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Document Version
Accepted author manuscript

Link to publication record in Manchester Research Explorer

Citation for published version (APA):

Published in:
Journal für die reine und angewandte Mathematik

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OPEN ACCESS
UNIFORM BOUNDS FOR BOUNDED GEODESIC IMAGE THEOREMS

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Abstract. We give a universal bound for the bounded geodesic image theorem of Masur–Minsky. The proof uses elementary techniques.

We also give a universal bound for a stronger version of subsurface projection, this demonstrates good control over many standard subsurface projections simultaneously.

1. Introduction

There are rich connections between the study of Teichmüller spaces, hyperbolic 3–manifolds, curve graphs (and other combinatorial complexes) and mapping class groups.

Determining how weak or strong these connections are with respect to the surface is an interesting problem. A widely used result is the distance estimate of Masur–Minsky and their hierarchy machinery [25]. Essential to the Masur–Minsky distance estimate is the bounded geodesic image theorem [25, Theorem 3.1], which involves the notion of subsurface projection (which goes back to Ivanov [20]). The distance estimate, hierarchies and the bounded geodesic image theorem has made a large impact in the study of mapping class groups [2], [3], [14], [21], [23], [24], [35], Teichmüller space [9], [10], [26], [32], [33] as well as hyperbolic 3–manifolds [12, 29], [11], [30], [31] - this list is by no means exhaustive. Additionally, the applications of subsurface projections in mapping class groups has inspired directions in the study of $Out(F_n)$, see for example [4], [13], [34] and [36].

We shall provide a universal bound for the bounded geodesic image theorem, using elementary techniques, and give a universal bound for a stronger version. We hope that this can be used to give insight into how connections between the geometry, algebra and combinatorics on surfaces depend on the underlying surface.

1.1. Preliminaries and statement of results. We write $S_{g,p}$ to denote the genus $g$ surface with $p$ points removed and $\xi(S) = 3g - 3 + p$ to denote the complexity of $S = S_{g,p}$. We say a simple closed curve on $S$ is essential if it does not bound a disc or once-punctured disc. In general, we say that an isotopy class of some subset of $S$ misses another isotopy class of some subset if they admit disjoint representatives, and otherwise we say that they cut. A curve is an isotopy class of essential simple closed curve. We write $C(S)$ to denote the curve graph of $S$, whose vertex set is the set of curves on $S$ with edges between non-equal curves that miss; this is the 1-skeleton of the curve complex which was introduced by Harvey [18]. Throughout,

This work was supported by the Engineering and Physical Sciences Research Council Doctoral Training Grant.
\( S = S_{g,p} \) with \( \xi(S) \geq 2 \). For the surfaces \( S_{0,4} \) and \( S_{1,2} \), one can use the Farey graph, the description of its geodesics and a lifting argument to prove Theorem 3.2.

We shall abuse notation by simply writing \( \gamma \) to mean both the simple closed curve \( \gamma \) and its isotopy class. We write \( d_{C(S)} \) to denote the path metric on \( C(S) \) with unit length edges. A sequence of curves \( g = (\gamma_i) \) is a geodesic if for all \( i \neq j \), we have \( d_{C(S)}(\gamma_i, \gamma_j) = |i - j| \). We say \( C(S) \) is \( \delta \)-hyperbolic if for all geodesic triangles \( g_1, g_2, g_3 \), we have \( g_1 \subset N_\delta(g_2 \cup g_3) \), where \( N_\delta \) is the metric closed \( \delta \)-neighbourhood.

**Theorem 1.1** (Masur–Minsky [28], see also [6, 17]). Fix \( S = S_{g,p} \) with \( \xi(S) \geq 2 \). There exists \( \delta \geq 0 \) such that \( C(S) \) is \( \delta \)-hyperbolic.

We write subsurface to denote a compact, connected, proper subsurface of \( S \) such that each component of its boundary is essential in \( S \). Throughout, we do not consider subsurfaces that are homotopy equivalent to \( S_{0,3} \); in this case Theorem 3.2 is straightforward.

