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Efficient Linear System Solution Techniques in the Simulation of Large Dense Mutually Inductive Circuits

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Abstract—The verification of Integrated Circuits (ICs) in deep submicron technologies requires that all mutual inductive effects are taken into account to properly validate the performance and reliable operation of the chip. However, the inclusion of all mutual inductive couplings results in a fully dense inductance matrix that renders the circuit simulation computationally prohibitive. In this paper, we present efficient techniques for the solution of the linear systems arising in transient analysis of large mutually inductive circuits. These techniques involve the compression of the dense inductance matrix block by low-rank products in hierarchical matrix format, as well as the development of a Schur-complement preconditioner for the iterative solution of the transient system (which comprises sparse blocks along its diagonal). Experimental results indicate that substantial compression rates of the inductance matrix can be achieved without compromising accuracy, along with considerable reduction in iteration counts and execution time of iterative solution methods.

Index Terms—RLC Simulation, mutual inductance, Hierarchical Matrices, Preconditioning, Schur Complement, Krylov methods

I. INTRODUCTION

In high-frequency integrated circuits, parasitic inductance becomes prominent and needs to be considered in simulation to accurately predict the behavior and verify the reliability of the chip. However, the simulation of large RLC parasitic networks can be extremely time-consuming or, in some cases, even infeasible due to their sheer size.

In the past, it has been sufficient to include only self-inductance in the RLC simulation, resulting in large linear systems with sparse coefficient matrices. For such systems, iterative solvers are the methods of choice offering small memory requirements and excellent scalability. However, in modern high-frequency ICs there is an increasing demand for modeling all mutual inductive couplings between the different circuit blocks, leading to a fully dense inductance matrix that affects the scaling properties and storage requirements of iterative solvers. As the convergence rate of iterative methods cannot be predicted a-priori, the use of appropriate preconditioners is necessary to ensure a robust and fast convergence across a wide range of problems.

As direct sparsification (by truncation) of the dense inductance matrix is well-known to lead to loss of circuit passivity and simulation stability [1], most previous attempts in the literature have focused on sparsifying its inverse (called the reluctance matrix), which is amenable to sparsification due to its diagonal dominance [2] [3] [4]. However, this requires prior inversion of a large dense matrix which is a very expensive computational procedure, making the reluctance-based approaches feasible only for problems of moderate sizes. An approach that avoids the expensive inversion of the inductance matrix, by computing selectively only some of the entries of the reluctance matrix up to a given sparsity ratio, has been proposed in [5]. However, this approach introduces error that is difficult to quantify in advance, thus limiting its application in practical settings.

In this paper, we propose the compression (instead of sparsification) of the actual inductance matrix via the approximation of large off-diagonal blocks by low-rank products, in a scheme known as hierarchical matrix (or H-matrix) format. This format conserves memory by storing only the low-rank factors of the blocks, while enabling the execution of basic matrix-vector operations required by iterative solvers in near optimal complexity. Moreover, we propose a block preconditioner based on an efficient approximation of the Schur complement, which improves considerably the convergence rate of iterative methods and has inexpensive application. Experimental results demonstrate significant reduction in memory footprint and execution time with negligible loss of accuracy, in comparison to existing dense linear solvers in typical simulation scenarios.

The rest of the paper is organized as follows: Section II offers an overview of transient RLC analysis, iterative solvers and hierarchical matrix format. Section III introduces hierarchical matrices as an approximation for the dense inductance matrix. Section IV presents the proposed preconditioning scheme for the simulation of mutually inductive RLC circuits. Section V provides experimental results that demonstrate the efficiency of the proposed methodology, while conclusions are drawn in section VI.

