gl}_p^i(1)$.

4.2.2 Nilpotent elements in $\mathfrak{l}_1(x)$

As an application of Lemma 4.2.3, it is possible to obtain a description of the nullcone $\mathcal{N}(\mathfrak{l}_1(x)(1))$.

Lemma 4.2.7. *The nullcone $\mathcal{N}(\mathfrak{l}_1^i(x)(1))$ is the union of p hyperplanes.*

Proof. An element $y = \sum_{j=0}^{p-1} \lambda_j e_{\alpha_j}$ belongs to $\mathcal{N}(\mathfrak{l}_1^i(x)(1))$ if and only if $\prod_{j=0}^{p-1} \lambda_j = 0$. Indeed, this is a consequence of $y^{[p]} = \left(\sum_{j=0}^{p-1} \lambda_j e_{\alpha_j}\right)^{[p]} = \left(\prod_{j=0}^{p-1} \lambda_j\right) \mathbf{1}_i$. □

The next lemma shows that elements of $\mathfrak{l}_1^i(x)(1)$ which are not nilpotent can be expressed in a canonical form, up to the action of $G(0)$.

Lemma 4.2.8. *Let $y = \sum_{j=0}^{p-1} \lambda_j e_{\alpha_j} \in \mathfrak{l}_1^i(x)(1) \setminus \mathcal{N}(\mathfrak{l}_1^i(x)(1))$. Then y is $G(0)$-conjugate to the element $y' = (\prod_{j=0}^{p-1} \lambda_j) e_{\alpha_0} + \sum_{j=1}^{p-1} e_{\alpha_j}$.*

Proof. Let C be the Cartan matrix in type A_{p-1}, so that $\det C = p$. If B is the adjugate matrix of C, then $BC = p \cdot \mathbf{Id}$. Let c_{ij} denote the (i,j)-entry of C, and likewise b_{ij} is the analogue (i,j)-entry of B. Consider the maximal toral subalgebra $\mathfrak{t}_i \subseteq \mathfrak{l}_1^i(x)$ (see Notation 4.2.6). Let $T(0) \subseteq L(x)(0)$ be a maximal torus whose Lie algebra contains $\mathfrak{t}_i$. Let $\mu_1, \ldots, \mu_{p-1}$ be coroots corresponding to the system of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_{p-1}\}$, i.e. $\mu_i : k^\times \to T(0)$ are algebraic morphisms such that the standard pairing verifies $\langle \alpha_i, \mu_j \rangle = c_{ij}$ for $i, j = 1, \ldots, p-1$. Consider the following system of equations with indeterminates $a_1, \ldots, a_p \in k^\times$:

$$\begin{cases} a_1^2 a_2^{-1} = \lambda_1 \\ a_1^{-1} a_2^2 a_3^{-1} = \lambda_2 \\ \vdots \\ a_{p-3}^{-1} a_{p-2}^2 a_{p-1}^{-1} = \lambda_{p-2} \\ a_{p-2}^{-1} a_{p-1}^2 = \lambda_{p-1} \end{cases} \tag{4.7}$$

Fix $1 \leq i \leq p-1$, then:

$$\prod_{j=1}^{p-1} \lambda_j^{b_{ij}} = \prod_{l=1}^{p-1} a_l^{\sum_{j=1}^{p-1} b_{ij} c_{jl}} = \prod_{l=1}^{p-1} a_l^{p\,\delta_{il}} = a_i^p.$$

Since by assumption $\lambda_j \neq 0$ for all j, one can recover $a_i \in k^\times$ from the previous expression. Consider the element $t = \prod_{j=1}^{p-1} \mu_j(a_j) \in T(0)$. The element t acts on the root subspace e_{α_0} as multiplication by $a_1^{-1} a_{p-1}^{-1} = \prod_{j=1}^{p-1} \lambda_j^{-1}$. In particular, one sees that $t \cdot ((\prod_{j=0}^{p-1} \lambda_j) e_{\alpha_0} + \sum_{j=1}^{p-1} e_{\alpha_j}) = \lambda_0 e_{\alpha_0} + \lambda_1 e_{\alpha_1} + \ldots + \lambda_{p-1} e_{\alpha_{p-1}}$.

□

Remark 4.2.9. The meaning of Lemma 4.2.8 can be summarized in the following way: if we take two elements in $\mathfrak{l}_1^i(x)(1) \setminus \mathcal{N}(\mathfrak{l}_1^i(x)(1))$, say $y = \lambda_0 e_{\alpha_0} + \ldots + \lambda_{p-1} e_{\alpha_{p-1}}$ and $y' = \lambda_0' e_{\alpha_0} + \ldots + \lambda_{p-1}' e_{\alpha_{p-1}}$, for which $\prod_{j=0}^{p-1} \lambda_j = \prod_{j=0}^{p-1} \lambda_j'$, then y and y' are $G(0)$-conjugate. Even more is true: they are conjugate under the adjoint action of an element belonging to a torus of $L_1(x)(0)$.

4.3 p-cyclic subspaces

We will now introduce the object that in this setting will play the role that in the classical theory belongs to the Cartan subspace.

Definition 4.3.1. Let $x \in \mathfrak{g}(1)_{reg}$. We call the subspace.

$$\mathfrak{c}_x := \mathfrak{l}_1(x)(1)$$

a *p-cyclic subspace* of $\mathfrak{g}(1)$.

Although the definition of $\mathfrak{c}_x$ seems to depend on the choice of a regular element x, we will prove shortly that any two regular elements produce $G(0)$-conjugate p-cyclic subspaces. Unlike a Cartan subspace, we stress that not every element belonging to $\mathfrak{c}_x$ has a closed $G(0)$-orbit, as it follows, for example, from Lemma 4.2.7. Still, we are going to prove that every closed $G(0)$-orbit meets $\mathfrak{c}_x$.

All these features heavily rely on the next result. As $\mathfrak{l}_2(x)(1)$ consists of nilpotent elements, after possibly replacing replacing $x \in \mathfrak{g}(1)_{reg}$ with its projection on $\mathfrak{l}_1(x)$, we can assume $x \in \mathfrak{l}_1(x)(1)$. Let $\mathfrak{c}_x^{reg} = \mathfrak{c}_x \cap \mathfrak{g}(1)_{reg}$ be the subset of regular elements of $\mathfrak{c}_x$ (note that this is a nonempty open subset of $\mathfrak{c}_x$ since it contains x). Pick an element $n \in \mathfrak{l}_2(x)(1)$ belonging to the dense open orbit $\mathcal{O}(x)$ (see Notation 4.2.6).

Proposition 4.3.2. *The morphism of irreducible varieties*

$$\begin{aligned} \Psi : \quad G(0) \times (\mathfrak{c}_x^{reg} + n) &\longrightarrow \mathfrak{g}(1) \\ (g, y + n) &\longmapsto Adg\,(y + n) \end{aligned} \tag{4.8}$$

is a submersion in $(1, x + n)$.

Proof. The differential of Ψ at the point $(1, x + n)$ is given by:

$$\begin{aligned} d\Psi_{(1,x+n)} : \quad \mathfrak{g}(0) \oplus \mathfrak{c}_x &\longrightarrow \mathfrak{g}(1) \\ (y, v) &\longmapsto [y, x + n] + v \end{aligned} \tag{4.9}$$

To show surjectivity of this map one has to prove $\mathfrak{g}(1) \subseteq [\mathfrak{g}(0), x + n] + \mathfrak{c}_x$. We can assume $\mathfrak{l}(x)$ is a standard Levi subalgebra, so that $L(x)$ is the Levi factor of a parabolic subgroup $P(x)$ corresponding to a subset of positive roots in Δ. In particular, there is a triangular decomposition of the form $\mathfrak{g}(1) = \mathfrak{n}_-(1) \oplus \mathfrak{l}(x)(1) \oplus \mathfrak{n}_+(1)$, where $\mathfrak{n}_+$ is the Lie algebra of the unipotent radical of $P(x)$, and $e_\alpha \in \mathfrak{n}_+ \Leftrightarrow e_{-\alpha} \in \mathfrak{n}_-$ for every $\alpha \in \Phi_+$.

As $n \in \mathcal{O}(x)$, the inclusion $\mathfrak{l}_2(x)(1) \subseteq [\mathfrak{l}_2(x)(0), n]$ holds. Therefore $\mathfrak{l}(x)(1) \subseteq [\mathfrak{g}(0), x+n] + \mathfrak{c}_x$, by recalling that $\mathfrak{c}_x = \mathfrak{l}_1(x)(1)$.

Now let $u \in \mathfrak{n}_+$ and suppose $[x, u] = 0$. Then $[x^{[p]^r}, u] = 0$ for all $r \geq 1$, thus $[\mathfrak{t}(x), u] = 0$ and so $u \in \mathfrak{l}(x) \Rightarrow u = 0$. The same holds for any element in $\mathfrak{n}_-$. As a consequence, $\operatorname{ad} x$ acts invertibly on $\mathfrak{n}_\pm$. Since for $r \gg 0$ one has $(x+n)^{[p]^r} = x^{[p]^r}$, also $\operatorname{ad}(x+n)$ acts invertibly on $\mathfrak{n}_\pm$. In particular, $[x+n, \mathfrak{n}_\pm] = \mathfrak{n}_\pm$, and by the properties of the grading $[x+n, \mathfrak{n}_\pm(0)] = \mathfrak{n}_\pm(1)$. This leads to $\mathfrak{n}_\pm(1) \subseteq [\mathfrak{g}(0), x+n]$, thus (4.8) is a submersion in $(1, x+n)$. □

Corollary 4.3.3. *Suppose that for $x \in \mathfrak{g}(1)$ (not necessarily regular) the inclusion $\mathfrak{l}_2(x)(1) \subseteq \mathcal{N}(\mathfrak{g}(1))$ holds, and assume n is an element belonging to the open $L_2(x)(0)$-orbit of $\mathfrak{l}_2(x)(1)$. Then the morphisms*

$$\begin{aligned} \Psi_1 : G(0) \times (\mathfrak{l}_1(x)(1) + n) &\longrightarrow \mathfrak{g}(1) \\ (g, y+n) &\longmapsto Adg(y+n) \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} \Psi_2 : G(0) \times \mathfrak{l}(x)(1) &\longrightarrow \mathfrak{g}(1) \\ (g, y) &\longmapsto Adg(y) \end{aligned} \tag{4.11}$$

are submersions in $(1, x+n)$.

Proof. Observe that the inclusion $\mathfrak{l}_2(x)(1) \subseteq \mathcal{N}(\mathfrak{g}(1))$ implies that $\mathfrak{l}_2(x)(1)$ admits a dense open $L_2(x)(0)$-orbit $\mathcal{O}(x)$. Then the proof of Proposition 4.3.2 applies verbatim by replacing $\mathfrak{c}_x$ with $\mathfrak{l}_1(x)(1)$. □

Corollary 4.3.4. *All the morphisms in Proposition 4.3.2 and Corollary 4.3.3 are separable.*

Corollary 4.3.5. *Suppose that for $x \in \mathfrak{g}(1)$ the inclusion $\mathfrak{l}_2(x)(1) \subseteq \mathcal{N}(\mathfrak{g}(1))$ holds and take $n \in \mathcal{O}(x)$. Then the sets $G(0) \cdot (\mathfrak{l}_1(x)(1) + n)$ and $G(0) \cdot \mathfrak{l}(x)(1)$ are dense in $\mathfrak{g}(1)$. If $x \in \mathfrak{g}(1)_{reg}$, the set $G(0) \cdot (\mathfrak{c}_x^{reg} + n)$ is dense in $\mathfrak{g}(1)$.*

Proof. Both Corollary 4.3.4 and Corollary 4.3.5 follow immediately from [Sp1, 4.3.6] combined with Proposition 4.3.2 and Corollary 4.3.3. □

4.3.1 Conjugacy of p-cyclic subspaces

We now possess all the machinery to prove that any two p-cyclic subspaces are $G(0)$-conjugate. Suppose $\mathfrak{c}_x, \mathfrak{c}_y$ are two p-cyclic subspaces corresponding to $x, y \in \mathfrak{g}(1)_{reg}$ respectively. As remarked above, up to replacing x and y with x' and y' respectively, we can assume $x \in \mathfrak{c}_x$ and $y \in \mathfrak{c}_y$.

Proposition 4.3.6. *Any two p-cyclic subspaces of $\mathfrak{g}(1)$ are $G(0)$-conjugate.*

Proof. Let $n \in \mathfrak{l}_2(x)(1)$ (resp. $m \in \mathfrak{l}_2(y)(1)$) whose $L(x)(0)$-orbit (resp. $L(y)(0)$-orbit) is open in $\mathfrak{l}_2(x)(1)$ (resp. $\mathfrak{l}_2(y)(1)$). The image of the morphism (4.8) is a constructible subset, so it contains an open dense subset of $\mathfrak{g}(1)$ by Corollary 4.3.5. This means that the intersection $G(0) \cdot (\mathfrak{c}_x^{reg} + n) \cap G(0) \cdot (\mathfrak{c}_y^{reg} + m)$ is nonempty, thus there exist $\bar{x} \in \mathfrak{c}_x^{reg}, \bar{y} \in \mathfrak{c}_y^{reg}$ and $g \in G(0)$ such that $g \cdot (\bar{x} + n) = \bar{y} + m$. As $\mathfrak{t}(x) = \mathfrak{t}(\bar{x})$ and $\mathfrak{t}(y) = \mathfrak{t}(\bar{y})$, we obtain $\mathfrak{l}_1(y) = \mathfrak{l}_1(\bar{y}) = \mathfrak{l}_1(g \cdot \bar{x}) = g \cdot \mathfrak{l}_1(\bar{x}) = g \cdot \mathfrak{l}_1(x)$. Moreover $g \in G(0)$ and therefore its adjoint action preserves graded components, so that $\mathfrak{c}_y = \mathfrak{l}_1(y)(1) = g \cdot \mathfrak{l}_1(x)(1) = g \cdot \mathfrak{c}_x$. □

Since any two p-cyclic subspaces coincide up to $G(0)$-conjugacy, we will often drop the subscript and simply use the notation $\mathfrak{c}$ instead of $\mathfrak{c}_x$.

4.4 Closed orbits

This part will be devoted to the description and characterization of elements of $\mathfrak{g}(1)$ whose $G(0)$-orbits are closed. For lack of a standard terminology, we will call these elements $G(0)$-*semisimple*. We will use notation as introduced in Notation 4.2.6. In this section x will denote a $G(0)$-semisimple element of $\mathfrak{g}(1)$, not necessarily regular. As the component of x on $\mathfrak{l}_2(x)$ is nilpotent, necessarily $x = x' \in \mathfrak{l}_1(x)$. In fact, x'' being nilpotent, its $L_2(x)(0)$-orbit closure contains 0, therefore the $G(0)$-orbit closure of $x = x' + x''$ contains x', and the equality $x = x'$ follows from $\overline{G(0)x} = G(0)x$.

Furthermore, $\mathfrak{l}_1(x) = \bigoplus_{i=1}^{d_x} \mathfrak{gl}_{r_i p}$ as in the comments after (4.5). Again, h'_i, x'_i will be the components of h and x on the subalgebra $\mathfrak{gl}_{r_i p}$. Let $\mathfrak{s}_i$ be the subalgebra of $\mathfrak{gl}_{r_i p}$ generated by h'_i and x'_i.

Lemma 4.4.1. *If x is $G(0)$-semisimple, the natural representation of the subalgebra $\mathfrak{s}_i$ on $k^{r_i p}$ is semisimple.*

Proof. We can restrict to those indices i for which $r_i > 1$. Let $L_1^i(x) \subseteq L(x)$ be the connected closed subgroup of G with Lie algebra $\mathfrak{gl}_{r_ip}$. The derived subgroup $L_1^i(x)'$ is isomorphic to $SL(r_ip)$ because G satisfies the standard hypothesis. Writing a composition series for the $\mathfrak{s}_i$-module k^{r_ip}, one can assume, by the proof of Lemma 4.2.3, that h_i' consists of a $r_ip \times r_ip$ block diagonal matrix of the form:

$$h_i' = \begin{pmatrix} H & 0 & \dots & 0 \\ 0 & H & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & H \end{pmatrix}$$

where H is the $p \times p$ matrix $\mathrm{diag}(p-1, p-2, \dots, 0)$.

On the other hand, x_i' is an upper block triangular matrix

$$x_i' = \begin{pmatrix} X_{11} & X_{12} & \dots & X_{1,r_i} \\ 0 & X_{22} & \ddots & \vdots \\ \vdots & & \ddots & X_{r_i-1,r_i} \\ 0 & \dots & 0 & X_{r_i,r_i} \end{pmatrix}$$

where each X_{ij} is a $p \times p$ matrix verifying $[H, X_{ij}] = X_{ij}$. In other words, $X_{ij} \in \mathfrak{gl}_{r_ip}(1)$. If e_{ij} denotes the matrix with 1 in the (i,j)-entry and zeroes elsewhere, then $\mathfrak{gl}_{r_ip}(l) = \mathrm{span}\langle e_{ij} \,|\, j - i \equiv l \mod p\rangle$, for every $l = 0, \dots, p-1$.

The diagonal matrices in this basis form a maximal toral subalgebra containing h_i'. Fix the base of simple roots $\{\alpha_1, \dots, \alpha_{r_ip-1}\}$ corresponding to the root subspaces $k \cdot e_{i,i+1}$, for $i = 1, \dots, r_ip-1$. The adjoint action of $L_1^i(x)$ on $\mathfrak{gl}_{r_ip}$ is induced by that of $GL(r_ip)$ ([PrSt], 2.1). Consider the 1-parameter subgroup $\mu : k^\times \to GL(r_ip)$ defined by:

$$\mu(t) = \begin{pmatrix} t^{r_i-1}\mathbf{1}_p & 0 & \dots & 0 \\ 0 & t^{r_i-2}\mathbf{1}_p & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{1}_p \end{pmatrix}$$

for $t \in k^\times$.

We can identify the adjoint action of $GL(r_ip)$ on $\mathfrak{gl}_{r_ip}$ with that of the group $PGL(r_ip) \simeq GL(r_ip)/Z(GL(r_ip))$; call μ' the image of μ under this quotient. Furthermore, $PGL(r_ip)$ is a homeomorphic image of $SL(r_ip) \simeq L_1^i(x)'$, namely $PGL(r_ip) \simeq$

$SL(r_ip)/Z(SL(r_ip))$. Consider the connected component of the preimage of μ' in $SL(r_ip)$. This is a 1-parameter subgroup, call it $\tilde{\mu}$, whose adjoint action on $\mathfrak{gl}_{r_ip}$ coincides with that of μ.