For a non-annular subsurface \( Y \subset S \). We write \( \partial Y \) for the boundary of \( Y \). We now define a map \( \pi_Y : C_0(S) \to P(\mathcal{AC}_0(Y)) \), where \( \mathcal{AC}(Y) \) is the arc and curve complex of \( Y \), and generally \( P(X) \) is the set of subsets of \( X \). Given a curve \( \gamma \in C(S) \), isotope \( \gamma \) so that it intersects \( Y \) minimally. We define \( \pi_Y(\gamma) \) to be the arcs and/or curves \( \gamma \cap Y \subset Y \). This is non-empty if and only if \( \gamma \) cuts \( Y \). The map \( \pi_Y \) is the subsurface projection to the arc and curve complex of \( Y \). We write \( \pi_Y(A) = \pi_Y(\gamma) \).

When \( Y \) is an annulus we write \( \partial Y \) for the core curve of \( Y \). This core curve represents a subgroup of \( \pi_1(S) \) and therefore there is an associated cover \( \rho_Y : Y_\gamma \to S_\gamma \), where \( Y_\gamma \) is homeomorphic to the interior of an annulus. There is a homeomorphic lift of \( Y \) to \( S_\gamma \), which we write \( Y \). One can compactify \( S_\gamma \) to a closed annulus by using a hyperbolic metric on \( S \). Let \( \mathcal{AC}_0(Y) \) be the set of arcs that connect one boundary component of \( S_\gamma \) to the other, modulo isotopies that fix the endpoints. Two arcs are adjacent if they admit disjoint representatives. We write \( \mathcal{AC}(Y) \) to denote this graph. Given a curve \( \gamma \) that cuts \( Y \), we define \( \pi_Y(\gamma) \) to be the set of arcs of the preimage \( \tilde{\gamma} = \rho_Y^{-1} \gamma \) that connect the two boundary components of \( S_\gamma \). Otherwise, \( \pi_Y(\gamma) = \emptyset \). This defines the subsurface projection \( \pi_Y : C_0(S) \to P(\mathcal{AC}_0(Y)) \) when \( Y \) is an annulus.

We write \( d_{\mathcal{AC}(Y)} \) to denote the standard metric on the graph \( \mathcal{AC}(Y) \). We write \( d_Y(A) = \text{diam}_{\mathcal{AC}(Y)}(\pi_Y(A)) \) and \( d_Y(A, B) = \text{diam}_{\mathcal{AC}(Y)}(\pi_Y(A) \cup \pi_Y(B)) \). The following lemma is immediate, see also [25, Lemma 2.2].

**Lemma 1.2.** Let \( Y \) be a subsurface of \( S \) and let \( \gamma_1, \gamma_2 \) be curves on \( S \). Suppose that \( \gamma_1 \) cuts \( Y \), \( \gamma_2 \) cuts \( Y \) and \( \gamma_1 \) misses \( \gamma_2 \). Then \( d_Y(\gamma_1, \gamma_2) \leq 1 \).

We shall give a proof of the bounded geodesic image theorem of Masur–Minsky [25, Theorem 3.1]. We shall give a bound that depends only on \( \delta \), where \( C(S) \) is \( \delta \)-hyperbolic.

**Theorem 3.2.** Given a surface \( S \) there exists \( M = M(\delta) \) such that whenever \( Y \) is a subsurface and \( g = (\gamma_i) \) is a geodesic such that \( \gamma_i \) cuts \( Y \) for all \( i \), then \( d_Y(g) \leq M \).

Recently, it has been shown that there exists \( \delta \) such that \( C(S) \) is \( \delta \)-hyperbolic for all surfaces \( S \) in Theorem 1.1, see Aougab [1], Bowditch [5], Clay–Rafi–Schleimer [15] and Hensel–Przytycki–Webb [19].

**Corollary 1.3.** There exists \( M \) independent of the surface \( S \) in Theorem 3.2.
We remark here that Chris Leininger has an unpublished proof of the bounded geodesic theorem that uses the ‘lines’ described in [6]. Bowditch’s work on uniform hyperbolicity [5] should give that Leininger’s argument recovers Corollary 1.3.