II. BACKGROUND

A. Overview of RLC transient analysis

Consider an RLC circuit composed of n nodes and m inductive branches with mutual inductive coupling between them, as well as its Modified Nodal Analysis (MNA) description [6]:

\[ G x(t) + C \dot{x}(t) = \ddot{e}(t) \]

(1)

where

\[
G = \begin{bmatrix} G & A_L \\ -A_L^T & 0 \end{bmatrix}, \quad C = \begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix}, \quad \dot{x}(t) = \begin{bmatrix} v(t) \\ i(t) \end{bmatrix}, \quad \ddot{e}(t) = \begin{bmatrix} e(t) \\ 0 \end{bmatrix}
\]

In the above, \( G \in \mathbb{R}^{n\times n} \) and \( C \in \mathbb{R}^{n\times n} \) are the node conductance and node capacitance matrices respectively, \( L \in \mathbb{R}^{m\times m} \) is the dense inductance matrix (with self-inductances as diagonal entries and mutual inductances as off-diagonal entries), \( A_L \in \mathbb{R}^{n\times m} \) is the corresponding node-to-rack connection matrix, \( v(t) \in \mathbb{R}^{n} \) and \( i(t) \in \mathbb{R}^{m} \) are the vectors of the unknown node voltages and branch currents, and \( \ddot{e}(t) \in \mathbb{R}^{m} \) is the vector of excitations from independent sources at the nodes (assuming, without loss of generality, that voltage sources have been transformed to Norton-equivalent current sources). Applying the Backward-Euler numerical integration method in (1), we arrive at the problem of solving a system of \( n + m \) linear algebraic equations at each discrete time \( t_k, k = 1, 2, \ldots \) (starting with initial values \( x(t_0) = [v(0) \quad i(0)]^T \)):

\[
J_x x(t_k) = b(t_k)
\]

(2)

where

\[
J_x = \begin{bmatrix} \frac{1}{\tau} C + G & A_L \\ -A_L^T & \frac{1}{\tau} L \end{bmatrix}, \quad b(t_k) = \ddot{e}(t_k) + \frac{C}{h_k} x(t_{k-1})
\]
and \( h_k = t_k - t_{k-1}, k = 1, 2, \ldots \) is the chosen time-step size (which can be either fixed or variable during the analysis). The above is a linear system of the form \( Ax = b \) that has to be solved at every discrete time \( t_k, k = 1, 2, \ldots \).

### B. Iterative linear solvers

Direct methods (based on matrix factorization) are not practically feasible for solving large dense systems or systems with a large dense block (like \( \frac{1}{h}L \) in (2)), due to their excessive runtime and memory requirements. The only viable option for such systems is the use of iterative methods, and particularly Krylov-subspace iterative methods like GMRES (which is suitable for general unsymmetric systems like (2)) [7]. The operation with the dominant cost inside the iteration loop of Krylov-subspace methods (and GMRES in particular) is the matrix-vector multiplication, which for the system (2) can be dealt efficiently by the hierarchical matrix framework that is introduced in the next subsection and applied to the inductance matrix \( L \) in Section III. Other operations of Krylov-subspace iterative methods like inner products, scalar-vector products and vector additions are not expensive computationally.

The convergence rate of Krylov-subspace methods is determined by the spread of the eigenvalues of the system matrix and their distance from 1 [8]. In particular, convergence is fast when the eigenvalues are tightly clustered together and slow when they are spread apart. A slow convergence rate can be alleviated by using a preconditioner matrix \( M \) and to solve an equivalent system \( M^{-1}Ax = M^{-1}b \).

An effective preconditioner needs to satisfy the following two prerequisites:

- Preconditioning should lead to a much tighter clustering of the eigenvalues of \( M^{-1}A \) than those of the original matrix \( A \), thus reducing significantly the iteration counts.
- The reduction in the number of iterations should offset the computational overhead introduced by the solution of \( Mz_j = r_j \) in every iteration.

### C. Low-rank product approximation and hierarchical matrices

If \( A \in \mathbb{R}^{n \times n} \) is a square matrix or matrix block with Singular Value Decomposition (SVD) \( A = U \Sigma V^T \), where \( \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n) \) and \( \sigma_1 > \cdots > \sigma_n \), then by introducing the following partition:

\[
U = [U_1 \ U_2], \quad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \quad V^T = \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}
\]

with \( U_1 \in \mathbb{R}^{n \times r}, \Sigma_1 \in \mathbb{R}^{r \times r} \) and \( V_1^T \in \mathbb{R}^{r \times n} \), the optimal low-rank product approximation of rank-\( r \) of \( A \) is defined as \( \tilde{A} = (U_1 \Sigma_1^{1/2}) (V_1 \Sigma_1^{1/2})^T = Z Y^T \). It has been proven in [9] that this approximation satisfies the optimization problem:

\[
\min \| A - \tilde{A} \| \quad \text{s.t. rank}(\tilde{A}) = r \quad \text{for any common matrix norm.}
\]

The benefit of the factorization \( \tilde{A} = Z Y^T \) with \( Z = U_1 \Sigma_1^{1/2} \) and \( Y = V_1 \Sigma_1^{1/2} \) is that only the factors \( Z \) and \( Y \) have to be kept in memory instead of the whole \( n \times n \) matrix \( A \) (see Fig. 1a). The above low-rank product approximation can be straightforwardly extended to rectangular matrix blocks.