The action of $\tilde{\mu}(t)$ on x_i' gives

$$\tilde{\mu}(t) \cdot x_i' = \begin{pmatrix} X_{11} & tX_{12} & \dots & t^{r_i-1}X_{1r_i} \\ 0 & X_{22} & & \vdots \\ \vdots & & \ddots & tX_{r_i-1r_i} \\ 0 & \dots & 0 & X_{r_ir_i} \end{pmatrix}$$

for all $t \in k^\times$. Taking the limit for $t \to 0$, one obtains that the element

$$y_i = \begin{pmatrix} X_{11} & 0 & \dots & 0 \\ 0 & X_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & X_{r_ir_i} \end{pmatrix}$$

belongs to $\overline{G(0) \cdot x_i'}$. Since x is $G(0)$-semisimple, it is $G(0)$-conjugate to $y = y_1 + \ldots + y_{d_x}$, and this element satisfies the requirements. □

Lemma 4.4.2. *Let $x \in \mathfrak{g}(1)$ and let $y \in \overline{G(0) \cdot x}$ whose $G(0)$-orbit is closed. Then the toral subalgebras $\mathfrak{t}(x)$ and $\mathfrak{t}(y)$ are $G(0)$-conjugate.*

Proof. The set $G(0) \cdot x$ is irreducible, therefore its closure in $\mathfrak{g}(1)$ is irreducible as well. Consider the $G(0)$-equivariant morphism:

$$\begin{aligned} \mathfrak{g}(1) &\longrightarrow \mathfrak{g}(0) \\ x &\longmapsto x^{[p]}. \end{aligned} \tag{4.12}$$

The image of the orbit $G(0) \cdot x$ under the map (2.1) is an irreducible $G(0)$-invariant subset, namely the orbit $G(0) \cdot x^{[p]}$. Consider $V = \overline{(G(0) \cdot x)^{[p]}} = \overline{G(0) \cdot x^{[p]}}$. We have that $x^{[p]} \in V$ and V is $G(0)$-invariant. The element x belongs to the preimage of V under the map (4.12), and then so does $y \in \overline{G(0) \cdot x}$. Therefore $G(0) \cdot y^{[p]} \subseteq \overline{G(0) \cdot x^{[p]}}$. Applying analogous considerations to higher p^i-th powers, we obtain $G(0) \cdot y^{[p]^i} \subseteq \overline{G(0) \cdot x^{[p]^i}}$ for all $i \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be big enough so that $x^{[p]^n}$ is a semisimple element of $\mathfrak{g}(0)$. Then $y^{[p]^n} \in \overline{G(0) \cdot x^{[p]^n}}$, and since the $G(0)$-orbit of $x^{[p]^n}$ is closed, there exists $g \in G(0)$ with $y^{[p]^n} = g \cdot x^{[p]^n}$. The analogue holds true for every higher power of x and y, so that $\mathfrak{t}(y) = g \cdot \mathfrak{t}(x)$. □

We provide also another proof of Lemma 4.4.2.

Proof. The reductive group $G(0)$ acts on its Lie algebra $\mathfrak{g}(0)$. The invariant subring $k[\mathfrak{g}(0)]^{G(0)}$ is a polynomial ring $k[\mathfrak{g}(0)]^{G(0)} = k[\varphi_1, \dots, \varphi_q]$ where the φ_i's are algebraically independent homogeneous generators. The morphism (4.12) is a $G(0)$-equivariant map. Consider the image $\mathfrak{g}(1)^{[p]}$ of this morphism, this is a $G(0)$-invariant subset of $\mathfrak{g}(0)$. Take $x \in \mathfrak{g}(1)$ and $y \in \overline{G(0) \cdot x}$ whose $G(0)$-orbit is closed.

Let $i \in \mathbb{N}$ be big enough so that both $x^{[p]^i}$ and $y^{[p]^i}$ are semisimple. The $G(0)$-orbit of $x^{[p]^i}$ is a closed subset of $\mathfrak{g}(0)$ and it is uniquely determined by the image of $x^{[p]^i}$ under the quotient morphism:

$$\begin{array}{rcl} \mathfrak{g}(0) & \longrightarrow & \mathfrak{g}(0)/\!\!/G(0) \simeq \mathbb{A}^q \\ z & \longmapsto & (\varphi_1(z), \dots, \varphi_q(z)). \end{array} \tag{4.13}$$

Consider the composite:

$$\begin{array}{rcl} \widetilde{\Psi} : \mathfrak{g}(1) \longrightarrow \mathfrak{g}(0) & \longrightarrow & \mathfrak{g}(0)/\!\!/G(0) \\ z \longmapsto z^{[p]^i} & \longmapsto & \left(\varphi_1(z^{[p]^i}), \dots, \varphi_q(z^{[p]^i})\right). \end{array}$$

Assume $\widetilde{\Psi}(x) = (a_1, \dots, a_q)$. The preimage of this point under $\widetilde{\Psi}$ is a closed $G(0)$-invariant subset of $\mathfrak{g}(1)$ which contains x, so in particular it contains y as well. It follows that $x^{[p]^i}$ and $y^{[p]^i}$ have the same image under the quotient map (4.13). Since they are semisimple their orbits are closed and therefore $x^{[p]^i}$ and $y^{[p]^i}$ are conjugate via an element $g \in G(0)$. As in the previous proof, $\mathfrak{t}(y) = g \cdot \mathfrak{t}(x)$. □

Remark 4.4.3. Let $x \in \mathfrak{g}(1)$ and $y \in \overline{G(0) \cdot x}$ whose orbit is closed. There exists $g \in G(0)$ such that $\mathfrak{t}(y) = g \cdot \mathfrak{t}(x)$. In particular, the centralizers $\mathfrak{l}(y) = c_{\mathfrak{g}}(\mathfrak{t}(y))$ and $\mathfrak{l}(x) = c_{\mathfrak{g}}(\mathfrak{t}(x))$ are $G(0)$-conjugate, and then isomorphic. Up to conjugation we can then assume that $\mathfrak{l}(y) = \mathfrak{l}(x)$ and both x, y belong to $\mathfrak{l}(x)$. This Levi subalgebra decomposes as $\mathfrak{l}(x) = \mathfrak{l}_1(x) \oplus \mathfrak{l}_2(x)$, and the component of y on $\mathfrak{l}_2(x)$ is $G(0)$-unstable. The fact that the orbit of y is closed means that y is only supported in $\mathfrak{l}_1(x)$.

Remark 4.4.4. From Lemma 4.4.2 it follows that there exist $G(0)$-semisimple regular elements.

The next result shows that a p-cyclic subspace contains a representative of every closed $G(0)$-orbit in $\mathfrak{g}(1)$.

Proposition 4.4.5. *Every $G(0)$-semisimple element belongs to a p-cyclic subspace.*

Proof. Let $x \in \mathfrak{g}(1)$ be $G(0)$-semisimple. If x is regular the statement holds trivially by considering $\mathfrak{c}_x \ni x$, thus we can assume $x \notin \mathfrak{g}(1)_{reg}$. Consider the Levi subalgebra $\mathfrak{l}(x)$, and more specifically its graded component $\mathfrak{l}(x)(1) = \mathfrak{l}_1(x)(1) \oplus \mathfrak{l}_2(x)(1)$. The component x'' of x on $\mathfrak{l}_2(x)(1)$ is nilpotent, so that $x \in \mathfrak{l}_1(x)$. Suppose that every element in $\mathfrak{l}_2(x)(1)$ is nilpotent. By Corollary 4.3.5 the set $G(0) \cdot (\mathfrak{l}_1(x)(1)+n)$ is dense in $\mathfrak{g}(1)$, where n is an element belonging to the dense open orbit $\mathcal{O}(x) \subseteq \mathfrak{l}_2(x)(1)$. As $G(0) \cdot (\mathfrak{l}_1(x)(1)+n)$ is constructible, it contains a nonempty open subset of $\mathfrak{g}(1)$. Chosen $y \in \mathfrak{g}(1)_{reg}$ and $m \in \mathcal{O}(y)$, the open $L(y)(0)$-orbit of $\mathfrak{l}_2(y)(1)$, again by Corollary 4.3.5 the intersection of $G(0) \cdot (\mathfrak{l}_1(x)(1) + n)$ and $G(0) \cdot (\mathfrak{c}_y^{reg} + m)$ is nonempty. Thus there exist $u \in \mathfrak{l}_1(x)(1), v \in \mathfrak{c}_y^{reg}$ and $g \in G(0)$ such that $g(u + n) = v + m$. But then $\dim \mathfrak{t}(u) = \dim \mathfrak{t}(v)$, so that $u \in \mathfrak{l}_1(x)$ is regular. Therefore, one of the following two conditions must hold for x: either $\mathfrak{l}_1(x) \cap \mathfrak{g}(1)_{reg} \neq \emptyset$, or $\mathfrak{l}_2(x)(1)$ contains nonzero elements which are not nilpotent.

As usual, let h' (resp. h'') be the component of h on $\mathfrak{l}_1(x)$ (resp. $\mathfrak{l}_2(x)$). We can apply the results of the previous sections to the reductive group $L_1(x)$ and its Lie algebra $\mathfrak{l}_1(x)$, graded by the toral element h'. It is easier to work with this Lie algebra since $\mathfrak{l}_1(x) \simeq \oplus_{i=1}^{d_x} \mathfrak{gl}_{r_i p}$. Thanks to Lemma 4.4.1, the projection x_i' of the element x on $\mathfrak{gl}_{r_i p}$ can be expressed as a block diagonal matrix in a suitable basis, with each block of size p. By Proposition 4.4.1 and Lemma 4.2.3 we can therefore choose $z \in \mathfrak{l}_1(x)(1)$ such that the p-cyclic subspace $\mathfrak{c}_z'$ of $\mathfrak{l}_1(x)$ is the direct sum of $(r_1 + \ldots + r_{d_x})$-copies of subalgebras isomorphic to $\mathfrak{gl}_p$ and $x \in \mathfrak{c}_z'$. If we are in the case $\mathfrak{l}_1(x) \cap \mathfrak{g}(1)_{reg} \neq \emptyset$ then we are done since $\mathfrak{c}_z'$ is a p-cyclic subspace of $\mathfrak{g}(1)$.

If this is not the case, then $\mathfrak{l}_2(x)(1) \nsubseteq \mathcal{N}(\mathfrak{g}(1))$ by the arguments above. Let us restrict our attention to the group $L_2(x)$ and its Lie algebra $\mathfrak{l}_2(x)$, which is graded by the toral element h''. Let $w \in \mathfrak{l}_2(x)(1)$ be a regular element for this grading; thanks to Remark 4.4.4 we can assume w to be $L_2(x)(0)$-semisimple. Consider $\mathfrak{t}(w)$

and its centralizer $C_{\mathfrak{l}_2(x)}(\mathfrak{t}(w))$. In analogy to (4.5), it decomposes as $C_{\mathfrak{l}_2(x)}(\mathfrak{t}(w)) = \mathfrak{l}_1(w)' \oplus \mathfrak{l}_2(w)'$, where $\mathfrak{l}_1(w)'$ is isomorphic to the direct sum of some copies of $\mathfrak{gl}_p$, while $\mathfrak{l}_2(w)'$ is a sum of ideals whose component of degree 1 consists entirely of nilpotent elements and admits an open dense $L_2(x)(0)$-orbit. As $[z, w] = 0$, $\mathfrak{t}(z) + \mathfrak{t}(w)$ is a toral subalgebra. Up to replacing w with a generic enough regular element of $\mathfrak{l}_2(x)(1)$, we can assume $\mathfrak{t}(z) + \mathfrak{t}(w) = \mathfrak{t}(z + w)$. But then $\mathfrak{l}_1(z + w)$ is a direct sum of subalgebras isomorphic to $\mathfrak{gl}_p$, $\mathfrak{l}_2(z+w) = \mathfrak{l}_2(w)'$ and $\mathfrak{l}_2(w)'(1)$ contains only nilpotent elements. It follows that $z + w \in \mathfrak{g}(1)_{reg}$ and $x \in \mathfrak{c}_{z+w}$ since $x \in \mathfrak{l}_1(z)(1)$, the component of degree 1 of the direct sum of $(r_1 + \ldots + r_{d_x})$-copies of subalgebras isomorphic to $\mathfrak{gl}_p$. □

We are now able to give a characterization of $G(0)$-semisimple elements. Assume again $x \in \mathfrak{g}(1)_{reg}$, and let $\mathfrak{c}_x$ be the corresponding p-cyclic subspace associated to it. Owing to the direct sum decomposition $\mathfrak{l}_1(x) = \bigoplus_{i=1}^{s} \mathfrak{gl}_p$, we can write $z = \sum_{i=1}^{s}(\lambda_0^i e_{\alpha_0}^i + \ldots + \lambda_{p-1}^i e_{\alpha_{p-1}}^i)$ as in Notation 4.2.6 for some scalars $\lambda_j^i \in k$.

Lemma 4.4.6. *Let $z \in \mathfrak{c}_x$, with $z = \sum_{i=1}^{s}(\lambda_0^i e_{\alpha_0}^i + \ldots + \lambda_{p-1}^i e_{\alpha_{p-1}}^i)$. Then z is $G(0)$-semisimple iff for every $i = 1, \ldots, s$ the following condition holds:*

$$\prod_{j=0}^{p-1} \lambda_j^i = 0 \quad \Longrightarrow \quad \lambda_j^i = 0 \quad \forall j = 0, \ldots, p-1. \tag{4.14}$$

Proof. The fact that (4.14) is necessary follows from Lemma 4.2.7, so we will verify that if z satisfies (4.14) then it is $G(0)$-semisimple.

Hence, suppose that (4.14) holds for z and let $y \in \overline{G(0) \cdot z}$ be a $G(0)$-semisimple element; y can be assumed to belong to $\mathfrak{c}_x$. Both $z^{[p]}, y^{[p]} \in \mathfrak{t}(x)$, more specifically $z^{[p]} = \sum_{i=1}^{s}(\prod_{j=0}^{p-1} \lambda_j^i)\mathbf{1}_i$.

Since both $z^{[p]}$ and $y^{[p]}$ are semisimple, by the proof of Lemma 4.4.2 they are $G(0)$-conjugate, so there exists $g \in G(0)$ with $z^{[p]} = g \cdot y^{[p]}$. Now $\mathfrak{t}(g \cdot y) = \mathfrak{t}(z)$. As $g \cdot y$ is $G(0)$-semisimple and belongs to $\mathfrak{l}_1(z)$, Proposition 4.3.6 and Proposition 4.4.5 combined imply that there exists $g' \in L(z)(0)$ such that $g'g \cdot y \in \mathfrak{c}_x$. Assume $g'g \cdot y = \sum_{i=1}^{s}(\mu_0^i e_{\alpha_0}^i + \ldots + \mu_{p-1}^i e_{\alpha_{p-1}}^i)$ for suitable scalars $\mu_j^i \in k^\times$. As $g'g \cdot y$ is $G(0)$-semisimple, the condition $\prod_{j=0}^{p-1} \mu_j^i = 0 \Rightarrow \mu_j^i = 0$ for all $j = 1, \ldots, p-1$ holds for every i. Since $g'g \cdot y^{[p]} = z^{[p]}$ we have $\prod_{j=0}^{p-1} \lambda_j^i = \prod_{j=0}^{p-1} \mu_j^i$ for every $i = 1, \ldots, s$. But then, by Lemma 4.2.8, the elements z and $g'g \cdot y$ are $G(0)$-conjugate, and so z is $G(0)$-semisimple. □

Corollary 4.4.7. *If $z \in \mathfrak{c}_x^{reg}$, the orbit $G(0) \cdot z$ is closed.* □

We introduce another piece of notation: $N_{G(0)}(\mathfrak{c})$ will indicate the normalizer of $\mathfrak{c} = \mathfrak{c}_x$ in $G(0)$, while $C_{G(0)}(\mathfrak{c})$ will stand for its centralizer.

Lemma 4.4.8. *Two $G(0)$-semisimple elements of $\mathfrak{c}$ are $G(0)$-conjugate iff they are $N_{G(0)}(\mathfrak{c})$-conjugate.*

Proof. Let $y, y' \in \mathfrak{c}$ be $G(0)$-semisimple conjugate elements, so that $y' = g \cdot y$ for a certain $g \in G(0)$. Restrict to the Lie subalgebra $\mathfrak{l}(y')$, which is the Lie algebra of a Levi subgroup $L(y')$. Then $\mathfrak{c}, g \cdot \mathfrak{c}$ are two p-cyclic subspaces of $\mathfrak{l}(y')$. By Proposition 4.3.6 there exists $g' \in L(y')(0)$ for which $\mathfrak{c} = g'g \cdot \mathfrak{c}$. But then $g'g \in N_{G(0)}(\mathfrak{c})$ and $g'g \cdot y^{[p]} = y'^{[p]}$. Say $\mathfrak{c} = \mathfrak{c}_x$ for a certain $x \in \mathfrak{g}(1)_{reg}$. By Lemma 4.4.6 and Remark 4.2.9, there exists an element $g'' \in L(x)(0) \subseteq N_{G(0)}(\mathfrak{c})$ such that $g''g'g \cdot y = y'$. □

Lemma 4.4.9. *An element $y \in \mathfrak{c} = \mathfrak{c}_x$ is $G(0)$-semisimple iff the orbit $N_{G(0)}(\mathfrak{c}) \cdot y \subseteq \mathfrak{c}$ is closed.*

Proof. If $y \in \mathfrak{c}$ is $G(0)$-semisimple, Lemma 4.4.8 shows that $N_{G(0)}(\mathfrak{c}) \cdot y = (G(0) \cdot y) \cap \mathfrak{c}$, a closed subset of $\mathfrak{c}$.

Conversely, assume $N_{G(0)}(\mathfrak{c}) \cdot y = \overline{N_{G(0)}(\mathfrak{c}) \cdot y}$ and suppose by contradiction that y is not $G(0)$-semisimple. Write $y = \sum_{i=1}^{s} y_i' \in \oplus_{i=1}^{s} \mathfrak{gl}_p^i(1)$ and $y^{[p]} = \sum_{i=1}^{s} \lambda_i \mathbf{1}_i$ for some scalars $\lambda_1, \ldots, \lambda_s \in k$. The characterization given in Lemma 4.4.6 implies that there exists an index $1 \leq j \leq s$ such that $\lambda_j = 0$ and $y_j' \neq 0$. Since ${y_j'}^{[p]} = 0$, the component y_j' is a nilpotent element of $\mathfrak{gl}_p^j$. Restrict to the group $L_1^j(x)(0) = (L_1^j(x) \cap G(0))^\circ$. By Lemma 4.1.3 there exists a cocharacter $\mu \in Y(L_1^j(x)(0))$ with $\lim_{t \to 0} \mathrm{ad}\mu(t)(y_j') = 0$. The subgroup $L_1^j(x)(0)$ is a maximal torus of $L_1^j(x)$ (the component h_j' of h on $\mathfrak{gl}_p^j$ is regular) and it acts trivially on the other summands $\mathfrak{gl}_p^i$ for $i \neq j$. Therefore $\mu(t) \in N_{G(0)}(\mathfrak{c})$ for all $t \in k^\times$, and since by assumption the $N_{G(0)}(\mathfrak{c})$-orbit of y is closed it must be that $y_j' = 0$. Thus y is $G(0)$-semisimple because it verifies the condition of Lemma 4.4.6. □

Chapter 5

Modular invariants: the little Weyl group and the invariant ring

5.1 An isomorphism of rings of invariants

In this section we will adapt some results contained in [Le1] to our setting. Fix a p-cyclic subspace $\mathfrak{c} = \mathfrak{c}_x$. The group $N_{G(0)}(\mathfrak{c})$ acts on the p-cyclic subspace and yields an action of the *little Weyl group* $W_\mathfrak{c} = N_{G(0)}(\mathfrak{c})/C_{G(0)}(\mathfrak{c})$ on $\mathfrak{c}$. Here is a summary of the main features obtained so far:

(i) Any two p-cyclic subspaces are $G(0)$-conjugate (Proposition 4.3.6).