1.2. Subsurface projection to filling multiarcs. Let $\alpha$ be a multicurve on $S$: an isotopy class of a collection of pairwise disjoint and non-isotopic essential simple closed curves. Let $\gamma$ be a curve that fills $S$ with $\alpha$, i.e. there are no curves that miss $\gamma$ and $\alpha$. We may isotope $\gamma$ so that it intersects $\alpha$ minimally. We write $n(\alpha)$ to denote a regular neighbourhood of $\alpha$, and $N(\alpha)$ for its closure. We define $\kappa(\gamma)$ to be the isotopy class of a maximal collection of pairwise non-isotopic arcs of $\gamma - n(\alpha) \subset S - n(\alpha)$. A multiarc is an isotopy class of pairwise disjoint, pairwise non-isotopic arcs, each of which is not isotopic into the boundary (i.e. essential). A filling multiarc is a multiarc that cuts every curve. Thus, $\kappa(\gamma)$ is a filling multiarc of $S - n(\alpha)$. We write $FMA(S, \alpha)$ to denote the filling multiarc graph with vertices corresponding to filling multiarcs with endpoints on $\partial N(\alpha)$, with edges between two non-equal multiarcs that miss. We write $d_{FMA(S, \alpha)}$ to denote the unit length edge metric on this graph.

We shall prove the following universal bound for this ‘rigid’ subsurface projection. The universal bound does not seem to follow directly from Corollary 1.3—for instance, a priori, for larger $\xi(S)$ there can be longer chains of nested subsurfaces such that each of the corresponding subsurface projection images has significant diameter. The following theorem, therefore, is demonstrating strong control over all subsurfaces of $S - n(\alpha)$.

**Theorem 4.2.** Suppose a multicurve $\alpha$ and a geodesic $g = (\gamma_i) \gamma_i, \alpha$ fill $S$ for each $i$. Then $\text{diam}_{FMA(S, \alpha)}(\kappa(\gamma)) \leq M$, where $M$ is a universal constant.

It is not difficult to see that $FMA(S, \alpha)$ is locally finite and quasi-isometric to the product of the mapping class groups of the components of $S - n(\alpha)$. The local finiteness of the filling multiarc graph is also useful in giving a new proof [37] of Bowditch’s bounds on the ‘slices’ of tight geodesics [7].

2. Loops and surgery

2.1. Loops. Throughout this section, $\alpha$ and $\beta$ are both collections of pairwise disjoint, essential, simple closed curves on $S$ such that $\alpha$ and $\beta$ intersect minimally (equivalent to $\alpha$ and $\beta$ do not share a bigon, see for example [16, Proposition 1.7]) and $\alpha, \beta$ fill $S$.

We say a collection of simple closed curves $\{\gamma_i\}$ is sensible if they are essential, pairwise in minimal position, and with no triple points, i.e. for distinct $i, j, k$, we have $\gamma_i \cap \gamma_j \cap \gamma_k = \emptyset$.

Let $\gamma, \alpha, \beta$ be sensible. Recall that whenever we orient $\gamma$ and $\beta$ arbitrarily, each point $\gamma \cap \beta$ has a sign of intersection $\pm 1$. We say a pair of such points have opposite signs if the signs of intersection are non-equal, and have the same sign otherwise. This notion does not depend on the orientation of $\gamma, \beta$ or $S$.

**Definition 2.1.** We say that $\gamma$ is an $(\alpha, \beta)$-loop if for each arc $b \subset \beta - \alpha$ we have $|\gamma \cap b| \leq 2$ with equality if and only if $\gamma \cap \beta$ have opposite signs.

**Definition 2.1** is inspired by Leasure’s $(\alpha \cup \beta)$-cycles [22, Definition 3.1.6]. These cycles allow one to construct quasigeodesics on closed surfaces with a combinatorial description. **Definition 2.1** is an adaptation, which allows one to work on punctured
Figure 1. The curve $\gamma'$ is dotted and $N(\beta)$ is shaded.