Hierarchical matrices or \( H \)-matrices [10] are a lossy compressed matrix format which relies on the partitioning of a dense matrix into a number of sub-matrix blocks that can be approximated efficiently and accurately by low-rank products. The special structure of \( H \)-matrices allows the development of algorithms for the basic operations of matrix-vector multiplication and matrix factorization with near optimal asymptotic complexity.

### III. APPROXIMATION OF THE INDUCTANCE MATRIX WITH HIERARCHICAL MATRICES

Because of the natural fact that interconnect segments which are farther apart exhibit weaker mutual inductive interactions, the inductance matrix \( L \) will be characterized by progressively smaller off-diagonal elements while moving away from the diagonal (assuming that segments in close physical proximity are enumerated consecutively - otherwise suitable permutation matrices can be applied). Then the matrix blocks that are away from the diagonal can be efficiently approximated by low-rank products and the whole inductance matrix by an appropriate \( H \)-matrix (see Fig. 1b). The size and the number of blocks, as well as the order of the low-rank approximation of each block, constitute trade-off parameters between the degree of compression and the quality of approximation.

It is noted that the usage of \( H \)-matrix format does not require the \( a \)-priori knowledge of the whole inductance matrix. Instead, only the spatial arrangement of the interconnects, which is available before the assembly of the inductance matrix, is required to perform the matrix blocking. Thus, provided that we integrate the \( H \)-matrix library with the inductance extraction tool, we can approximate a matrix block by a low-rank product immediately after its computation, and next store it directly in \( H \)-matrix format rather than as part of a dense inductance matrix.

### IV. SOLUTION AND PRECONDITIONING OF THE TRANSIENT LINEAR SYSTEM

#### A. Multiplication of transient system matrix with a vector

The matrix \( J_k \) is composed of different storage formats, with \( G + \frac{1}{h}C \) and \( A_L \) being in sparse format (compressed row or column form) and \( L \) in \( H \)-matrix format. Thus, the multiplication of \( J_k \) with a vector inside the iteration loop of a Krylov-subspace method like GMRES, can be performed in a block fashion by calling the appropriate sparse and \( H \)-matrix subroutines.

#### B. Preconditioner formulation

Consider the block LU factorization of \( J_k \) of (2):

\[
J_k = L_k U_k = \begin{bmatrix} -A_L^T (G + \frac{1}{h_k}C)^{-1} & 0 \\ A_L & S \end{bmatrix}
\]

where

\[
S = \frac{1}{h_k} L_k + A_L^T (G + \frac{1}{h_k} C)^{-1} A_L
\]

is the Schur complement of \( J_k \). Because \( L_k \) is a block lower triangular matrix with identity blocks on the main diagonal, and thus has all eigenvalues equal to 1, the block upper triangular matrix

\[
U_k = \begin{bmatrix} G + \frac{1}{h_k} C & A_L \\ 0 & S \end{bmatrix}
\]

constitutes an ideal preconditioner for \( J_k \) (meaning that GMRES will converge in only one iteration).
C. Preconditioner application

The preconditioning step in the body of Krylov methods involves the solution of the following linear system:

$$ U_{21} x = r \Rightarrow \begin{bmatrix} G + \frac{1}{h_k} C & A_L \\ 0 & S \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \tag{4} $$

The system (4) has a block upper triangular coefficient matrix and its solution can be achieved by block back substitution. Specifically, we first solve $S z_2 = r_2$ (details are given in the next paragraph), then update the right hand side as $r_1 = r_1 - A_L z_2$, and finally solve $(G + \frac{1}{h_k} C) z_1 = r_1$. The latter is a $n \times n$ sparse linear system which can be solved by a direct or iterative sparse linear solver. Since it can be demonstrated that $G + \frac{1}{h_k} C$ is a symmetric diagonally dominant matrix with non-positive off-diagonal elements [11], it is recommended to use iterative methods for which very efficient preconditioners have been developed [12], [13].