(ii) Every closed $G(0)$-orbit meets $\mathfrak{c}$ (Proposition 4.4.5).

(iii) Two $G(0)$-semisimple elements of $\mathfrak{c}$ are $G(0)$-conjugate if and only if they are $N_{G(0)}(\mathfrak{c})$-conjugate (Lemma 4.4.8).

(iv) For any $y \in \mathfrak{c}$, the orbit $N_{G(0)}(\mathfrak{c}) \cdot y$ is closed if and only if the orbit $G(0) \cdot y$ is closed (Lemma 4.4.9).

We will now work with the quotient variety $\mathfrak{c}/\!\!/W_\mathfrak{c}$. A priori, it is by no means obvious that it even makes sense to consider this object. Yet, in Section 5.2 we will show that $W_\mathfrak{c}$ is always a reductive group, so as to insure the existence of the quotient variety $\mathfrak{c}/\!\!/W_\mathfrak{c}$. The results in Section 5.2 are independent of those contained in this section, still we decided to leave the discussion on the structure of the group $W_\mathfrak{c}$ for a subsequent section since it becomes immediately very technical. For now we will just claim that the quotient variety $\mathfrak{c}/\!\!/W_{\mathfrak{c}}$ is well defined; the reasons for this fact will emerge later in this chapter.

Lemma 5.1.1. *The inclusion* $j : \mathfrak{c} \hookrightarrow \mathfrak{g}(1)$ *induces a bijective morphism of varieties* $j' : \mathfrak{c}/\!\!/W_{\mathfrak{c}} \to \mathfrak{g}(1)/\!\!/G(0)$.

Proof. Take the composite $\mathfrak{c} \hookrightarrow \mathfrak{g}(1) \to \mathfrak{g}(1)/\!\!/G(0)$. Since it is constant on $N_{G(0)}(\mathfrak{c})$-orbits, there exists a morphism j' that fits into the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{c} & \xrightarrow{\ j\ } & \mathfrak{g}(1) \\ \downarrow & & \downarrow \\ \mathfrak{c}/\!\!/W_{\mathfrak{c}} & \xrightarrow{\ j'\ } & \mathfrak{g}(1)/\!\!/G(0). \end{array} \qquad (\star)$$

Any closed $G(0)$-orbit meets $\mathfrak{c}$ (property (ii) above). By Lemma 4.4.8, two $G(0)$-semisimple elements of $\mathfrak{c}$ are $G(0)$-conjugate iff they are $N_{G(0)}(\mathfrak{c})$-conjugate. In particular, $N_{G(0)}(\mathfrak{c})$-orbits on $\mathfrak{c}$ of $G(0)$-semisimple elements are closed, hence j' is surjective. Injectivity of j', on the other hand, is a consequence of property (iii) above combined with property (iv) (Lemma 4.4.9). It follows that the morphism j' is bijective. □

It is worth stressing that a bijective morphism of varieties *need not* be an isomorphism of varieties; we will prove shortly that this is actually the case for the morphism j' (Theorem 5.1.4) but this will require some further work.

Thanks to Lemma 5.1.1 and [Hu1, Theorem 4.6], the field $\mathrm{Frac}(k[\mathfrak{c}]^{W_{\mathfrak{c}}})$ is a finite, purely inseparable extension of $(j')^*\mathrm{Frac}(k[\mathfrak{g}(1)]^{G(0)})$.

The next results follow very closely [Le1] and [Ri]. The following Lemma is adapted from [Le1, Lemma 2.17] and represents a crucial step towards proving separability of the quotient morphism $\mathfrak{g}(1) \to \mathfrak{g}(1)/\!\!/G(0)$.

Lemma 5.1.2. $k(\mathfrak{g}(1))^{G(0)} = \mathrm{Frac}(k[\mathfrak{g}(1)]^{G(0)})$.

Proof. Consider $\mathfrak{c} = \mathfrak{c}_x$, so that $\mathfrak{l}(x)(1) = \mathfrak{c} + \mathfrak{l}_2(x)(1)$. Let $R(\mathfrak{c}) = \mathfrak{c}^{reg} + \mathfrak{l}_2(x)(1)$ denote the open dense subset of regular elements of $\mathfrak{l}(x)(1)$. Recall the Levi subgroup $L(x) \subseteq G$ with Lie algebra $\mathfrak{l}(x)$. Consider an element $f \in k(\mathfrak{g}(1))^{G(0)}$. Its domain $\mathrm{dom}f$ is an open subset of $\mathfrak{g}(1)$, thus it intersects $\mathfrak{g}(1)_{reg}$ nontrivially by Lemma 4.2.2. Due to $G(0)$-invariance, $\mathrm{dom}f$ intersects nontrivially with $R(\mathfrak{c})$; this also follows from Corollary 4.3.5 by noticing that $G(0) \cdot R(\mathfrak{c})$ contains a dense open subset of $\mathfrak{g}(1)$. Thus f restricts to a rational invariant function $f|_{\mathfrak{l}(x)(1)} \in k(\mathfrak{l}(x)(1))^{L(x)(0)}$. Pick $y+n \in R(\mathfrak{c})$, where $y \in \mathfrak{c}^{reg}$ and $n \in \mathfrak{l}_2(x)(1)$. Using arguments almost identical to those in [Le1], we prove that if the intersection $y+\mathfrak{l}_2(x)(1) \cap \mathrm{dom} f$ is nonempty then $y+\mathfrak{l}_2(x)(1) \subseteq \mathrm{dom} f$. Assume indeed $y+n \in \mathrm{dom} f$. The ring $k[\mathfrak{c} \oplus \mathfrak{l}_2(x)(1)]$ is a unique factorization domain and so the restriction of f can be written as g_2/g_1, for $g_1, g_2 \in k[\mathfrak{c} \oplus \mathfrak{l}_2(x)(1)]$ coprime polynomials. By assumption g_1 is not identically 0 on $y + \mathfrak{l}_2(x)(1)$. In particular, its nonzero locus intersects $y + \mathcal{O}(x)$, where $\mathcal{O}(x)$ is the open $L_2(x)(0)$-orbit on $\mathfrak{l}_2(x)(1)$ (observe moreover that $y + \mathcal{O}(x)$ is dense in $y + \mathfrak{l}_2(x)(1)$). Then the domain of f intersects $y + \mathcal{O}(x)$ nontrivially, and by invariance f is constant on this subset, and equal to a certain $c \in k$. As a consequence, $y + \mathcal{O}(x) \subseteq \mathrm{dom} f$.

It follows that f restricts to a rational function $f^y = f|_{y+\mathfrak{l}_2(x)(1)} \in k(y + \mathfrak{l}_2(x)(1))$. Again, $k[y + \mathfrak{l}_2(x)(1)]$ being a unique factorization domain we can write $f^y = f_2/f_1$ for $f_1, f_2 \in k[y + \mathfrak{l}_2(x)(1)]$ coprime polynomials. Let $D(f_i) = \{z \in y + \mathfrak{l}_2(x)(1) \mid f_i(z) \neq 0\}$ for $i = 1, 2$. The set $D(f_1)$ is an open subset of $y + \mathfrak{l}_2(x)(1)$ contained in the domain of f. Since f^y is constant on $y + \mathcal{O}(x)$, it is constant on $D(f_1)$ as well, and so $(f_2/f_1)|_{D(f_1)} = c \in k$. If $c = 0$ then $f_2 = 0 \in k[y + \mathfrak{l}_2(x)(1)]$ and so $f^y = 0$ and it is defined at every point of $y + \mathfrak{l}_2(x)(1)$. If $c \in k^\times$ then $D(f_1) \subseteq D(f_2)$, thus there exist $n \in \mathbb{N}$ and $q \in k[y + \mathfrak{l}_2(x)(1)]$ with $(f_1)^n = qf_2$. Since f_1, f_2 were assumed to be coprime, this gives a contradiction unless f_1, f_2 are both constant, so that again f is defined (and constant) on the whole of $y + \mathfrak{l}_2(x)(1)$.

Let us call $X = \mathfrak{g}(1) \setminus \mathfrak{g}(1)_{reg}$ and $Y = X \cup (\mathfrak{g}(1) \setminus \mathrm{dom} f)$; by Lemma 4.2.2 they are $G(0)$-invariant Zariski closed subsets of $\mathfrak{g}(1)$. Notice that $\mathfrak{g}(1) \setminus Y = \mathfrak{g}(1)_{reg} \cap \mathrm{dom} f$. Let $v \in \mathfrak{g}(1) \setminus Y$, so v can be written as $v = z + m$, where $z \in \mathfrak{l}_1(v)(1)$ (a p-cyclic subspace) and is regular, while $m \in \mathfrak{l}_2(v)(1)$ is nilpotent. Up to replacing v, z and m with $G(0)$-conjugates, we can assume $\mathfrak{l}_1(v)(1) = \mathfrak{c}$. By the above and given that $v \in \mathrm{dom} f$, f is defined on the whole of $z + \mathfrak{l}_2(v)(1)$ since it is defined at $z+m$. Let $U = G(0) \cdot (z + \mathfrak{l}_2(v)(1))$. Thanks to Corollary 4.4.7, $U = \pi^{-1}(\pi(z))$, thus $U \subseteq \mathfrak{g}(1)$ is closed and $G(0)$-stable. Moreover, f is defined at each point of U and so $U \cap Y = \emptyset$. In particular, there exists $h_1 \in k[\mathfrak{g}(1)]^{G(0)}$ such that $h_1(u) = 1 \quad \forall u \in U$ and $h_1(y) = 0 \quad \forall y \in Y$. It follows that $\mathrm{dom} f$ contains $D(h_1) = \{w \in \mathfrak{g}(1) \mid h_1(w) \neq 0\}$. As a consequence, $f = h_2/h_1^l$ for some $l \geq 0$ and for a certain $h_2 \in k[\mathfrak{g}(1)]$. As both f and h_1 are $G(0)$-invariant, h_2 is as well, so that $f \in \mathrm{Frac}(k[\mathfrak{g}(1)]^{G(0)})$. □

Corollary 5.1.3. *The quotient morphism $\mathfrak{g}(1) \to \mathfrak{g}(1) /\!\!/ G(0)$ is separable.*

Proof. The fraction field $\mathrm{Frac}(k[\mathfrak{g}(1)]^{G(0)})$ is the field of rational functions of the irreducible variety $\mathfrak{g}(1)/\!\!/G(0)$. By [Bo, AG 2.4] together with Lemma 5.1.2, one sees that $\mathrm{Frac}(k[\mathfrak{g}(1)]^{G(0)}) = k(\mathfrak{g}(1))^{G(0)} \subseteq k(\mathfrak{g}(1))$ is a separable extension. Therefore the quotient morphism is separable. □

Corollary 5.1.3 allows us to adapt an argument of Richardson ([Ri, proof of 11.3]) to prove that the morphism j' in $(\star)$ is an isomorphism of varieties.

Theorem 5.1.4. *The embedding $j : \mathfrak{c} \hookrightarrow \mathfrak{g}(1)$ induces an isomorphism of varieties $j' : \mathfrak{c}/\!\!/W_{\mathfrak{c}} \to \mathfrak{g}(1)/\!\!/G(0)$.*

Proof. As both $\mathfrak{c} = \mathfrak{c}_x$ and $\mathfrak{g}(1)$ are normal varieties, the quotients $\mathfrak{c}/\!\!/W_{\mathfrak{c}}$ and $\mathfrak{g}(1)/\!\!/G(0)$ are normal as well (Section 2.4). The goal is to show that j' is a separable morphism, so that one can apply Zariski's Main Theorem (see for example [Di, Corollaire 2, pag 137]) to prove that j' is an isomorphism of varieties. By Corollary 4.3.4, the morphism $\Psi_2 : G(0) \times \mathfrak{l}(x)(1) \to \mathfrak{g}(1)$ given by $(g, y) \mapsto \mathrm{Ad}g(y)$ is separable. Hence we can apply the argument in the proof of [Ri, 11.3], obtaining that the morphism $\psi : \mathfrak{l}(x)(1) \to \mathfrak{g}(1)/\!\!/G(0)$, induced by the inclusion $\mathfrak{l}(x)(1) \subseteq \mathfrak{g}(1)$, is separable. Observe that the subspace $\mathfrak{l}(x)(1)$ is $N_{G(0)}(\mathfrak{c})$-invariant; indeed $N_{G(0)}(\mathfrak{c})$ stabilizes $\mathfrak{c}$ and $\mathfrak{l}(x)(1) = C_{\mathfrak{g}(1)}(\mathfrak{c}^{[p]})$. As a consequence, the morphism ψ factors through

$$\begin{array}{ccc} \mathfrak{l}(x)(1) & \xrightarrow{\ \psi\ } & \mathfrak{g}(1)/\!\!/G(0) \\ \downarrow & \nearrow & \\ \mathfrak{l}(x)(1)/\!\!/N_{G(0)}(\mathfrak{c}) & & \end{array}$$

It follows that the morphism $\mathfrak{l}(x)(1)/\!\!/N_{G(0)}(\mathfrak{c}) \to \mathfrak{g}(1)/\!\!/G(0)$ is separable.

Recall that $\mathfrak{l}(x)(1) = \mathfrak{c} \oplus \mathfrak{l}_2(x)(1)$, where the second summand consists only of nilpotent elements. The group $L_2(x)(0)$ has an open orbit on $\mathfrak{l}_2(x)(1)$ and stabilizes $\mathfrak{c}$, so that $L_2(x)(0) \subseteq N_{G(0)}(\mathfrak{c})$. Therefore we obtain the following isomorphisms of rings:

$$k[\mathfrak{l}(x)(1)]^{N_{G(0)}(\mathfrak{c})} = k[\mathfrak{c} \oplus \mathfrak{l}_2(x)(1)]^{N_{G(0)}(\mathfrak{c})} \simeq k[\mathfrak{c}]^{N_{G(0)}(\mathfrak{c})} \simeq k[\mathfrak{c}]^{W_{\mathfrak{c}}}. \tag{5.1}$$

As a result, there exists an isomorphism of varieties $\mathfrak{c}/\!\!/W_{\mathfrak{c}} \simeq \mathfrak{l}(x)(1)/\!\!/N_{G(0)}(\mathfrak{c})$ induced by the inclusion $\mathfrak{c} \hookrightarrow \mathfrak{l}(x)(1)$. Thus, the morphism $j' : \mathfrak{c}/\!\!/W_{\mathfrak{c}} \to \mathfrak{g}(1)/\!\!/G(0)$ is separable, and it is an isomorphism of varieties. □

Corollary 5.1.5. *The ring $k[\mathfrak{g}(1)]^{G(0)}$ is isomorphic to $k[\mathfrak{c}]^{W_{\mathfrak{c}}}$.* □

5.2 The little Weyl group

In both the classical case and the graded cases studied in [Vi] and [Le1], the Weyl group and the little Weyl group always turn out to be finite groups generated by pseudoreflections. This is not the case in our setting, as the little Weyl group $W_{\mathfrak{c}}$ need not be finite. Still, it is the semidirect product of a (normal) torus and a finite group.

5.2.1 Preliminary discussion

Assume $x \in \mathfrak{g}(1)_{reg}$ is a $G(0)$-semisimple element and let $\mathfrak{c} = \mathfrak{c}_x$ be the corresponding p-cyclic subspace, so that we can write $\mathfrak{l}(x)(1) = \mathfrak{c} \oplus \mathfrak{l}_2(x)(1)$. Since $\mathfrak{t}(x) = \mathfrak{c}^{[p]}$ and using that the p-th power map is a G-invariant morphism, the toral subalgebra $\mathfrak{t}(x)$ is $N_{G(0)}(\mathfrak{c})$-stable, so that $N_{G(0)}(\mathfrak{c}) \subseteq N_{G(0)}(\mathfrak{t}(x))$.

Moreover $N_{G(0)}(\mathfrak{t}(x)) \subseteq N_{G(0)}(c_{\mathfrak{g}}(\mathfrak{t}(x))) \subseteq N_{G(0)}(\mathfrak{l}(x))$. Looking at the structure of $\mathfrak{l}(x)$ clearly $N_{G(0)}(\mathfrak{l}(x)) \subseteq N_{G(0)}(\mathfrak{l}_1(x))$. As the action of $G(0)$ preserves graded components we have the inclusion $N_{G(0)}(\mathfrak{t}(x)) \subseteq N_{G(0)}(\mathfrak{l}_1(x)(1)) = N_{G(0)}(\mathfrak{c})$, but then equality holds:

$$N_{G(0)}(\mathfrak{c}) = N_{G(0)}(\mathfrak{t}(x)). \tag{5.2}$$

Let us now look at the centralizer $C_{G(0)}(\mathfrak{c})$. Using the same argument as above $C_{G(0)}(\mathfrak{c}) \subseteq C_{G(0)}(\mathfrak{t}(x))$. In particular we get a surjective morphism of groups:

$$\phi : N_{G(0)}(\mathfrak{c})/C_{G(0)}(\mathfrak{c}) \twoheadrightarrow N_{G(0)}(\mathfrak{t}(x))/C_{G(0)}(\mathfrak{t}(x)). \tag{5.3}$$

Define $W_{\mathfrak{t}} \coloneqq N_{G(0)}(\mathfrak{t}(x))/C_{G(0)}(\mathfrak{t}(x))$, this is a finite group since it is the Weyl group of the toral subalgebra $\mathfrak{t}(x)$.

We now analyze the kernel of the map ϕ in (5.3). Since $G(0)$ satisfies the standard hypothesis, $C_{G(0)}(\mathfrak{t}(x))^{\circ} = (L(x) \cap G(0))^{\circ}$; moreover, $L_2(x) \cap G(0) \subseteq C_{G(0)}(\mathfrak{c})$ because $L_2(x)$ acts trivially on $\mathfrak{c}$. Now consider $(L_1(x) \cap C_{G(0)}(\mathfrak{c}))^{\circ}$, and call this group $\widetilde{T}(x)$. The Lie algebra $\widetilde{\mathfrak{t}}(x)$ of $\widetilde{T}(x)$ is included in $\mathfrak{l}_1(x) = \oplus_{i=1}^{s} \mathfrak{gl}_p$. Looking at a component $\mathfrak{l}_1^i(x) \simeq \mathfrak{gl}_p$, an element in the projection of $\widetilde{\mathfrak{t}}(x)$ on $\mathfrak{gl}_p$ is a diagonal matrix commuting with all $p \times p$ matrices in the p-cyclic subspace. Assuming as usual that the projection of h is $h'_i = \mathrm{diag}(p-1,\dots,1,0)$, matrices in $\mathfrak{c}$ are of the form:

$$\begin{pmatrix} 0 & a_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & 0 \\ 0 & & & \ddots & a_{p-1} \\ a_p & 0 & \dots & & 0 \end{pmatrix}$$

with $a_i \in k$ for all $i = 1,\dots,p$. As a consequence, $\widetilde{\mathfrak{t}}(x) = Lie(\widetilde{T}(x)) = \mathfrak{t}(x)$, and so $\widetilde{T}(x)$ is a torus with Lie algebra $\mathfrak{t}(x)$.

Observe that:

$$\ker\phi \simeq C_{G(0)}(\mathfrak{t}(x))/C_{G(0)}(\mathfrak{c}).$$

The little Weyl group is the semidirect product $W_{\mathfrak{c}} = W_{\mathfrak{t}} \ltimes \ker\phi$. This implies

$$k[\mathfrak{c}]^{W_{\mathfrak{c}}} \simeq \left(k[\mathfrak{c}]^{\ker\phi}\right)^{W_{\mathfrak{t}}}. \tag{5.4}$$

5.2.1.1 Invariants for $\ker\phi$**.** Let $T \subseteq L_1(x)$ be a maximal torus of $L_1(x)$ whose Lie algebra contains h' and such that $\widetilde{T}(x) \subseteq T$. Then the quotient $T/\widetilde{T}(x)$ embeds into $\ker\phi$ since the only elements of T acting trivially on $\mathfrak{c}$ are those belonging to $\widetilde{T}(x)$.

The coordinate ring $k[\mathfrak{c}]$ is a polynomial ring in sp indeterminates:

$$k[\epsilon_0^1,\dots,\epsilon_j^i,\dots,\epsilon_{p-1}^s] \qquad \text{for } 1 \le i \le s, 0 \le j \le p-1,$$

where $\epsilon_j^i(e_{\alpha_l}^m) = \delta_{jl}\delta_{im}$, and, retaining notation from Section 2.7, $e_{\alpha_i}^m = e_{i,i+1}$ for $i = 1,\dots,p-1$ and $e_{\alpha_0}^m = e_{p,1}$ as elements of $\mathfrak{gl}_p$.

The adjoint action of the subgroup $L_1^i(x)'$ on $\mathfrak{l}(x)$ is induced by that of $GL(p)$, as seen in the proof of Lemma 4.4.1. Thus, the adjoint action of T on $\mathfrak{l}_1^i(x)$ can be described as conjugation by diagonal matrices. Take $t = \mathrm{diag}(t_1,\dots,t_p) \in T \cap L_1^i(x)'$, where $t_i \in k^\times$ and $t_1 \cdot \ldots \cdot t_p = 1$. The element t acts on generators of the ring $k[\mathfrak{c}]$ as $t \cdot \epsilon_j^l = \delta_{i,l} \cdot t_j^{-1} t_{j+1} \cdot \epsilon_j^l$ for $j = 1,\dots,p-1$ and $t \cdot \epsilon_0^l = \delta_{i,l} \cdot t_p^{-1} t_1 \cdot \epsilon_0^l$.

This implies that $k[\mathfrak{c}]^T \simeq k[\varepsilon_1,\dots,\varepsilon_s]$, a polynomial ring generated by the algebraically independent functions $\varepsilon_i = \prod_{j=0}^{p-1} \epsilon_j^i$ for $i = 1,\dots,s$.

In particular, $k[\mathfrak{c}]^{\ker\phi} \subseteq k[\varepsilon_1,\dots,\varepsilon_s]$. In order to show equality, and therefore polynomiality of $k[\mathfrak{c}]^{\ker\phi}$, it suffices to prove that each ε_i is $\ker\phi$-invariant.

Always following Notation 4.2.6, we call $\mathfrak{gl}_p^i = \mathfrak{l}_1^i(x)$ for $i = 1, \ldots, s$ one of the direct summands of $\mathfrak{l}_1(x)$, and $\mathbf{1}_i$ is the identity matrix in $\mathfrak{gl}_p^i$. Every element $v \in \mathfrak{t}(x)$ can be expressed as $v = \mathbf{1}_1 v_1 + \ldots + \mathbf{1}_s v_s$, for suitable scalars $v_1, \ldots, v_s \in k$. The ring of regular functions of $\mathfrak{t}(x)$ is a polynomial ring $k[\mathfrak{t}(x)] = k[y_1, \ldots, y_p]$, where $y_i(v) = v_i$ for all $v \in \mathfrak{t}(x)$. Since $\ker\phi$ is a quotient of the group $C_{G(0)}(\mathfrak{t}(x))$, it acts trivially on $\mathfrak{t}(x)$ and thus on $k[\mathfrak{t}(x)]$.

Take an element $z \in \mathfrak{c}$, so that $z^{[p]} \in \mathfrak{t}(x)$. For any $g \in \ker\phi$ one has $g \cdot z^{[p]} = z^{[p]}$. Moreover, $\varepsilon_i^p(z) = y_i(z^{[p]})$. But then, for all $z \in \mathfrak{c}$:

$$(g \cdot \varepsilon_i^p)(z) = \varepsilon_i^p(g^{-1} \cdot z) = y_i(g^{-1} \cdot z^{[p]}) = y_i(z^{[p]}) = \varepsilon_i^p(z).$$

This implies that ε_i^p is $\ker\phi$-invariant. But the value of ε_i at each point of $\mathfrak{c}$ is uniquely determined by the value of ε_i^p at that point, due to uniqueness of p^{th} roots in characteristic $p > 0$. As a result, ε_i is $\ker\phi$-invariant for all $i = 1, \ldots, s$ and we have polynomiality of the ring of invariant functions for $\ker\phi$ acting on $\mathfrak{c}$:

$$k[\mathfrak{c}]^{\ker\phi} = k[\varepsilon_1, \ldots, \varepsilon_s]. \tag{5.5}$$

5.2.2 The group $W_{\mathfrak{t}}$

The arguments in Section 5.2.1.1 combined with (5.4) imply that the ring of invariants $k[\mathfrak{c}]^{G(0)}$ is isomorphic to $k[\varepsilon_1, \ldots, \varepsilon_s]^{W_{\mathfrak{t}}}$, where the generators $\varepsilon_1, \ldots, \varepsilon_s$ are algebraically independent and were introduced in Section 5.2.1.1. Recall that s is the dimension of $\mathfrak{t}(x)$ for $x \in \mathfrak{g}(1)_{reg}$ and $W_{\mathfrak{t}} = N_{G(0)}(\mathfrak{t}(x))/C_{G(0)}(\mathfrak{t}(x))$. We will simply write $\mathfrak{t} = \mathfrak{t}(x)$ to shorten notation.

Recall that h', the component of h on $\mathfrak{l}_1(x)$, is regular semisimple in $\mathfrak{l}_1(x)$. Let $\mathfrak{t}'$ be its centralizer in $\mathfrak{l}_1(x)$, and let T_1 be the maximal torus of $L_1(x)(0)$ with Lie algebra $\mathfrak{t}'$. Notice that $\mathfrak{t} \subseteq \mathfrak{t}'$. If we choose a maximal torus T_2 of $L_2(x)(0)$, then $T = T_1 \cdot T_2$ is a maximal torus of $G(0)$ (and in particular of G). Call W_0 the subgroup of $W = N_G(T)/T$ of elements that fix h, and let $C_{W_0}(\mathfrak{t})$ be the subgroup of W_0 consisting of elements fixing $\mathfrak{t}$ pointwise.

The following Lemma appears in a similar form in [Le1] (Lemma 4.2) and will be very handy over the next sections.

Lemma 5.2.1. *The group $W_{\mathfrak{t}}$ embeds into $W_0/C_{W_0}(\mathfrak{t})$.*

Proof. Let $g \in N_{G(0)}(\mathfrak{t})$; it fixes h and normalizes both $\mathfrak{t}$ and $\mathfrak{l}_1(x)$, and so it fixes also h'. Then g must fix $\mathfrak{t}'$, hence it stabilizes T_1.

The element g also leaves $L_2(x)(0)$ stable. There exists $g' \subseteq L_2(x)(0)$ with $gg' \cdot T_2 = T_2$. But $g' \in L_2(x)(0) \subseteq C_{G(0)}(\mathfrak{t})$, so gg' and g represent the same class in $W_\mathfrak{t}$ and $gg' \in N_G(T) \cap G(0)$.

Hence we can associate an element of W_0 to the coset $g\, C_{G(0)}(\mathfrak{t}) \in W_\mathfrak{t}$. Two elements in this coset have the same adjoint action on $\mathfrak{t}$, namely that of g. Therefore this gives a well-defined group homomorphism $\zeta : W_\mathfrak{t} \to W_0/C_{W_0}(\mathfrak{t})$. To see that it is injective, suppose $\zeta(w) = \zeta(w')$ for $w, w' \in W_\mathfrak{t}$. Then $\zeta(w^{-1}w')$ is the identity on T_1, and hence the same holds for $w^{-1}w'$. □

Remark 5.2.2. Once again, we know that the subalgebra $\mathfrak{l}_1(x)$ is isomorphic to the direct sum of s copies of $\mathfrak{gl}_p$. Every element of $W_\mathfrak{t}$ permutes these blocks. Indeed, any $w \in W_\mathfrak{t}$ acts as an element of W, and in particular it fixes the torus T_1. But then it permutes the s components of type A_{p-1}.

Now suppose $w \in W_\mathfrak{t}$ fixes each of the $\mathfrak{l}_1^i(x)$. Its action on the i-th component leaves the projection of $\mathfrak{t}$ on $\mathfrak{l}_1^i(x)$ stable; this is a 1-dimensional toral subalgebra $\mathfrak{t}_i \subseteq \mathfrak{gl}_p$ (see Notation 4.2.6), namely the scalar multiples of the identity matrix $\mathbf{1_i}$. Call $T^i \subseteq T$ the 1-dimensional subtorus of T with Lie algebra $\mathfrak{t}_i$. The action of w on $\mathfrak{t}_i$ is determined by that on T^i. As the action of w on T^i is an algebraic isomorphism of T^i, it follows that w acts on $\mathfrak{t}_i$ either as $Id_{\mathfrak{t}_i}$ or $-Id_{\mathfrak{t}_i}$.

Until the end of Section 5.2 we will be concerned with studying the group $W_\mathfrak{t}$ in the case where $\mathfrak{g}$ is a simple Lie algebra. Observe that this will suffice to obtain the result in the general case. Indeed, assume G is a connected reductive algebraic group. Then the centre $Z(G)$ acts trivially on $\mathfrak{g}$, so that every polynomial function is an invariant for $Z(G)$; moreover its identity component is a torus. The group G can be written as the almost-direct product $G = G_1 \times \ldots \times G_n \times Z(G)^\circ$ according to the decomposition of the root system Φ into irreducible components, where $G_1, \ldots, G_n$ are almost simple groups (see for example [Hu1, Ch. 27]), and correspondingly $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_n \oplus z(\mathfrak{g})$. The group G_i acts trivially on $\mathfrak{g}_j$ for $j \neq i$. As a result $k[\mathfrak{g}]^G \simeq k[\mathfrak{g}_1]^{G_1} \otimes \ldots k[\mathfrak{g}_n]^{G_n} \otimes k[z(\mathfrak{g})]$, and therefore it is enough to look at the case of G almost simple.

5.2.3 Exceptional types

As hinted in Remark 4.2.5, there are not many possibilities to embed subalgebras of type A_{p-1} in an exceptional Lie algebra when the characteristic of the field is a good prime for the root system. This happens in type E_6 in characteristic 5 ($\mathfrak{l}_1(x)$ of type A_4), type E_7 in characteristics 5 and 7 ($\mathfrak{l}_1(x)$ of type A_4 and A_6 respectively) and finally type E_8 in characteristic 7 ($\mathfrak{l}_1(x)$ is a subalgebra of type A_6). As a result, for exceptional Lie algebras $\dim \mathfrak{t}(x) = 1$, and thanks to Remark 5.2.2 $W_\mathfrak{t}$ is a subgroup of $\boldsymbol{\mu}_2$, more specifically each of its elements act as $\pm Id$ on the toral subalgebra $\mathfrak{t}(x)$.

5.2.4 Type A

Generalities. We will be consistent with notation from Section 2.7.1. Here $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and $G = SL(n+1)$. They naturally act on $V \simeq k^{n+1}$. Once a basis $\{v_1, \ldots, v_{n+1}\}$ of V is fixed, a maximal torus is given by diagonal matrices in this basis. We can assume that our toral element is of the form $h = \operatorname{diag}(a_1, \ldots, a_{n+1})$ where each $a_i \in \mathbb{F}_p$. Let $V(\lambda) = \{v \in V \mid h \cdot v = \lambda v\}$, for $\lambda \in \mathbb{F}_p$, be the eigenspace of h on V of eigenvalue λ.

In this case we have

$$s = \min_{\lambda \in \mathbb{F}_p} \dim V(\lambda).$$

This is because the projection of h on each $\mathfrak{gl}_p^i$ must have spectrum $\mathbb{F}_p$. Up to G-conjugation we can assume h is in the form

$$\begin{pmatrix} H_1 & 0 & \ldots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & H_s & 0 \\ 0 & \ldots & 0 & F \end{pmatrix}$$

where $H_i = \operatorname{diag}(0, 1, \ldots, p-1)$ for any $i = 1, \ldots, s$ is a $p \times p$ diagonal matrix and $F = \operatorname{diag}(a_{sp+1}, \ldots, a_{n+1})$ is a $(n - sp + 1) \times (n - sp + 1)$ diagonal matrix. Observe that the matrix H presents the eigenvalues in reversed order compared to earlier notation (e.g. the proof of Lemma 4.4.1). This is equivalent to our previous choice (Remark 2.7.2) up to G-conjugation, and will be notationally convenient later.

For a suitable $x \in \mathfrak{g}(1)_{reg}$, we can take as subalgebra $\mathfrak{l}_1(x)$ the direct sum of $\mathfrak{l}_1^i(x)$, for $i = 1, \dots, s$, where each $\mathfrak{l}_1^i(x)$ is the subalgebra of linear endomorphisms leaving $\mathrm{span}_k\langle v_{(i-1)p+1}, \dots, v_{ip}\rangle$ stable and sending v_j to 0 for $j \notin \{(i-1)p+1, \dots, ip\}$. The p-cyclic subspace can be identified with the degree 1 component of this subalgebra.

We can reformulate this concept as follows. Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be a base of simple roots for the root system. As in Section 2.7.1, we can regard $\alpha_i = \gamma_i - \gamma_{i+1}$, where γ_i is the function that associates to a diagonal matrix in $\mathfrak{sl}_{n+1}$ its i-th eigenvalue, for $i = 1, \dots, n+1$. The description given above amounts to saying that $\gamma_{pi+j}(h) = j-1$ for $0 \le i < s$ and $1 \le j \le p$. Moreover, each subspace $\mathfrak{l}_1^i(x)(1)$ is spanned by root vectors $\{e_{-\alpha_{p(i-1)+1}}, \dots, e_{-\alpha_{pi-1}}, e_{\alpha_0^i}\}$, where α_0^i is the highest root in the subsystem of Φ spanned by $\{\alpha_{p(i-1)+1}, \dots, \alpha_{pi-1}\}$.

5.2.4.1 $W_{\mathfrak{t}}$ in type A. The Weyl group W of G is isomorphic to S_{n+1} and each element can be regarded as a permutation of the indices $1, 2, \dots, n+1$ of the γ_i's.

We claim that every permutation of the $\mathfrak{l}_1^i(x)$, for $i = 1, \dots, s$, has a representative in $W_{\mathfrak{t}}$. In order to see this, it is enough to show it for a transposition in S_s of the form $(i,\ i+1)$.

Consider the following permutation in W:

$$\varsigma = (p(i-1)+1,\ pi+1)(p(i-1)+2,\ pi+2)\dots(pi,\ p(i+1)).$$

Clearly $\varsigma \in W_0$ as it fixes h. Moreover it stabilizes $\mathfrak{t}$ by switching $\mathfrak{t}_i$ and $\mathfrak{t}_{i+1}$ and leaving all the other components pointwise fixed. This is exactly the transposition we were looking for.

Thanks to Lemma 5.2.1 and Remark 5.2.2, we obtain $W_{\mathfrak{t}} \simeq S_s$ as the Weyl group in type A does not contain sign changes. Summarizing:

Proposition 5.2.3. *If G is of type A, then $W_{\mathfrak{t}}$ is isomorphic to the symmetric group S_s.*

5.2.5 Types B, C and D

Here we will stick to the notation in Section 2.7 for types B, C and D. Depending on the relevant type, we will suppose $h = \mathrm{diag}(b_1, \dots, b_n, -b_n, \dots, -b_1)$ or $h = \mathrm{diag}(b_1, \dots, b_n, 0, -b_n, \dots, -b_1)$ for suitable $b_1, \dots, b_n \in \mathbb{F}_p$. Let $\tilde{h} = \mathrm{diag}(b_1, \dots, b_n)$

be the diagonal $n \times n$ matrix acting naturally on the vector space $V = k^n$. As for the case of type A, we will call $V(\lambda)$ the eigenspace of eigenvalue $\lambda \in \mathbb{F}_p$ for $\tilde{h}$.

Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be a base of simple roots. The description of the functionals α_i for each classical type has been given in Section 2.7 in terms of the functionals $\gamma_1, \dots, \gamma_n$ and we will use it in the sequel without further comment.

Remark 5.2.4. The subalgebra $\mathfrak{l}_1(x)$ can always be embedded in a Levi subalgebra of $\mathfrak{g}$ of type A_m, for m big enough. In type B and C this depends on the fact that $\mathrm{char} k > 2$ and so I must contain simple roots all of the same length. As for type D, it is enough to notice that the simple roots α_{n-1} and α_n cannot be simultaneously part of a subset of Δ defining a Levi subalgebra of type $(A_{p-1})^s$. Indeed, this would only be the case for $p = 4$, which is not prime. In what follows, we will assume that in type D $\alpha_n \notin I$; the case $\alpha_{n-1} \notin I$ is absolutely analogous.

As the projection of h on each $\mathfrak{l}_1^i$ must have all eigenvalues $0, \dots, p-1$, in all cases the dimension of a p-cyclic subspace is given by:

$$s = \min_{\lambda \in \mathbb{F}_p} \left\lceil \frac{\dim V(\lambda) + \dim V(p - \lambda)}{2} \right\rceil .$$

The discussion at the end of Remark 2.7.2 implies that the toral element h can be expressed, up to G-conjugacy, as:

$$h = \mathrm{diag}\,(H_1, \dots, H_s, F, F', H'_s, \dots, H'_1) \tag{5.6}$$

for types C and D, and

$$h = \mathrm{diag}\,(H_1, \dots, H_s, F, 0, F', H'_s, \dots, H'_1) \tag{5.7}$$

for type B. Here $H_1, \dots, H_s$ are just as in the argument for type A (Section 5.2.4), while $F = \mathrm{diag}(a_{sp+1}, \dots, a_n)$ is a diagonal $(n - sp) \times (n - sp)$-matrix. The matrices $H'_1, \dots, H'_s$ and F' are respectively the negative of $H_1, \dots, H_n$ and F transposed about the antidiagonal.

This description of h amounts to saying that $\gamma_{pi+j}(h) = j - 1$ for $0 \le i < s$ and $1 \le j \le p$.

5.2.5.1 Permutations in $W_{\mathfrak{t}}$. The Weyl group W in types B and C is the semidirect product $(\boldsymbol{\mu}_2)^n \rtimes S_n$, where S_n is intended as permutations of the functions γ_i, and $(\boldsymbol{\mu}_2)^n$ are sign changes of them. We will identify each γ_i with its index i. The Weyl group in type D is the subgroup of that of type BC consisting of elements involving only an even number of sign changes.