The Schur complement $m \times m$ system $S z_2 = r_2$ has coefficient matrix with two additive terms, $S_1 \equiv \frac{1}{h_k} L$ and $S_2 \equiv A_L^T (G + \frac{1}{h_k} C)^{-1} A_L$, both of which are dense and in different matrix formats ($S_1$ is stored as an $H$-matrix). However, since the relative contributions of $S_1$ and $S_2$ depend on the time step size $h_k$, we can choose one term over the other for the typical range of step sizes in a variable-step simulation. Specifically, for small step sizes the term $S_1 = \frac{1}{h_k} L$ dominates and the system $\frac{1}{h_k} L z_2 = r_2$ can be solved by LU factorization of $L$ in $H$-matrix format. For larger step sizes, the term $S_2 = A_L^T (G + \frac{1}{h_k} C)^{-1} A_L$ is dominant and the block preconditioner becomes

$$ \begin{bmatrix} G + \frac{1}{h_k} C & A_L \\ 0 & A_L^T (G + \frac{1}{h_k} C)^{-1} A_L \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} $$

which can be straightforwardly derived to be equivalent to the system

$$ \begin{bmatrix} G + \frac{1}{h_k} C & A_L \\ -A_L^T & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \tag{5} $$

The latter is an $(n + m) \times (n + m)$ sparse linear system which can be solved by any direct or iterative linear solver.

D. Selection of the dominant term of Schur complement

The choice of the approximation of $S$ as $S_1$ or $S_2$, for a given timestep, can be guided by computing the relative magnitude of each term in its spectral norm (see Algorithm 1). While there exist efficient routines to compute the spectral norm of $\frac{1}{h_k} L$ in $H$-matrix format, the spectral norm of $A_L^T (G + \frac{1}{h_k} C)^{-1} A_L$ can be estimated by the inequality

**Algorithm 1** Choice of suitable approximation of $S$ for a given timestep $h_k$

```plaintext
1: function S = APPROX (L, A_L^T (G + \frac{1}{h_k} C)^{-1} A_L)
2: if $\|\frac{1}{h_k} L\|_F > \|A_L^T (G + \frac{1}{h_k} C)^{-1} A_L\|_F$ then
3: $S \simeq \frac{1}{h_k} L$
4: else
5: $S \simeq A_L^T (G + \frac{1}{h_k} C)^{-1} A_L$
6: end if
7: end function
```

$$ \|A_L^T (G + \frac{1}{h_k} C)^{-1} A_L\|_F \leq \|A_L\|_F \|G + \frac{1}{h_k} C\|^{-1}_F \|A_L\|_F \| \|A_L^T\|_2 \| \tag{6} $$

Due to the specific structure of the interpolation matrix $A_L$ the inequality (6) is fairly sharp and the estimate is getting better as $h_k$ increases. With $\|A_L\|_2 = \sqrt{2}$ we have

<table>
<thead>
<tr>
<th>Benchmark</th>
<th>Number of RL branches ($m$)</th>
<th>Number of circuit nodes ($n$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>bus1</td>
<td>71.24 MIB</td>
<td>1.26 GB</td>
</tr>
<tr>
<td>bus2</td>
<td>125.53 MIB</td>
<td>2.30 GB</td>
</tr>
<tr>
<td>bus3</td>
<td>147.44 MIB</td>
<td>2.75 GB</td>
</tr>
<tr>
<td>bus4</td>
<td>139.28 MIB</td>
<td>2.26 GB</td>
</tr>
<tr>
<td>bus5</td>
<td>111.14 MIB</td>
<td>1.75 GB</td>
</tr>
</tbody>
</table>

TABLE II: Computational impact of the approximation of dense inductance matrix $L$ with $H$-matrix $L_H$. 