Remark 5.2.5. For the three types considered, all the permutations of the s blocks $\mathfrak{l}_1^i(x)$ belong to $W_{\mathfrak{t}}$. This follows immediately from the case of type A combined with Remark 5.2.4.

Hence, up to conjugating by a suitable permutation, we can assume that a fixed element of $W_{\mathfrak{t}}$ leaves each $\mathfrak{l}_1^i(x)$ invariant; by Remark 5.2.2 it must act as a sign change on each $\mathfrak{t}_i$.

5.2.5.2 $W_{\mathfrak{t}}$ in types B and C. In type B and C all sign changes of s elements belong to $W_{\mathfrak{t}}$, so that this group is isomorphic to a Weyl group of type BC. It is enough to show that there exists an element fixing h and acting as $-Id$ on $\mathfrak{t}_i$ and as Id on $\mathfrak{t}_j$ for all $j \neq i$.

Consider the following permutation $\rho \in W$, written as a product of disjoint transpositions (recall that $p \neq 2$ here):

$$\rho = (p(i-1)+2,\ pi)(p(i-1)+3,\ pi-1)\dots(\ p(i-1)+\frac{p+1}{2},\ pi-\frac{p-3}{2}) \in W,$$

and the following sign change $\sigma \in W$:

$$\sigma(j) = \begin{cases} -j & \text{for}\, p(i-1)+1 \leq j \leq pi; \\ j & \text{otherwise.} \end{cases}$$

The composite $\sigma\rho$ fixes h and sends the i-th component $\mathfrak{t}_i \subseteq \mathfrak{t}$ to its negative while fixing all the others. Indeed, $\sigma\rho$ fixes all elements in $\mathfrak{gl}_p^j$ for $j \neq i$ since neither σ nor ρ involves indices other than $p(i-1)+1, \dots, pi$. On an element of $\mathfrak{gl}_p(0)$ of the form $A = \mathrm{diag}(a_1, a_2, a_3, \dots, a_{p-1}, a_p)$ the composite $\sigma\rho$ acts as:

$$\sigma(\rho(A)) = \sigma(\rho(\mathrm{diag}(a_1, a_2, a_3, \dots, a_{p-1}, a_p))) =$$
$$= \sigma(\mathrm{diag}(a_1, a_p, a_{p-1}, \dots, a_3, a_2)) = \mathrm{diag}(-a_1, -a_p, -a_{p-1}, \dots, -a_3, -a_2).$$

It is then clear that $\sigma\rho$ fixes H_i and acts as $-Id$ on $\mathfrak{t}_i$, so that is the element we were looking for.

We have just proved the following:

Proposition 5.2.6. *If G is of type B or C, then $W_\mathfrak{t}$ is isomorphic to $S_s \ltimes \mathbb{Z}_2^s$.*

Notice that this argument fails for type D as the element σ involves an odd number of sign changes and therefore does not belong to W. We will analyse this case separately as it turns out to be slightly more involved.

5.2.5.3 $W_\mathfrak{t}$ in type D. As in [Le1, 4.2], we will denote by $\overline{G} = O(2n)$ the full orthogonal group and by $\overline{W} = N_{\overline{G}}(T)/T$ the analogue of the Weyl group in $\overline{G}$; this a Weyl group of type BC. Let $\overline{W}_0$ stand for the subgroup of elements of $\overline{W}$ fixing h. As $W_0 \subseteq \overline{W}_0$ and $C_{W_0}(\mathfrak{t}) = C_{\overline{W}_0}(\mathfrak{t}) \cap W_0$, we obtain an injective map $W_0/C_{W_0}(\mathfrak{t}) \hookrightarrow \overline{W}_0/C_{\overline{W}_0}(\mathfrak{t})$. Thanks to Lemma 5.2.1, $W_\mathfrak{t}$ embeds in $\overline{W}_0/C_{\overline{W}_0}(\mathfrak{t})$.

Proposition 5.2.7. *Let G be of type D. If $\dim V(0) > s$ then $W_\mathfrak{t}$ is isomorphic to $S_s \ltimes \mathbb{Z}_2^s$, otherwise it is isomorphic to $S_s \ltimes \mathbb{Z}_2^{s-1}$.*

Proof. Thanks to Remark 5.2.5, it is enough to consider elements of $W_\mathfrak{t}$ which stabilize every $\mathfrak{l}_1^i(x)$. The element $\sigma\rho$ from 5.2.5.2 belongs to $\overline{W}_0$ and stabilizes $\mathfrak{t}$. Then $W_\mathfrak{t}$ is isomorphic to a Weyl group of type BC if and only if there exists a coset $\phi C_{W_0}(\mathfrak{t})$, with $\phi \in W_0$, such that $\phi C_{\overline{W}_0}(\mathfrak{t}) = \sigma\rho C_{\overline{W}_0}(\mathfrak{t})$. Otherwise stated, if and only if there exists $\eta \in C_{\overline{W}_0}(\mathfrak{t})$ with $\sigma\rho\eta \in W_0$. If such η exists, it fixes h and $\mathfrak{t}$ pointwise, so that it acts trivially on the toral subalgebra $\mathfrak{t}'$ (see beginning of Section 5.2 for the notations). It follows that η is a product of permutations and sign changes involving only the indices $ps+1, \ldots, n$. Furthermore, since $\sigma\rho$ changes an odd number of signs, $\eta = \eta_1\eta_2$ must be the composite of a permutation $\eta_2 \in S_{n-ps}$ and an odd number of sign changes η_1. As η fixes h, it is not hard to see that if $\dim V(0) > s$ such an element exists: say $\gamma_i(h) = 0$ for a certain $ps+1 \leq i \leq n$, then we can take η to be the sign change $i \mapsto -i$. As a result, in this case $W_\mathfrak{t}$ is a Weyl group of type BC.

Assume therefore $\dim V(0) = s$. For $j \in \{ps+1, \ldots, n\}$, recall that we wrote $b_j = \gamma_j(h) \neq 0$. Break η_2 into a product of cyclic permutations, and consider one of its cycles $(j\ \eta_2(j)\ \eta_2^2(j) \ldots \eta_2^l(j))$. Then for every $m = 1, \ldots, l$ we have $b_{\eta_2^m(j)} = \pm b_{\eta_2^{m-1}(j)}$ and $\eta_1(\eta_2^m(j)) = +\eta_2^m(j)$ iff $b_{\eta_2^m(j)} = b_{\eta_2^{m-1}(j)}$, otherwise $\eta_1(\eta_2^m(j)) = -\eta_2^m(j)$. But since

$\eta_2^{l+1}(j) = j$, we have that η_1 consists only of an even number of sign changes, so that $\eta \in C_{\overline{W0}}(\mathfrak{t})$ with the properties required does not exist. It follows that in this case $W_{\mathfrak{t}}$ is isomorphic to a Weyl group of type D, as each of its elements must involve an even number of sign changes on $\mathfrak{t}$. □

5.3 Polynomiality of the ring of invariants

Knowing the groups $W_{\mathfrak{t}}$ for simple Lie algebras allows us to understand many properties of the ring $k[\mathfrak{g}(1)]^{G(0)}$. The main tools we shall repeatedly apply in this section are the Chevalley-Shephard-Todd Theorem (see [Be, Theorem 7.2.1], for example) and the Fundamental Theorem of Symmetric Polynomials (an elementary proof of which can be found in [CLO, Chapter 7, Theorem 3]). We include these two theorems for the sake of completeness.

Theorem 5.3.1 (Chevalley-Shephard-Todd)**.** *Suppose that V is a finite dimensional faithful representation of a finite group H over a field K of characteristic coprime to the order of H. The following are equivalent:*

(i) H is generated by elements acting on V as pseudoreflections;

(ii) $K[V]^H$ is a polynomial ring;

(iii) $K[V]$ is a free $K[V]^H$-module.

We recall that a *pseudoreflection* is a linear endomorphism of finite order of a vector space V which fixes pointwise a subspace of V of codimension 1.

Theorem 5.3.2 (Fundamental Theorem of Symmetric Polynomials)**.** *Let K denote a field. Every symmetric polynomial in $K[x_1, \ldots, x_n]$ can be written uniquely as a polynomial in the elementary symmetric functions $\sigma_1, \ldots, \sigma_n$.*

Here the j-th elementary symmetric polynomial is

$$\sigma_j = \sum_{i_1 < i_2 < \ldots < i_j} x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_j}, \qquad \forall\, j = 1, \ldots, n.$$

Theorem 5.3.2 states that $\sigma_1, \ldots, \sigma_n$ are algebraically independent and generate the ring of invariants $K[x_1, \ldots, x_n]^{S_n}$ for the natural action of the symmetric group S_n on $K[x_1, \ldots, x_n]$ by permutation of subscripts.

5.3.1 Case by case proof of polynomiality

Retaining notation used at the beginning of section 5.2, it is worth stressing that if $w \in W_{\mathfrak{t}}$ acts as $-Id$ on $\mathfrak{t}_i$, then $w \cdot \varepsilon_i = -\varepsilon_i$.

5.3.1.1 As for exceptional Lie algebras, Remark 4.2.5 implies that the order of $W_{\mathfrak{t}}$ divides 2. Since 2 is a bad characteristic for these types, the Chevalley-Shephard-Todd Theorem applies, entailing polynomiality of $k[\mathfrak{g}(1)]^{G(0)}$; this latter is therefore a ring in $s = 1$ indeterminate.

5.3.1.2 For type A, combining Proposition 5.2.3 with Theorem 5.3.2, again $k[\mathfrak{g}(1)]^{G(0)}$ is a polynomial ring in s indeterminates.

5.3.1.3 For types B and C, we use the description of $W_{\mathfrak{t}}$ given in Proposition 5.2.6. The normal subgroup $\mathbb{Z}_2^s \subseteq W_{\mathfrak{t}}$ has order coprime with $p = \operatorname{char} k$ as 2 is a bad prime for these types. Moreover, $\mathbb{Z}_2^s$ acts as a group generated by reflections, its invariants are therefore polynomial. To be more explicit:

$$k[\mathfrak{g}(1)]^{G(0)} \simeq k[\varepsilon_1, \dots, \varepsilon_s]^{W_{\mathfrak{t}}} \simeq (k[\varepsilon_1, \dots, \varepsilon_s]^{\mathbb{Z}_2^s})^{S_s} \simeq k[\varepsilon_1^2, \dots, \varepsilon_s^2]^{S_s}.$$

At this point one can resort to Theorem 5.3.2 again.

5.3.1.4 The case of type D is a little bit different as Proposition 5.2.7 gives $W_{\mathfrak{t}} \simeq S_s \ltimes \mathbb{Z}_2^{s-1}$, but the normal subgroup $\mathbb{Z}_2^{s-1} \subseteq W_{\mathfrak{t}}$ is not generated by pseudoreflections. In fact, each of its elements consists of an even number of sign changes on $\mathfrak{t}$, and therefore every $w \in \mathbb{Z}_2^{s-1}$ has an even number of eigenvalues equal to -1 on $\mathfrak{t}$. Polynomiality still holds, but since Theorem 5.3.1 does not apply, we give a proof of this fact hereafter.

We look at the ring $(k[\varepsilon_1, \dots, \varepsilon_s]^{\mathbb{Z}_2^{s-1}})^{S_s}$. Since 2 is a bad prime in type D, we have that $p = \operatorname{char} k$ and the order of the group $\mathbb{Z}_2^{s-1}$ are coprime, thus the Chevalley-Shephard-Todd Theorem implies that $k[\varepsilon_1, \dots, \varepsilon_s]^{\mathbb{Z}_2^{s-1}}$ is *not* a polynomial ring. Indeed, it is not hard to prove that $k[\varepsilon_1, \dots, \varepsilon_s]^{\mathbb{Z}_2^{s-1}} \simeq k[\varepsilon_1^2, \dots, \varepsilon_s^2, \varepsilon_1 \cdot \varepsilon_2 \cdot \dots \cdot \varepsilon_s]$. To be more explicit, take a $\mathbb{Z}_2^{s-1}$-invariant polynomial

$$R(\varepsilon_1, \dots, \varepsilon_s) = \sum_{I=(i_1,\dots,i_s)\in\mathbb{Z}_{\geq 0}^s} a_I \cdot \varepsilon_1^{i_1} \dots \varepsilon_s^{i_s} \in k[\varepsilon_1, \dots, \varepsilon_s],$$

where I stands for a multiindex and $a_I \in k$. Suppose that for a multiindex $I = (i_1, \ldots, i_s)$ we have that $a_I \neq 0$ and i_j is even for all $j = 1, \ldots s$, then the corresponding monomial $a_I \cdot \varepsilon_1^{i_1} \ldots \varepsilon_s^{i_s}$ is $\mathbb{Z}_2^{s-1}$-invariant. If i_j is odd for a certain $1 \leq j \leq s$, assume there exists $l \neq j$ such that i_l is even. Take the element $w \in \mathbb{Z}_2^{s-1}$ that acts by changing signs only to the j-th and the l-th entries; then w acts on the corresponding monomial as $w \cdot a_I \varepsilon_1^{i_1} \ldots \varepsilon_s^{i_s} = -a_I \varepsilon_1^{i_1} \ldots \varepsilon_s^{i_s}$, and therefore this monomial is not invariant. It follows that if a certain i_j is odd, then all indices in $I = (i_1, \ldots, i_s)$ are, in particular $\varepsilon_1 \cdot \ldots \cdot \varepsilon_s$ divides $\varepsilon_1^{i_1} \ldots \varepsilon_s^{i_s}$. Upon dividing the monomial by the maximum (odd) power of $\varepsilon_1 \cdot \ldots \cdot \varepsilon_s$, we obtain a monomial $\varepsilon_1^{j_1} \ldots \varepsilon_s^{j_s}$ in which every j_l is even for $l = 1, \ldots, s$, so that the description of the invariant ring follows.

Now consider $k[\varepsilon_1^2, \ldots, \varepsilon_s^2, \varepsilon_1 \cdot \varepsilon_2 \cdot \ldots \cdot \varepsilon_s]^{S_s}$, where S_s acts by permuting the ε_i's. Observe that the monomial $\varepsilon_1 \cdot \ldots \cdot \varepsilon_s$ is fixed by this action. Take a polynomial $P \in k[\varepsilon_1^2, \ldots, \varepsilon_s^2, \varepsilon_1 \cdot \varepsilon_2 \cdot \ldots \cdot \varepsilon_s]^{S_s}$. By looking at the highest powers of $\varepsilon_1 \cdot \ldots \cdot \varepsilon_s$ dividing each monomial in P, one can uniquely write

$$P = P_0 + (\varepsilon_1 \cdot \ldots \cdot \varepsilon_s) P_1 + \ldots + (\varepsilon_1 \cdot \ldots \cdot \varepsilon_s)^m P_m,$$

where $P_i \in k[\varepsilon_1^2, \ldots, \varepsilon_s^2]$ for $i = 0, \ldots, m$ and none of the monomials in each P_i is divisible by $\varepsilon_1 \cdot \ldots \cdot \varepsilon_s$. As P is S_s-invariant, all the P_i's are, so that $P_i \in k[\varepsilon_1^2, \ldots, \varepsilon_s^2]^{S_s}$. But $k[\varepsilon_1^2, \ldots, \varepsilon_s^2]^{S_s}$ is a polynomial ring, generated by the elementary symmetric polynomials in $\varepsilon_1^2, \ldots, \varepsilon_s^2$. Call e_i the elementary symmetric polynomial of degree i in the $\varepsilon_1^2, \ldots, \varepsilon_s^2$ (of course, e_i has degree $2i$ as a polynomial in $\varepsilon_1, \ldots, \varepsilon_s$). We have $e_s = (\varepsilon_1 \cdot \ldots \cdot \varepsilon_s)^2$, so call $\tilde{e}_s = \varepsilon_1 \cdot \ldots \cdot \varepsilon_s$. Then, the functions $e_1, \ldots, e_{s-1}, \tilde{e}_s$ generate the ring of invariants, and they are also algebraically independent. Indeed, assume that there exists a polynomial in s indeterminates $Q \in k[y_1, \ldots, y_s]$ with $Q(e_1, \ldots, e_{s-1}, \tilde{e}_s) = 0$. Write $Q(y_1, \ldots, y_s) = \sum_{I=(i_1,\ldots,i_s) \in \mathbb{Z}_{\geq 0}^s} b_I \cdot y_1^{i_1} \ldots y_s^{i_s}$, for some scalars $b_I \in k$. We cannot have for every multiindex I the implication $b_I \neq 0 \Rightarrow i_s$ is even, otherwise we would get a relation of algebraic dependence for $e_1, \ldots, e_s$, contradicting Theorem 5.3.2. Therefore there exists a multiindex I for which $b_I \neq 0$ and i_s is odd. Then, as all $e_1, \ldots, e_{s-1}$ have even degrees in each $\varepsilon_1, \ldots, \varepsilon_s$, we can assume that an odd power of y_s divides each monomial in Q. But then $\tilde{e}_s Q(e_1, \ldots, e_{s-1}, \tilde{e}_s) = 0$, and this gives a nontrivial relation of algebraic dependence between $e_1, \ldots, e_s$. As a consequence, $Q = 0$ and the ring of invariants is polynomial.

5.3.2 The Main Theorem

The discussion in this section implies our Main Theorem 5.3.3. To summarize the hypothesis, we assume that G is reductive and defined over an algebraically closed field k of characteristic $p > 0$, and G satisfies the standard hypothesis. The toral element $h \in \mathfrak{g}$ endows the Lie algebra with an $\mathbb{F}_p$-grading $\mathfrak{g} = \sum_{i\in\mathbb{F}_p} \mathfrak{g}(i)$. We set $G(0) = C_G(h)^\circ$ and $s = \dim \mathfrak{t}(x)$ for $x \in \mathfrak{g}(1)_{reg}$.

Theorem 5.3.3. *Let G be a connected reductive algebraic group satisfying the standard hypothesis. Then the ring $k[\mathfrak{g}(1)]^{G(0)}$ is a polynomial ring in s indeterminates.*

Chapter 6

Consequences of polynomiality

6.1 Flatness of the quotient morphism

Thanks to results in 5.3, we can express the ring of invariant functions as $k[\mathfrak{g}(1)]^{G(0)} = k[\psi_1, \dots, \psi_s]$, where $\psi_1, \dots, \psi_s \in k[\mathfrak{g}(1)]$ are homogeneous algebraically independent polynomials and $s = \dim \mathfrak{t}(x)$ for $x \in \mathfrak{g}(1)_{reg}$. The variety $\mathfrak{g}(1)/\!\!/G(0)$ is isomorphic to an affine space of dimension s $\mathbb{A}^s$, and it comes with the surjective morphism:

$$\begin{aligned} \Psi : \mathfrak{g}(1) &\longrightarrow \mathfrak{g}(1)/\!\!/G(0) \simeq \mathbb{A}^s \\ x &\longmapsto (\psi_1(x), \dots, \psi_s(x)). \end{aligned} \tag{6.1}$$

Ψ is independent of the choice of generators.

Proposition 6.1.1. *All fibres of Ψ have the same dimension, that equals $\dim \mathfrak{g}(1) - s$. In particular, Ψ is a flat morphism.*

Proof. It is enough to prove the first half of the statement as flatness of Ψ, under our assumptions, is equivalent to all fibers of the quotient morphism having the same dimension ([Ma, Corollary to Theorem 23.1]).

Let $\xi = (\xi_1, \dots, \xi_s) \in \mathfrak{g}(1)/\!\!/G(0)$ and let X be an irreducible component of the fibre $\Psi^{-1}(\xi)$. Consider the cone $\mathbb{K}X$ (see Section 2.6 for definitions). Since $\Psi^{-1}(\xi) = \{x \in \mathfrak{g}(1) \mid \psi_i(x) = \xi_i \ \forall\, i = 1, \dots, s\}$ and the generators $\psi_1, \dots, \psi_s$ are homogeneous, the ideal defining $\mathbb{K}X$ in $k[\mathfrak{g}(1)]$ is $gr((\psi_i - \xi_i)_{i=1,\dots,s}) \supseteq (\psi_1, \dots, \psi_s)$ and therefore $\mathbb{K}X \subseteq \mathcal{N}(\mathfrak{g}(1))$. As $\dim X = \dim \mathbb{K}X$, the maximal dimension of a fibre of Ψ is less than or equal to the maximal dimension of an irreducible component of $\mathcal{N}(\mathfrak{g}(1)) = \mathcal{N}(\mathfrak{g}) \cap \mathfrak{g}(1)$. Recall that this subset consists of finitely many $G(0)$-orbits; for this reason any of its irreducible components is the closure of a certain orbit $G(0) \cdot e$, for $e \in \mathcal{N}(\mathfrak{g}(1))$.

Let $y \in \mathfrak{g}(1)$, and take a basis $\{x_1, \ldots, x_r\}$ of $\mathfrak{g}(0)$. We have $\dim C_{G(0)}(y) = \dim C_{\mathfrak{g}(0)}(y) = \dim \mathfrak{g}(0) - \operatorname{rank} M_y$, where M_y is the matrix whose columns are the coordinates of the vectors $[x_1, y], \ldots, [x_r, y]$ expressed in a basis of $\mathfrak{g}(1)$. In particular, there exists a Zariski open subset $Y \subseteq \mathfrak{g}(1)$ such that for $y \in Y$ one has that $\operatorname{rank} M_y$ is maximal, and therefore $\dim C_{G(0)}(y)$ is minimal. As a result, for $y \in Y$ one has that $\dim G(0) \cdot y$ is maximal and equal to $\dim \mathcal{N}(\mathfrak{g}(1))$.

The morphisms Ψ and the varieties $\mathfrak{g}(1)$ and $\mathfrak{g}(1)/\!\!/G(0)$ satisfy the hypothesis of [Hu1, Th. 4.3], hence there exists an open subset $U \subseteq \mathfrak{g}(1)/\!\!/G(0)$ such that if $u \in U$ then $\dim \Psi^{-1}(u) = \dim \mathfrak{g}(1) - s$. The subset $\Psi^{-1}(U) \subseteq \mathfrak{g}(1)$ is open and nonempty, so it intersects Y nontrivially. It follows that $\dim \mathcal{N}(\mathfrak{g}(1)) = \dim \mathfrak{g}(1) - s$, and this is the maximal dimension of fibres. Applying [Hu1, Th. 4.1] to the morphisms Ψ, we also obtain that for $\xi \in \mathfrak{g}(1)/\!\!/G(0)$ $\dim \Psi^{-1}(\xi) \geq \dim \mathfrak{g}(1) - s$, which then is the minimal dimension of fibres. This implies that all the fibres of Ψ have the same dimension $\dim \mathfrak{g}(1) - s$ and the morphism Ψ is flat. □

6.1.1 The module of covariants

The inclusion $k[\mathfrak{g}(1)]^{G(0)} \subseteq k[\mathfrak{g}(1)]$ gives $k[\mathfrak{g}(1)]$ the structure of a $k[\mathfrak{g}(1)]^{G(0)}$-module, called the *module of covariants* of the $G(0)$-module $\mathfrak{g}(1)$. Both $k[\mathfrak{g}(1)]^{G(0)}$ and $k[\mathfrak{g}(1)]$ admit a natural $\mathbb{Z}_{\geq 0}$-grading; in particular, $k[\mathfrak{g}(1)]$ is a graded $k[\mathfrak{g}(1)]^{G(0)}$-module. Flatness of the quotient morphism Ψ, proven in Proposition 6.1.1, entails freeness of the module of covariants.

Proposition 6.1.2. *$k[\mathfrak{g}(1)]$ is a free $k[\mathfrak{g}(1)]^{G(0)}$-module.*

The proof of this fact relies on some general results on graded algebras. We refer to [CM, Section 2.1] for the proofs; however, we will include the statement for completeness.

Let A be a finitely generated $\mathbb{Z}_{\geq 0}$-graded k-algebra. Call A_+ the ideal of A generated by homogeneous elements of strictly positive degree; assume moreover $A = k + A_+$. If

M is a graded A-module, let $M_+ = A_+M$.

Lemma 6.1.3 ([CM], Lemma 2.2). *Let M be a graded A-module and assume that $V \subseteq M$ is a graded subspace with $M = V \oplus M_+$. The canonical map $A \otimes_k V \to M$ is surjective, and it is bijective iff M is a flat A-module. Moreover, any homogeneous k-basis of V is an A-basis of M.*

Remark 6.1.4. Even though in [CM] the authors work over fields of characteristic 0, the proof of the lemma above does not require any restrictions on the characteristic of k.

Proof of Proposition 6.1.2. This is a consequence of Lemma 6.1.3 applied by replacing A and M with $k[\mathfrak{g}(1)]^{G(0)}$ and $k[\mathfrak{g}(1)]$ respectively. Indeed, $k[\mathfrak{g}(1)]_+^{G(0)}k[\mathfrak{g}(1)]$ is a graded vector subspace of $k[\mathfrak{g}(1)]$. Choose graded $V \subseteq k[\mathfrak{g}(1)]$ satisfying the hypothesis of Lemma 6.1.3. It follows that $k[\mathfrak{g}(1)] \simeq k[\mathfrak{g}(1)]^{G(0)} \otimes_k V$ and every k-basis of V is a $k[\mathfrak{g}(1)]^{G(0)}$-basis for $k[\mathfrak{g}(1)]$.

6.2 Sections for the action of $G(0)$ on $\mathfrak{g}(1)$

Most of the definitions in this section are sourced from [Po].

Let H be a connected reductive algebraic group acting on a finite dimensional vector space V, both defined over a field $k = \overline{k}$.

Definition 6.2.1. A linear subvariety $S \subseteq V$ is a *Weierstrass section* for the action of H on V if the inclusion $S \subseteq V$ induces a ring isomorphism

$$k[S] \simeq k[V]^H.$$

This in particular implies that S intersects each fibre of the morphism $V \to V/\!\!/H$ in exactly one point.

Remark 6.2.2. One of the most remarkable and best-known examples of a Weierstrass section is the existence of such a subvariety for the adjoint action of a connected reductive algebraic group G on its Lie algebra $\mathfrak{g}$ in the characteristic 0 case (however this can be generalized, see Remark 6.2.3).

Let $e \in \mathfrak{g}$ be a regular nilpotent element. As $\operatorname{char} k = 0$, the element e can be embedded in an $\mathfrak{sl}_2$-triple (e, h_e, f_e). Consider the subspace $C_{\mathfrak{g}}(f_e)$, the centralizer of

f_e in $\mathfrak{g}$. The variety $S = e + C_{\mathfrak{g}}(f_e)$ is a Weierstrass section for the action of G on $\mathfrak{g}$. It is also called *Kostant section* since it was first discovered by Kostant ([Ko2, Th 0.10]).

Remark 6.2.3. When p is a good prime for the root system, Veldkamp describes a section for the adjoint action of G on $\mathfrak{g}$ ([Ve]). For this purpose, he chooses the two following regular nilpotent elements:

$$X_+ = \sum_{\alpha\in\Delta} e_\alpha, \qquad X_- = \sum_{\alpha\in\Delta} e_{-\alpha}.$$

In [Ve, Section 6] he shows that the affine subspace $V = X_- + c_{\mathfrak{g}}(X_+)$ is a section for the adjoint action of G on $\mathfrak{g}$. Moreover, this subspace satisfies the property $\mathfrak{g} = V \oplus [\mathfrak{g}, e]$.

Notice that in [Ve, Section 6] Veldkamp works under the assumption that $\operatorname{char} k = 0$ or $p > 0$ with p not dividing the order of W. This is more restrictive than the standard hypothesis, still Veldkamp's results are valid in our more general setting as well. Indeed, the arguments in [Ve, (6.3) to (6.5)] go through provided $k[\mathfrak{g}]^G$ is a polynomial ring generated by $l = \operatorname{rank}\mathfrak{g}$ homogeneous polynomials $f_1, \ldots, f_l$, and the differentials $df_1, \ldots, df_l$ at a regular element $x \in \mathfrak{g}$ are linearly independent. These facts hold for groups satisfying the standard hypothesis. A proof in the case p odd follows, for example, from [BG, Lemma 3.3] and [BG, Corollary 3.4]. The only case left out is therefore G comprising only components of type A; [PrSt, Section 2] entails that the components of $\mathfrak{g}$ of type A_r with $p|r$ are isomorphic to $\mathfrak{gl}_r$. Hence, the facts above hold also in this case thanks to [PrTa, Lemma 1 §3.1] and [PrTa, Corollary §3.2].

6.2.1 The N-regular case

At the beginning of Section 2.2 we defined the regular nilpotent orbit $\mathcal{O}_{reg} \subseteq \mathfrak{g}$. The $\mathbb{F}_p$-grading given by h is called *N-regular* if $\mathcal{O}_{reg} \cap \mathfrak{g}(1) \neq \emptyset$; equivalently, if $\mathfrak{g}(1)$ contains a regular nilpotent element of $\mathfrak{g}$.

Panyushev ([Pa]) proved the existence of sections for Vinberg's θ-groups in characteristic 0, in the N-regular case. More specifically, he works with a graded $\mathfrak{g} = \oplus_{i\in\mathbb{Z}_m}\mathfrak{g}(i)$, where $m \in \mathbb{Z}_{\geq 0}$ and the grading is given by an automorphism $\theta \in \operatorname{Aut}(\mathfrak{g})$ of finite order $m \geq 0$, and $\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid \theta(x) = \zeta^i x\}$, where ζ is a fixed primitive m^{th} root of unity. The group $G(0) \subseteq G$ is the closed connected subgroup with Lie algebra $\mathfrak{g}(0)$, and it acts on each subspace $\mathfrak{g}(i)$.

Like Panyushev, we will use the terminology *Kostant-Weierstrass section* (KW-section). His construction and Kostant's ([Ko2]) are somewhat analogous. Let $e \in \mathfrak{g}(1)$ be regular nilpotent; then the element e can be embedded in an *adapted* $\mathfrak{sl}_2$-triple (e, h_e, f_e), that is, verifying $h_e \in \mathfrak{g}(0)$ and $f_e \in \mathfrak{g}(-1)$. The affine subspace $e + C_{\mathfrak{g}(1)}(f_e)$ is a KW-section for the action of $G(0)$ on $\mathfrak{g}(1)$.

Our goal is to prove the existence of a section for the grading given by our toral element $h \in \mathfrak{g}$. For the rest of 6.2.1 we will assume that the grading given by h is N-regular. To begin with, we show that this assumption leads to polynomiality of invariants without making use of most of the machinery developed over the last two chapters. Indeed, we will do this by finding a suitable section for the action of $G(0)$ on $\mathfrak{g}(1)$ and exploiting only Lemma 4.1.3. A crucial observation is that the affine subspace V described in 6.2.3 is graded, since it decomposes as $V = \oplus_{i \in \mathbb{F}_p} V(i)$, where $V(i) = \mathfrak{g}(i) \cap V$. This follows directly from [Ve, Prop. 5.2] or [Ko1, Th. 6.7].

Proposition 6.2.4. *If* $\mathfrak{g}(1)$ *contains a regular nilpotent element,* $k[\mathfrak{g}(1)]^{G(0)}$ *is a polynomial ring.*

Proof. Let $e \in \mathfrak{g}(1)$ be regular nilpotent. By Lemma 4.1.3 we can choose an optimal cocharacter λ for e (see Section 2.5) such that $Ad(\lambda(t)) \cdot h = h$, so that the action of λ on $\mathfrak{g}$ commutes with the adjoint action of h. Thanks to the observation above, the variety V in remark 6.2.3 can be chosen to be graded $V = \oplus_{i \in \mathbb{F}_p} V(i)$, where the subspaces $V(i)$ introduced above are common eigenspaces for both $\operatorname{ad} h$ and $Ad(\lambda(t))$. Recall that there is an isomorphism $\chi : e + V \xrightarrow{\sim} \mathfrak{g}/\!\!/G \simeq k^l$, and let $l(1)$ be the dimension of $V(1)$.
The fact that e is regular in $\mathfrak{g}$ implies that the following morphism is dominant:

$$\begin{aligned} \pi : \quad G \times (e+V) &\longrightarrow \mathfrak{g} \\ (g, e+v) &\longmapsto Ad(g)(e+v). \end{aligned}$$

In fact, its differential in $(1, e)$ is given by

$$\begin{aligned} d_{1,e}\pi : \quad \mathfrak{g} \oplus V &\longrightarrow \mathfrak{g} \\ (x, v) &\longmapsto [x, e] + v. \end{aligned}$$

Recall that $V \oplus [e, \mathfrak{g}] = \mathfrak{g}$; this yields surjectivity of the differential $d_{1,e}\pi$, so that we can apply [Sp1, Th. 4.3.6], since both $G \times (e + V)$ and $\mathfrak{g}$ are irreducible varieties. The element e belongs to $\mathfrak{g}(1)$, and therefore the following equality holds:

$$[\mathfrak{g}(0), e] \oplus V(1) = \mathfrak{g}(1). \tag{6.2}$$

By restricting π we can consider the map:

$$\begin{aligned} \bar{\pi}: \quad G(0) \times (e + V(1)) &\longrightarrow \mathfrak{g}(1) \\ (g, e + v) &\longmapsto Ad(g)(e + v). \end{aligned}$$

Just as above, the differential of this morphism in $(1, e)$ is given by $d_{1,e}\bar{\pi}(x, v) = [x, e] + v$. Thanks to the identity $Lie\, G(0) = \mathfrak{g}(0)$ and the fact that (6.2) holds, the map $\bar{\pi}$ is dominant as well.
The composite $e + V(1) \hookrightarrow \mathfrak{g}(1) \twoheadrightarrow \mathfrak{g}(1)/\!\!/G(0)$ yields a ring homomorphism:

$$k[\mathfrak{g}(1)]^{G(0)} \longrightarrow k[e + V(1)]. \tag{6.3}$$

The morphism $\bar{\pi}$ being dominant, we have that the map (6.3) is injective.

The existence of the isomorphism of varieties $e + V \simeq \mathfrak{g}/\!\!/G$ implies that every regular function in $k[e+V]$ is the restriction to $e+V$ of a polynomial in $k[f_1, \ldots, f_l] = k[\mathfrak{g}]^G$. Take coordinate functions $\varsigma_1, \ldots, \varsigma_{l(1)} \in k[e + V]$ for the subvariety $e + V(1) \subseteq e + V$. These are restrictions of some elements in $k[f_1, \ldots, f_l]$. But $f_1, \ldots, f_l$ are G-invariant, thus their restrictions to $\mathfrak{g}(1)$ are $G(0)$-invariant. As a consequence, every function in $k[e + V(1)]$ is the restriction of a polynomial belonging to $k[\mathfrak{g}(1)]^{G(0)}$. This gives surjectivity of the map (6.3). But then $k[\mathfrak{g}(1)]^{G(0)} \simeq k[\varsigma_1, \ldots, \varsigma_{l(1)}]$ is a polynomial ring. □

Retain notation from the proof of Proposition 6.2.4. In particular, the proof implies:

Corollary 6.2.5. *Under the assumptions of Proposition 6.2.4, the affine subspace $e + V(1)$ is a KW-section for the action of $G(0)$ on $\mathfrak{g}(1)$.*

6.2.2 Existence of KW-sections for classical Lie algebras

Here we will prove the existence of KW-sections in the general case for classical Lie algebras. This will be done by resorting to the N-regular case, similarly to [Le1, Section 5].

Assume $\mathfrak{g}$ is a simple Lie algebra of classical type. We are going to find in each case a Levi subgroup $L \subseteq G$ satisfying:

(i) the toral element h belongs to the Lie algebra $\mathfrak{l} = LieL$;

(ii) the grading $\mathfrak{l} = \oplus_{i \in \mathbb{F}_p} \mathfrak{l}(i)$ induced by h on $\mathfrak{l}$ is N-regular;

(iii) there exists a ring isomorphism $k[\mathfrak{l}(1)]^{L(0)} \simeq k[\mathfrak{g}(1)]^{G(0)}$.

For property (iii) to hold, it is enough that a p-cyclic subspace of $\mathfrak{g}(1)$ is contained in $\mathfrak{l}(1)$ (this shall be clear in each case), and moreover that the little Weyl group for the action of $L(0)$ on $\mathfrak{l}(1)$ is isomorphic to $W_{\mathfrak{c}}$. Thanks to the discussion at the beginning of Section 5.2, and more specifically the isomorphism (5.4), it suffices to prove that $W_{\mathfrak{t}}$ embeds into the Weyl group of L (Lemma 5.2.1).

6.2.3 Type A

Retain notations from Section 2.7.1 and Section 5.2.4, in particular assume that h is in the form given in Section 5.2.4. The group L can be chosen to be the standard Levi subgroup defined by the subset of simple roots $\{\alpha_1, \ldots, \alpha_{sp-1}\} \subseteq \Delta$. Then L is a subgroup of type A_{sp-1} and its Lie algebra contains the p-cyclic subspace described in 5.2.4. There is a Lie algebra isomorphism $[\mathfrak{l}, \mathfrak{l}] \simeq \mathfrak{sl}_{sp}$, so that we can identify this derived subalgebra with a subset of $sp \times sp$ matrices using the same notation as in 2.7.1. Moreover, $W_{\mathfrak{t}}$ embeds into the Weyl group of L because the permutations in 5.2.4.1 do, and as a consequence $\mathfrak{l}(1) /\!\!/ L(0) \simeq \mathfrak{g}(1) /\!\!/ G(0)$. The grading on $\mathfrak{l}(1)$ is N-regular since for example the regular nilpotent element:

$$\sum_{i=1}^{sp-1} e_{i+1,i}$$

belongs to $\mathfrak{l}(1)$.