<table>
<thead>
<tr>
<th>Bench.</th>
<th>Compr. Time to $H$ matrix</th>
<th>Storage</th>
<th>Mat-vec Product Time</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Dense</td>
<td>$H$ matrix</td>
<td>Mem.</td>
</tr>
<tr>
<td>bus1</td>
<td>20ms</td>
<td>512 MB/1 KB</td>
<td>144.28 KB</td>
</tr>
<tr>
<td>bus2</td>
<td>99ms</td>
<td>832 MB/1 KB</td>
<td>148.22 KB</td>
</tr>
<tr>
<td>bus3</td>
<td>150ms</td>
<td>1255 MB/1 KB</td>
<td>183.04 KB</td>
</tr>
<tr>
<td>bus4</td>
<td>157ms</td>
<td>1595 MB/1 KB</td>
<td>195.85 KB</td>
</tr>
<tr>
<td>bus5</td>
<td>52.02 ms</td>
<td>320 MB/1 KB</td>
<td>50.64 MB</td>
</tr>
</tbody>
</table>

TABLE I: Details of the experimental benchmarks
parameters of HLIBpro. Each block $A_k$ was approximated by low-rank product $A_k = Z_k Y_k^T$ such that $||A_k - A_k||_F < 10^{-4}$. Table II reports the time required to perform $H$-matrix compression of $L$, as well as the memory savings and the speedups of matrix-vector multiplication required in the Krylov loop.

**C. Preconditioner efficiency analysis**

The efficiency of the proposed preconditioner depends on the timestep $h_k$ used in the simulation. Table III reports the iteration count to solve (2) with the proposed preconditioner, which uses two different approximations of the Schur complement ($S = S_1$ and $S = S_2$), and with no preconditioner for the range of time step sizes that is of practical interest. Also, it reports the spectral norm of $S_1$ and $S_2$ used to approximate $S$ in (4) for a given timestep, as in Algorithm 1. We adopt $h_k \in [10fs, 1ps, 100ps]$, where the lower range is used to compute detailed waveforms and the upper range is used to either examine the general trend in waveforms or to compute steady state. From the iteration count we can observe that for small steps $h_k \sim 10$ fs the choice $S = S_1$ is superior, and for $h_k \sim [1, 100]$ ps the choice $S = S_2$ leads to smaller iteration count. This is in agreement with the discussion in Subsection IV-C on different weightings of the two terms $S_1$ and $S_2$ to the Schur complement of $S$. The cross-over point between the two preconditioners occurs approximately in the interval [10fs 1Ps]. In the case with no preconditioning, the Krylov-iterative method failed to converge for larger step sizes.

![Fig. 2: Voltage response at randomly chosen nodes of bus1 (a) and bus3 (b) with $h_k = 1ps$ (the choice of $h_k = 1ps$ shows all the details in the response waveform)](image)

D. Transient analysis results

Combining the benefits of the storage of the dense inductance matrix as an $H$-matrix and the reduced number of Krylov iterations through the proposed preconditioner, Table IV reports the speedups of the simulation of RLC benchmarks for $h_k = 100$ ps and 30 time points, in comparison to standard SPICE simulation (dense $L$) and simulation with $H$-matrix $L$ but without preconditioner. For this simulation scenario, the proposed methodology leads to speedups up to 2139× for the largest circuit. Regarding the accuracy of the $H$-matrix approximation, the relative rms error of the simulation compared to exact SPICE was less than 0.01 in all nodes of every benchmark. Simulation waveforms for random nodes of two benchmark circuits are graphically displayed in Fig. 2, where it can be observed that the waveforms for $H$-matrix with the proposed preconditioner are indistinguishable from exact SPICE. Superimposed in the same figures are waveforms from sparse reluctance-based simulation, obtained by inversion and truncation of (with sparsity ratio leading to same memory footprint as the $H$-matrix approximation), which can be observed to exhibit a clear deviation from both exact SPICE and the proposed methodology. Note that for the largest benchmark circuit, the inversion of the dense $L$ (with size 16.3K) took an additional 111s.

**VI. CONCLUSION**

In this paper, we use the $H$-matrices for the compression of the dense inductance matrix, arising in the modelling of all mutual inductive couplings between the interconnects in ICs and present an efficient preconditioner for the system to be solved during the time integration. With these two ingredients, we can tackle much larger problems than previously possible on a given computing platform, both in terms of storage and wall clock time. Our experimental results indicate that a very good compression ratio of inductance matrix can be attained without compromising accuracy, while the proposed preconditioner reduces the iterations count of Krylov iterative method in simulation, effectively leading to a significant speedup of the simulation.

**REFERENCES**