6.2.4 Type B

Here we will assume that h is in the form $\mathrm{diag}(F, H_1, \ldots, H_s, 0, H_s', \ldots, H_1', F')$ where the square matrices H_i, H_i', F and F' are the same as in 5.2.5 (see (5.7)). This toral element is G-conjugate to the one fixed in (5.7) thanks to the discussion at the end of Remark 2.7.2. This different choice has been made for notational convenience, so that the subgroup L can be easily described as a standard Levi subgroup. Indeed, it is given by the choice of simple roots $I = \{\alpha_{n-ps}, \dots, \alpha_n\} \subseteq \Delta$. Recall that the root α_n coincides with the functional γ_n. This Levi subgroup is of type B as its Lie algebra $\mathfrak{l}$ is a central extension of its derived subalgebra $[\mathfrak{l}, \mathfrak{l}]$, which is isomorphic to $\mathfrak{so}(2ps+1)$. A p-cyclic subspace for the action of $L(0)$ on $\mathfrak{l}(1)$ is the 1-degree component of the standard Levi subalgebra of type A_{p-1}^s given by the subset $I \setminus \{\alpha_{n-ip} \,|\, i = 0, \dots, s\}$. Notice that this is a p-cyclic subspace for the action of $G(0)$ on $\mathfrak{g}(1)$ as well. Moreover, $W_\mathfrak{t}$ is a subgroup of the Weyl group of L thanks to the discussion in 5.2.5.2. Finally, this grading is N-regular since $[\mathfrak{l}, \mathfrak{l}]$, in the isomorphism with $\mathfrak{so}(2ps+1)$, contains the regular nilpotent element $\sum_{\alpha \in \Delta} e_{-\alpha} = \sum_{i=1}^{ps} e_{i+1,i} - e_{2ps+2-i,2ps+1-i}$

6.2.5 Type C

In order to describe easily the Levi subalgebra in type C, the toral element h will be taken in a slightly different form from that in (5.6). Indeed, we will assume $h = \operatorname{diag}(F, \widetilde{H}_1, \dots, \widetilde{H}_s, \widetilde{H}'_s, \dots, \widetilde{H}'_1, F')$, where F and F' are the same as in (5.6), whereas $\widetilde{H}_i = \operatorname{diag}(\frac{p+2p-1}{2}, \frac{p+2p-3}{2}, \dots, \frac{p+3}{2}, \frac{p+1}{2})$ for all i and $\widetilde{H}'_i$ is the negative of $\widetilde{H}_i$ transposed about its antidiagonal. Once again, this choice is perfectly equivalent to that in (5.6) thanks to Remark 2.7.2.

Here we take L to be the standard Levi subgroup given by the following choice of simple roots: $\{\alpha_{n-sp}, \dots, \alpha_n\} \subseteq \Delta$. Recall that α_n coincides with the functional $2\gamma_n$. The derived subalgebra $[\mathfrak{l}, \mathfrak{l}]$ is isomorphic to $\mathfrak{sp}(2ps)$, so that we can identify it with a subalgebra of the space of $2ps \times 2ps$ matrices. By considerations similar to those in 6.2.4, $\mathfrak{l}(1) /\!\!/ L(0) \simeq \mathfrak{g}(1) /\!\!/ G(0)$ and the grading is N-regular as

$$\sum_{\alpha \in \Delta} e_\alpha = e_{ps,ps+1} + \sum_{i=1}^{ps-1} e_{i,i+1} - e_{2ps-i,2ps-i+1}.$$

belongs to $\mathfrak{l}(1)$ and is regular nilpotent in $\mathfrak{l}$.

6.2.6 Type D

In 5.2.5.3 it was proved that the group $W_\mathfrak{t}$ in type D is isomorphic either to $S_s \ltimes \mathbb{Z}_2^{s-1}$ or $S_s \ltimes \mathbb{Z}_2^s$.

Assume we are in the first case, and $W_{\mathfrak{t}} \simeq S_s \ltimes \mathbb{Z}_2^{s-1}$. Then h can be taken in the form $\mathrm{diag}(F, H_1, \dots, H_s, H_s', \dots, H_1', F')$, an element that is G-conjugate to that in (5.6). The subset of simple roots is $\{\alpha_{n-sp+1}, \dots, \alpha_n\} \subseteq \Delta$. In this way we get a Levi subgroup of type D, since $[\mathfrak{l}, \mathfrak{l}] \simeq \mathfrak{so}(2ps)$. The toral element h belongs to $\mathfrak{l}$ and $W_{\mathfrak{t}}$ embeds into the Weyl group of L. Finally, this grading is N-regular since the element

$$\sum_{\alpha\in\Delta} e_{-\alpha} = e_{ps+1,ps-1} - e_{ps+2,ps} + \sum_{i=1}^{ps-1} e_{i+1,i} - e_{2ps-i+1,2ps-i}.$$

belongs to $\mathfrak{l}(1)$.

Assume now $W_{\mathfrak{t}} \simeq S_s \ltimes \mathbb{Z}_2^s$. In this case $\dim V(0) > s$, in particular $n > ps$. Suppose that h is in the form $\mathrm{diag}(H_1, \dots, H_s, F, F', H_s', \dots, H_1')$ just as in (5.6). Then F is a square matrix of size greater than or equal to 1, and moreover we can assume up to conjugacy that $F = \mathrm{diag}(\lambda_{ps-1}, \dots, \lambda_n)$ with $\lambda_i \in \mathbb{F}_p$ for all $i = ps-1, \dots, n$ and $\lambda_n = 0$.

Consider again the full orthogonal group $O(2n)$, that contains $G = SO(2n)$. The element

$$g = \left(\begin{array}{c|cc|c} Id_{n-1} & & & \\ \hline & 0 & 1 & \\ & 1 & 0 & \\ \hline & & & Id_{n-1} \end{array}\right)$$

belongs to $O(2n) \setminus SO(2n)$ (the entries left empty in the matrix above are intended to equal 0). Because of the description of F above, $h \in Lie(C_G(g))$.

Since the characteristic of the field k is a good prime for the root system, one has the identity $Lie\, C_G(g)^\circ = C_{\mathfrak{g}}(g)$, a Lie subalgebra of type B isomorphic to $\mathfrak{so}(2n-1)$. In order to se this fact directly, pick $X \in C_{\mathfrak{g}}(g)$. We write $X \in \mathfrak{so}(2n)$ as

$$X = \left(\begin{array}{ccc|cc|ccc} & & & e_1 & e'_1 & & & \\ & A & & \vdots & \vdots & & B & \\ & & & e_n & e'_n & & & \\ \hline g_1 & \dots & g_n & a_1 & a_2 & h_1 & \dots & h_n \\ g'_1 & \dots & g'_n & a'_1 & a'_2 & h'_1 & \dots & h'_n \\ \hline & & & f_1 & f'_1 & & & \\ & C & & \vdots & \vdots & & D & \\ & & & f_n & f'_n & & & \end{array}\right),$$

where A, B, C, D are $(n-2)\times(n-2)$ matrices and the entries of X satisfy the conditions given in 2.7.4. Then it is just a matter of computations to check that the condition $gXg^{-1} = X$ leads to:

- $a_1 = a_2 = a'_1 = a'_2 = 0$;
- $e_i = e'_i$, $f_i = f'_i$, $g_i = g'_i$, $h_i = h'_i$ for all $i = 1, \dots, n$;
- $e_i = -h_{n-i+1}$ for all $i = 1, \dots, n$;
- $g_i = -f_{n-i+1}$ for all $i = 1, \dots, n$.

By comparing these expressions with the conditions in 2.7.5 for a matrix to belong to $\mathfrak{so}(2n-1)$, the isomorphism is clear.

It follows that we can restrict to the group $\widetilde{G} = C_G(g)^\circ$. The element g being semisimple, $\widetilde{G}$ is a closed connected reductive subgroup of G of type B whose Lie algebra contains h and the groups of the form $W_\mathfrak{t}$ for the gradings on $\mathfrak{g}$ and $Lie\,\widetilde{G}$ are isomorphic. We can then resort to 6.2.4 to prove the existence of a Levi subgroup with the properties required.

6.2.7

The constructions above allow to prove the result sought for the existence of KW-sections.

Theorem 6.2.6. *Let G be a connected reductive algebraic group of classical type, and let h, $\mathfrak{g}(1)$ and $G(0)$ be as in Section 4.1. Then the action of $G(0)$ on $\mathfrak{g}(1)$ admits a KW-section.*

Proof. The discussion in 6.2.3, 6.2.4, 6.2.5 and 6.2.6 implies that for classical groups there exists a Levi subalgebra $\mathfrak{l} = Lie(L)$ containing h, such that $\mathfrak{l}(1)/\!\!/L(0) \simeq \mathfrak{g}(1)/\!\!/G(0)$ and the grading on $\mathfrak{l}$ is N-regular. Thanks to Corollary 6.2.5, the subspace $\mathfrak{l}(1)$ admits a KW-section for the action of $L(0)$, and the isomorphism above ensures that it is a KW-section also for the action of $G(0)$ on $\mathfrak{g}(1)$. □

6.2.8 Existence of KW-sections for exceptional Lie algebras

When $\mathfrak{g}$ is a $\mathbb{Z}_m$-graded simple Lie algebra of exceptional type, the existence of KW-sections in the case $\gcd(\operatorname{char} k, m) = 1$ has been proven in [RLYG] and [Le2]. We shall exploit some of the results in [RLYG], therefore a brief digression on the setting therein appears indispensable.

The gradings considered in [RLYG] come from finite order automorphisms of $\mathfrak{g}$, so assume $\theta \in Aut(\mathfrak{g})$ of order m is fixed. Following [Le1], the Cartan subspace is defined as a maximal abelian subspace $\mathfrak{c} \subseteq \mathfrak{g}(1)$ consisting of semisimple elements; all such subspaces are $G(0)$-conjugate. The *rank* of the automorphism θ (or, equivalently, of the grading), is defined as the dimension of a Cartan subspace. The rank of a grading is positive iff $\mathfrak{g}(1) \nsubseteq \mathcal{N}(\mathfrak{g}(1))$, otherwise θ is said to have rank 0. In [RLYG] gradings of positive rank of exceptional Lie algebras are classified.

Thanks to Remark 4.2.5, we only need to consider Lie algebras of type E_6, E_7 and E_8 with the relevant restrictions on p. Moreover, for any given good characteristic p, we can assume without loss of generality that there exists a subsystem of type A_{p-1} embedded in the Dynkin diagram of $\mathfrak{g}$. Assume this is given by the subset $J = \{\alpha_1, \dots, \alpha_{p-1}\} \subseteq \Delta$, ordered according to Bourbaki's numbering, then the Kac coordinates of h corresponding to simple roots in J are all equal to 1. We recall that this can only happen for $p = 5$ if $\mathfrak{g}$ is of type E_6, $p = 5$, 7 in type E_7 and $p = 7$ for E_8, hence for the remainder of this section we shall assume p is one of those primes.

All Levi subalgebras of type A_{p-1} of a Lie algebra of type E are conjugate. Indeed, this is the case for any two isomorphic Levi subalgebras in types E_6 and E_8; in type E_7 there are three pairs of non-conjugate isomorphic Levi subalgebras but none of them is of type A_{p-1}. As a consequence, it suffices to consider any embedding of A_{p-1} into the Dynkin diagram. As it emerges from [RLYG, Tables 16, 17 and 18], the corresponding gradings in non-torsion characteristic are of positive rank, more specifically they are all of rank 1. We will reproduce and combine hereafter the relevant rows from Tables $16, 17, 18, 19, 20$ and 21 in [RLYG] in order to make the discussion clearer.

The meaning of the entries in each table can be summarized as follows ([RLYG, Section 9]). The integer m is the order of the automorphism θ, hence $\mathfrak{g}$ is $\mathbb{Z}_m$-graded. Since $\gcd(\operatorname{char} k, m) = 1$, the automorphism θ is conjugate to an element w of the Weyl group of G, so that the grading is given by the adjoint action of w on $\mathfrak{g}$. The diagram $\operatorname{Kac}(w)_{un}$ contains the un-normalized Kac coordinates of w, that is, coordinates of a point in $\mathbb{R} \otimes Y(T)$ not necessarily belonging to the fundamental alcove. The symbol $*$ indicates that the corresponding Kac coordinate can take any integer value.

The diagrams in the column $\operatorname{Kac}(w)$ represent all the possible normalized Kac coordinates of the diagram $\operatorname{Kac}(w)_{un}$. The entry labeled $W(\mathfrak{c}, \theta)$ is the little Weyl group for the grading given by $\operatorname{Kac}(w)$; it is worth stressing that all these little Weyl groups are finite cyclic groups of order divisible by m.

The column θ' contains un-normalized Kac coordinates of an automorphism θ' conjugate to θ, with the property that θ' is N-regular on a Levi subalgebra L_θ whose type is given in the rightmost column. Indeed, in each case θ' has Kac coordinates equal to 1 on a standard Levi subalgebra of the type described. In [RLYG, Theorem 10.3] the authors prove that, upon restricting to the Levi subalgebra L_θ, one obtains an isomorphism $W_L(\mathfrak{c}, \theta) \simeq W(\mathfrak{c}, \theta)$, where $W_L(\mathfrak{c}, \theta)$ is the little Weyl group for the restriction of the grading to L_θ.

6.2.8.1 Type E_6

Case $m = 5$:

$\operatorname{Kac}(w)_{un}$	1 1 1 1 * * *

$\operatorname{Kac}(w)$	$W(\mathfrak{c}, \theta)$	θ'	L_θ
0 1 0 1 0 0 1	$\boldsymbol{\mu}_5$	1 1 1 1 1 1 -6	A_5

1 0 1 0 1 0 0	$\boldsymbol{\mu}_5$	1 1 1 1 1 2 -8	A_5
1 0 0 0 1 1 1	$\boldsymbol{\mu}_5$	1 1 1 1 1 3 -10	A_5

Table 6.2

6.2.8.2 Type E_7

Case $m = 7$:

Kac $(w)_{un}$	* 1 1 1 1 1 1 *

Kac(w)	$W(\mathfrak{c}, \theta)$	θ'	L_θ
0 1 0 1 0 0 1 0	$\boldsymbol{\mu}_{14}$	-10 1 1 1 1 1 1 1	E_7

Table 6.4

Case $m = 5$:

Kac $(w)_{un}$	* * 1 1 1 1 * *

Kac(w)	$W(\mathfrak{c}, \theta)$	θ'	L_θ
0 0 0 1 0 0 1 0	$\boldsymbol{\mu}_{10}$	-12 1 1 1 1 1 1 1	D_6

0 0 1 0 0 1 0 0	$\boldsymbol{\mu}_{10}$	-14 2 1 1 1 1 1 1	D_6
0 1 0 0 0 1 1 0	$\boldsymbol{\mu}_{10}$	-10 0 1 1 1 1 1 1	D_6
0 1 0 0 0 0 1 1	$\boldsymbol{\mu}_5$	-11 1 1 1 1 1 0 0	A_4
1 0 0 0 1 0 1 0	$\boldsymbol{\mu}_5$	-11 1 1 1 1 0 2 1	A_4

Table 6.6

6.2.8.3 Type E_8

Case $m = 5$:

Kac $(w)_{un}$	1 1 1 1 1 1 * * *

Kac(w)	$W(\mathfrak{c}, \theta)$	θ'	L_θ
0 0 0 1 0 0 1 0 0	$\boldsymbol{\mu}_{14}$	1 1 1 1 1 1 1 -22 1	E_7
0 0 1 0 0 0 0 1 0	$\boldsymbol{\mu}_{14}$	1 1 1 1 1 1 3 -26 1	E_7
0 1 0 0 0 1 0 0 0	$\boldsymbol{\mu}_{14}$	1 1 1 1 1 1 2 -24 1	E_7

1 0 0 0 1 0 0 1 0	$\boldsymbol{\mu}_{14}$	1 1 1 1 1 1 4 -28 1	E_7

Table 6.8

We remark that the tables above are valid also when the characteristic is not coprime with the order of the grading. This is because conjugacy of elements corresponding to certain Kac coordinates (finite order automorphisms or toral elements) is independent of the characteristic of the field of definition.

6.2.8.4 Action of the group $W_{\mathfrak{t}}$. The difficulty in applying the arguments of [RLYG] directly to our setting arises from the fact that the descriptions of little Weyl groups, p-cyclic subspaces and Cartan subspaces differ significantly.

The discussion in section 5.2.3 provides very strict limitations on the structure of the group $W_{\mathfrak{t}}$ for exceptional Lie algebras. Indeed, this turns out to be a subgroup of $\boldsymbol{\mu}_2$, the cyclic group of order 2.

In case $W_{\mathfrak{t}}$ is the trivial group and $x \in \mathfrak{g}(1)_{reg}$, one can restrict to the subalgebra $\mathfrak{l}_1(x) \simeq \mathfrak{gl}_p$. It follows immediately that a section for the action of $G(0)$ on $\mathfrak{g}(1)$ is given by the subspace $\lambda e_{\alpha_0} + \sum_{j=1}^{p-1} e_{\alpha_j}$, where $\lambda \in k$, and the notations are the same as in Lemma 4.2.8.

If this is not the case and $W_{\mathfrak{t}} \simeq \boldsymbol{\mu}_2$, we need to analyze further the action of the element of order 2, call it $\omega \in W_{\mathfrak{t}}$. The element ω fixes h and acts as $-Id$ on the 1-dimensional torus $\mathfrak{t}$. It fixes $\mathfrak{l}_1(x) \simeq \mathfrak{gl}_p$ (and each of its graded components); it acts therefore as an automorphism of $\mathfrak{gl}_p$.

Let $\mathfrak{g}_A$ be a Lie algebra of type A_n for $n \in \mathbb{N}$ defined over k. Assume we are in one of our cases of interest for this chapter, so that $\mathfrak{g}_A$ is one of the subalgebras $\mathfrak{l}_1(x)$ arising from a graded Lie algebra of type E, $p \geq 5$ and p divides $n+1$. Under these conditions, the automorphism group of $\mathfrak{g}_A$ consists of the union of two connected components (see [Ho] and [St1]). The component containing the identity, let us call it G_A, is the adjoint group with Lie algebra $\mathfrak{g}_A$, whereas the other component is the coset $G_A\nu$, where ν is the nontrivial graph automorphism of the Dynkin diagram.

We will let $W_1 \subseteq L_1(x)$ stand for the Weyl group of the Lie algebra $\mathfrak{l}_1(x) \simeq \mathfrak{gl}_p$, this is isomorphic to S_p, the symmetric group on p elements. We saw already that the projection h' of the element h on $\mathfrak{l}_1(x)$ is regular in $\mathfrak{gl}_p$, and its centralizer in $L_1(x)$ is a maximal torus T_1. If ω is an inner automorphism of $\mathfrak{l}_1(x)$, we can assume up to conjugation that its action coincides with that of an element of T_1. Since ω stabilizes both h and $C_{\mathfrak{l}_1(x)}(h) = \mathfrak{t}'$, a maximal toral subalgebra of $\mathfrak{gl}_p$, but it does not act trivially on $\mathfrak{t}'$, it follows that $\omega \in G_A\nu$ is an outer automorphism of $\mathfrak{gl}_p$. The element ν can be chosen to be the automorphism of $\mathfrak{gl}_p$ that acts on an $p \times p$ matrix B as:

$$\nu(B) = -J_p \cdot B^T \cdot J_p^{-1}.$$

Here the superscript ^{T} indicates matrix transposition and J_p has been defined in 2.7. This fact follows, for example, by taking differentials in the description from [Le1, Section 4.6, Lemma 4.15]. In particular, ν acts on each coroot as $\nu(h_{\alpha_i}) = h_{\alpha_{p-i}}$.

Assume the projection of h on $\mathfrak{gl}_p$ is $H = \operatorname{diag}(p-1, p-2, \ldots, 1, 0)$. The element ω can be written in the form $g\nu$ for $g \in G_A$. As both ω and ν preserve a fixed maximal toral subalgebra $\mathfrak{t}'$ of $\mathfrak{gl}_p$ (diagonal matrices under these assumptions), we can write $\omega = tw\nu$, where $w \in W_1$ and t is an element of a maximal torus of G_A whose Lie algebra contains $\mathfrak{t}'$. Since elements of $W_{\mathfrak{t}}$ are defined up to the action of a maximal torus (see 5.2), it is not restrictive to assume $\omega = w\nu$ with the same meaning of symbols.

Since h is regular in $\mathfrak{gl}_p$ and the Weyl group of $\mathfrak{gl}_p$ acts by permuting eigenvalues of a diagonal matrix, there exists a unique element $w \in W_1$ such that $w\nu \cdot h = h$. Using the form of h above, w must be the permutation $(1,\ p,\ p-1, \ldots, 3,\ 2) \in S_p$. This can be realized on $\mathfrak{gl}_p$ as conjugation by the matrix:

$$g_w = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & 0 \\ 0 & & & & 1 \\ 1 & 0 & \ldots & & 0 \end{pmatrix}.$$

The action of the element ω on the p-cyclic subspace can be therefore visualized as follows:

$$-g_w \cdot J_p \cdot \begin{pmatrix} 0 & \lambda_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & 0 \\ 0 & & & & \lambda_{p-1} \\ \lambda_p & 0 & \dots & & 0 \end{pmatrix}^{\mathrm{T}} \cdot J_p \cdot g_w^{-1} =$$

$$\begin{pmatrix} 0 & -\lambda_{p-2} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ & & \ddots & -\lambda_1 & 0 \\ 0 & & & & -\lambda_p \\ -\lambda_{p-1} & 0 & \dots & & 0 \end{pmatrix}.$$

Call α_0 the negative of the highest root in the root system of type A_{p-1}. By the above, and because $\omega \in W$ (Lemma 5.2.1), it is possible to choose suitable vectors $X_{\alpha_0}, X_{\alpha_1}, \dots, X_{\alpha_{p-1}} \in \mathfrak{gl}_p(\mathbb{Z})$ defined over the integers, such that $\omega X_{\alpha_0} = -X_{\alpha_{p-1}}$, $\omega X_{\alpha_{p-1}} = -X_{\alpha_0}$ and $\omega X_{\alpha_i} = -X_{\alpha_{p-1-i}}$ for $i = 1, \dots, p-2$ (see for example [St2, Lemma 19]).

6.2.8.5 Base change to non-torsion characteristic. Now consider an algebraically closed field K of good characteristic for the root system of G and such that $\operatorname{char} K \neq p$. Let G_K denote the algebraic group of the same type as G defined over K, and let $\mathfrak{g}_K$ be its Lie algebra. Fix a maximal torus T_K and a Borel subgroup $B_K \supseteq T_K$, this amounts to choosing a base of simple roots; notice that $W \simeq W_K = N_{G_K}(T_K)/C_{G_K}(T_K)$. In order to avoid a proliferation of subscripts, we will use a superscript as in L^K to refer to a Levi subgroup of G_K. Consider the automorphism θ of $\mathfrak{g}_K$ of order p defined by the same Kac coordinates as h, so that the action of θ on $\mathfrak{g}_K$ coincides with the adjoint action of an element of T_K. This gives an $\mathbb{F}_p$-grading of $\mathfrak{g}_K$, where each root subspace for T_K is included in a graded component. One can look at the action of $G_K(0) = C_{G_K}(\theta)^\circ$ on $\mathfrak{g}_K(1)$. Call $W_K(\mathfrak{c})$ the little Weyl group for the action of $G_K(0)$ on $\mathfrak{g}_K(1)$.

Lemma 6.2.7. *If θ is of rank 1, the group $W_{\mathfrak{t}}$ embeds into $W_K(\mathfrak{c})$.*

Proof. Consider the element ω belonging to W, the Weyl group of G_K. It commutes with θ since it fixes its Kac coordinates (recall that ω stabilizes $h \in \mathfrak{g}$), hence $\omega \in G_K(0)$. Restrict to the standard Levi subalgebra of $\mathfrak{g}_K$ of type A_{p-1} corresponding to the subset $J \subseteq \Delta$ defined in the third paragraph of Section 6.2.8. Let $e = X_{\alpha_0} + X_{\alpha_1} + \ldots + X_{\alpha_{p-1}} \in \mathfrak{g}_K$ as introduced at the end of Section 6.2.8.4. Then $\omega e = -e$ because the action of the Weyl group on this element is independent of the characteristic of the field (see for example [St2, Lemma 19]). But the element e is a semisimple element of $\mathfrak{g}_K$ since $\operatorname{char} K \neq p$. In particular, e spans a Cartan subspace of $\mathfrak{g}_K(1)$ because θ has rank 1 in this case. This entails that ω commutes with θ and acts as $-Id$ on a Cartan subspace of $\mathfrak{g}_K(1)$, hence the result. □

It is worth stressing that this result does not require $\mathfrak{g}$ to be of exceptional type, but holds more generally for any grading of rank 1. For our cases of interest, the groups $W_K(\mathfrak{c})$ are those denoted $W(\mathfrak{c}, \theta)$ in Tables 6.2, 6.4, 6.6 and 6.8. This, combined with Lemma 6.2.7, implies immediately that $W_{\mathfrak{t}}$ is the trivial group for type E_6 in characteristic 5 and for some gradings in type E_7 and characteristic 5. More generally, we are now able to prove the existence of KW-sections for exceptional Lie algebras.

Theorem 6.2.8. *Let G be a connected reductive algebraic group of exceptional type, and let h, $\mathfrak{g}(1)$ and $G(0)$ be as in 4.1. Then the action of $G(0)$ on $\mathfrak{g}(1)$ admits a KW-section.*

Proof. The possible gradings for these Lie algebras are summarized in Tables 6.2, 6.4, 6.6 and 6.8. Lemma 6.2.7 shows that $W_{\mathfrak{t}} \hookrightarrow W_K(\mathfrak{c})$. Moreover, thanks to the results in [RLYG] discussed at the beginning of Section 6.2.8, one obtains an isomorphism $W_{L^K_\theta}(\mathfrak{c}) \simeq W(\mathfrak{c}, \theta)$ upon restricting to the Levi subgroup L^K_θ (in Tables 6.2, 6.4, 6.6 and 6.8 this was called L_θ); here $W_{L^K_\theta}(\mathfrak{c})$ is the little Weyl group for the grading on $\mathfrak{l}^K_\theta = Lie L^K_\theta$. We recall that L^K_θ was chosen so that the grading on $\mathfrak{l}^K_\theta$ given by a conjugate θ' of θ is N-regular ([RLYG, Theorem 10.3]) and θ' is again of rank 1; in particular $\mathfrak{l}^K_\theta(1)$ contains a Cartan subspace of $\mathfrak{g}_K(1)$. We can take L^K_θ to be in the form of a standard Levi subgroup given by a certain subset $I \subsetneq \Delta$. When the field of definition is k, we can restrict to the Levi subgroup of G corresponding again to $I \subseteq \Delta$, call it L_h. The translation of the argument above to $\operatorname{char} k = p$ is the following: up to replacing h with a conjugate h' (having the same Kac coordinates as θ'), we can restrict to a Levi subgroup L_h whose Lie algebra admits an N-regular grading for h'. Moreover, the Lie algebra $\mathfrak{l}_h = Lie L_h$ contains a p-cyclic subspace of $\mathfrak{g}(1)$ since a Lie subalgebra of type A_{p-1} can be embedded in its Dynkin diagram (see column labeled L_θ in Tables 6.2, 6.4, 6.6 and 6.8). As in the discussion preceding 6.2.8.5, we can assume that this subsystem is defined by simple roots $\alpha_1, \ldots, \alpha_{p-1}$, with α_0 the negative of the highest root. Once vectors $X_{\alpha_0}, X_{\alpha_1}, \ldots, X_{\alpha_{p-1}} \in \mathfrak{g}(\mathbb{Z})$ defined over the integers have been chosen (see discussion at the end of Section 6.2.8.4), we can consider the element $e = X_{\alpha_0} + X_{\alpha_1} + \ldots + X_{\alpha_{p-1}} \in \mathfrak{l}_h$ as in the proof of Lemma 6.2.7. We will let $e_K \in \mathfrak{l}_\theta^K$ be the analogue of the element e when the field of definition is K.

The K-span of the element $e_K \in \mathfrak{l}_\theta^K$ is a Cartan subspace of $\mathfrak{g}_K(1)$, while the subspace $\text{span}_k\langle X_{\alpha_0}, X_{\alpha_1}, \ldots, X_{\alpha_{p-1}}\rangle \subseteq \mathfrak{l}_h(1)$ is a p-cyclic subspace of $\mathfrak{g}(1)$. Therefore, we can always find a Cartan subspace and a p-cyclic subspace, denoted $\mathfrak{c}$ in both cases, belonging to $\mathfrak{l}_\theta^K$ and $\mathfrak{l}_h$ respectively.

If $W_\mathfrak{t}$ is the trivial group, the existence of a KW-section follows immediately by restricting to the group L_h and applying Corollary 6.2.5.

Therefore, assume this is not the case and $W_\mathfrak{t} \simeq \boldsymbol{\mu}_2$. This means, thanks to Lemma 5.2.1, that there exists an element $w \in W$ acting on e_K as multiplication by -1. Owing to the isomorphism $W_{L_\theta^K}(\mathfrak{c}) \simeq W_K(\mathfrak{c})$, we can assume $w \in W_{L_\theta^K}$, the Weyl group of L_θ^K. The Weyl groups $W_{L_\theta^K}$ and W_{L_h} of L_θ^K and L_h respectively are isomorphic, so we can consider the corresponding $w \in W_{L_h}$, that exists thanks to Lemma 6.2.7 and for which we use the same symbol. Since the X_{α_i}'s are defined over the integers, as discussed in the proof of Lemma 6.2.7, the action of the Weyl group on these element is independent of the characteristic of the field of definition. As a consequence, $we = -e \in \mathfrak{l}_h$. Hence, $(W_{L_h})_\mathfrak{t} \simeq \boldsymbol{\mu}_2 \simeq W_\mathfrak{t}$ because we can take $\mathfrak{t} = k\, e^{[p]}$. This in turn provides an isomorphism of varieties $\mathfrak{l}_h(1)/\!\!/L_h(0) \simeq \mathfrak{g}(1)/\!\!/G(0)$. Since the grading on $\mathfrak{l}_h$ is N-regular, the result follows from Corollary 6.2.5. □

6.2.9 Existence of KW-sections: statement of results

Combining the results in Section 6.2.2 and Section 6.2.8 we obtain the existence of KW-sections in the general case. Here we will retain assumptions from Section 5.3.2. Theorem 6.2.6 and Theorem 6.2.8 together imply:

Theorem 6.2.9. *If G is a connected reductive algebraic group satisfying the standard hypothesis, the action of $G(0)$ on $\mathfrak{g}(1)$ admits a KW-section.*

Appendix A

Example of code (type F_4)

This stems from the discussion in Section 3.1.5. We write down the full code for the case of a root system of type F_4.

```
#include <stdio.h>
#include <math.h>


int main(){


int s0=1;
int s1=2;
int s2=3;
int s3=4;
int s4=2; //s0,..., s4 coefficients of highest root
int M[24][4];
int a[5]={0}; //Kac coordinates initialized at 0
int v[49]={0};
int p[13];
int O[5][15];
int d;
int i=0,j=0,k,l=0,m;
int r0=1, r1=1, r2=1;
```

```
M[0][0]=1;  M[0][1]=0;  M[0][2]=0;  M[0][3]=0;
M[1][0]=0;  M[1][1]=1;  M[1][2]=0;  M[1][3]=0;
M[2][0]=0;  M[2][1]=0;  M[2][2]=1;  M[2][3]=0;
M[3][0]=0;  M[3][1]=0;  M[3][2]=0;  M[3][3]=1;
M[4][0]=1;  M[4][1]=1;  M[4][2]=0;  M[4][3]=0;
M[5][0]=0;  M[5][1]=1;  M[5][2]=1;  M[5][3]=0;
M[6][0]=0;  M[6][1]=0;  M[6][2]=1;  M[6][3]=1;
M[7][0]=1;  M[7][1]=1;  M[7][2]=1;  M[7][3]=0;
M[8][0]=0;  M[8][1]=1;  M[8][2]=2;  M[8][3]=0;
M[9][0]=0;  M[9][1]=1;  M[9][2]=1;  M[9][3]=1;
M[10][0]=1; M[10][1]=1; M[10][2]=2; M[10][3]=0;
M[11][0]=1; M[11][1]=1; M[11][2]=1; M[11][3]=1;
M[12][0]=0; M[12][1]=1; M[12][2]=2; M[12][3]=1;
M[13][0]=1; M[13][1]=2; M[13][2]=2; M[13][3]=0;
M[14][0]=1; M[14][1]=1; M[14][2]=2; M[14][3]=1;
M[15][0]=0; M[15][1]=1; M[15][2]=2; M[15][3]=2;
M[16][0]=1; M[16][1]=2; M[16][2]=2; M[16][3]=1;
M[17][0]=1; M[17][1]=1; M[17][2]=2; M[17][3]=2;
M[18][0]=1; M[18][1]=2; M[18][2]=3; M[18][3]=1;
M[19][0]=1; M[19][1]=2; M[19][2]=2; M[19][3]=2;
M[20][0]=1; M[20][1]=2; M[20][2]=3; M[20][3]=2;
M[21][0]=1; M[21][1]=2; M[21][2]=4; M[21][3]=2;
M[22][0]=1; M[22][1]=3; M[22][2]=4; M[22][3]=2;
M[23][0]=2; M[23][1]=3; M[23][2]=4; M[23][3]=2;


O[0][0]=0;    O[1][0]=0;    O[2][0]=0;    O[3][0]=2;    O[4][0]=30;
O[0][1]=2;    O[1][1]=0;    O[2][1]=0;    O[3][1]=0;    O[4][1]=30;
O[0][2]=2;    O[1][2]=0;    O[2][2]=0;    O[3][2]=2;    O[4][2]=40;
O[0][3]=0;    O[1][3]=0;    O[2][3]=2;    O[3][3]=0;    O[4][3]=40;
O[0][4]=0;    O[1][4]=2;    O[2][4]=0;    O[3][4]=0;    O[4][4]=40;
O[0][5]=0;    O[1][5]=0;    O[2][5]=2;    O[3][5]=2;    O[4][5]=42;
O[0][6]=2;    O[1][6]=2;    O[2][6]=0;    O[3][6]=0;    O[4][6]=42;
```

```
O[0][7]=0;    O[1][7]=2;    O[2][7]=2;    O[3][7]=0;    O[4][7]=44;
O[0][8]=0;    O[1][8]=2;    O[2][8]=0;    O[3][8]=2;    O[4][8]=44;
O[0][9]=2;    O[1][9]=0;    O[2][9]=2;    O[3][9]=0;    O[4][9]=44;
O[0][10]=0;   O[1][10]=2;   O[2][10]=2;   O[3][10]=2;   O[4][10]=46;
O[0][11]=2;   O[1][11]=0;   O[2][11]=2;   O[3][11]=2;   O[4][11]=46;
O[0][12]=2;   O[1][12]=2;   O[2][12]=2;   O[3][12]=0;   O[4][12]=46;
O[0][13]=2;   O[1][13]=2;   O[2][13]=0;   O[3][13]=2;   O[4][13]=46;
O[0][14]=2;   O[1][14]=2;   O[2][14]=2;   O[3][14]=2;   O[4][14]=48;


p[0]=5; p[1]=7; p[2]=11; p[3]=13; p[4]=17; p[5]=19; p[6]=23; p[7]=27;
p[8]=31; p[9]=37; p[10]=41; p[11]=43; p[12]=47;


for(l=0;l<12;l++){ //loop on l=characteristic of the field


  for(m=0;m<15;m++){ //loop on Richardson orbit



  if(!(O[4][m]%(p[l]-1))){ //if orbit dimension is divisible by l-1

    k=O[4][m]/(p[l]-1); //required dimension
    for(i=0;i<49;i++){v[i]=0;}
    a[0]=0;
    a[1]=0;
    a[2]=0;
    a[3]=0;
    a[4]=0;
    r0=1; r1=1; r2=1;


for(i=0;i<4;i++){if(O[i][m]){a[i+1]=1;}} //if a simple root is not
//in the centralizer, the corresponding Kac coordinate is >0


while(s1*a[1]+s2*a[2]+s3*a[3]+s4*a[4]<p[l]){
 while((s1*a[1]+s2*a[2]+s3*a[3]+s4*a[4]<p[l])&&(r2)){
```

```
while((s1*a[1]+s2*a[2]+s3*a[3]+s4*a[4]<p[l])&&(r1)){
 while((s1*a[1]+s2*a[2]+s3*a[3]+s4*a[4]<p[l])&&(r0)){

   a[0]=p[l]-s1*a[1]-s2*a[2]-s3*a[3]-s4*a[4];

  for(i=0;i<24;i++){ //compute dimension eigenspaces
             //for given Kac coordinates a[1],...,a[4]
   j=M[i][0]*a[1]+M[i][1]*a[2]+M[i][2]*a[3]+M[i][3]*a[4];
   v[j]=v[j]+1;
   v[p[l]-j]=v[p[l]-j]+1;
  }

  j=1;

  for(i=1;i<p[l];i++){

   if(v[i]>k){ j=0; }
   if(!v[j]){ j=0; } //all eigenespaces relative to strictly positive
                  eigenvalues have to be nontrivial
     }

      if(j){printf("Characteristic %d\nKac coordinates\n",p[l]);

      for(i=0;i<5;i++){printf("%d\n",a[i]);}

      printf("\n\n");

           }

      for(i=0;i<p[l];i++){v[i]=0;}

 if(O[0][m]){a[1]++;}else{r0=0;} //if root alpha_1 is
       //in the centralizer, the fourth loop runs only once
```

```
      } //end fourth while loop
     r0=1;

     if((s1*(O[0][m]/2)+s2*(a[2]+1)+s3*a[3]+s4*a[4]<p[l])
         &&(O[1][m])){ //if a[2] can be incresed further
          if(O[0][m]){a[1]=1;} }

      if(O[1][m]){a[2]++;}else{r1=0;}

     } //end third while loop
r1=1;

    if((s1*(O[0][m]/2)+s2*(O[1][m]/2)+s3*(a[3]+1)+s4*a[4]
         <p[l])&&(O[2][m])){
          if(O[0][m]){a[1]=1;}
          if(O[1][m]){a[2]=1;}
            }

       if(O[2][m]){a[3]++;}else{r2=0;}

   } //end second while loop

r2=1;

if(O[3][m]){

   if((s1*(O[0][m]/2)+s2*(O[1][m]/2)+s3*(O[2][m]/2)
         +s4*(a[4]+1)<p[l])&&(O[3][m])){
          if(O[0][m]){a[1]=1;}
          if(O[1][m]){a[2]=1;}
          if(O[2][m]){a[3]=1;}
          }
```

```
        }
     a[4]++;
    } /end first while loop

    } //end if in loop on m
   } //loop on m
  } //loop on l
 }
```

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