Figure 2. Surgery in Case 2 on the left, and Case 3 on the right.

surfaces. Both $(\alpha, \beta)$-loops and Leasure’s cycles satisfy some variant of Lemma 2.5, however cycles a priori require larger constants for Lemma 2.5 and a more careful proof since they are not necessarily in minimal position with $\alpha$ and $\beta$. Our surgery argument to construct $(\alpha, \beta)$-loops from curves is necessarily more technical, but they will intersect $\alpha$ and $\beta$ minimally.

2.2. Surgery. Suppose that $\gamma, \alpha, \beta$ are sensible. We shall describe a surgery process on $\gamma$ to construct an $(\alpha, \beta)$-loop which will be written $\gamma'$. If $\gamma$ is an $(\alpha, \beta)$-loop then we set $\gamma' = \gamma$. If $\gamma$ is not an $(\alpha, \beta)$-loop then let $c$ be a minimal (with respect to inclusion) connected subarc $c \subseteq \gamma$ such that there exists an arc $b \subseteq \beta - \alpha$ with either

- $c \cap b$ is a pair of points with the same sign
- $c \cap b$ has cardinality at least 3

Since $c$ is minimal we have that $c$ has endpoints on $b$, $b$ is the unique arc with properties described above, and $|c \cap b| \leq 3$. Thus, for each arc $b' \subset \beta - \alpha$ such that $b' \neq b$, we have $|c \cap b'| \leq 2$ with equality only if $c \cap b'$ have opposite signs.

In what follows, we write $N = N(\beta)$ to denote a closed regular neighbourhood of $\beta$. Write $R \subset N - \alpha$ to denote the connected component of $N - \alpha$ with $b \subset R$. We now describe how to construct $\gamma'$, in each case of how $c$ intersects $b$.

Case 1: $|c \cap b| = 2$ and $c \cap b$ have the same sign. See Figure 2. Let $\{p_1, p_2\} = c \cap \partial R$. Connect $p_1$ to $p_2$ by an arc $a \subset R$ that intersects $b$ once and intersects $c$ only at the endpoints of $a$. We let $\gamma'$ be the simple closed curve $a \cup (c - R)$. 
Case 2: $|c \cap b| = 3$ with alternating signs of intersection with respect to some order on $b$. See Figure 2. Let $p_1, p_2, p_3$ be the points $c \cap b$ in some order along $b$. Let $c_1, c_2 \subset c$ be arcs such that $c_1 \cup c_2 = c$, $\partial c_1 = \{p_1, p_2\}$ and $\partial c_2 = \{p_2, p_3\}$. Connect $c_1 \cap \partial R$ to $c_2 \cap \partial R$ by two disjoint arcs $a_1, a_2 \subset R$ so that $a_1$ intersects $c_1, c_2$ only at its endpoints and intersects $b$ once, and similarly $a_2$. We let $\gamma' = a_1 \cup (c_1 - R) \cup a_2 \cup (c_2 - R)$.

Case 3: $|c \cap b| = 3$ with non-alternating signs of intersection. See Figure 2. We define $\gamma'$ in a similar fashion as Case 1.

We call $R$ the surgery rectangle of $\gamma'$ and the arc $a$ or the arcs $a_1, a_2$ (depending on the case analysis above) the surgery arc(s) of $\gamma'$.

**Lemma 2.2.** In each case above, $\gamma'$ is in minimal position with $\beta$ and with $\alpha$. Furthermore, $\gamma'$ is essential and an $(\alpha, \beta)$-loop.

**Proof.** Any arc of $\gamma' - \beta$ is isotopic in $S - \beta$ to some arc of $\gamma - \beta$. Therefore, $\gamma'$ and $\beta$ cannot share a bigon since $\gamma$ and $\beta$ do not, thus $\gamma'$ is essential. We now show that $\gamma'$ and $\alpha$ do not share a bigon in all the cases of the surgery process described above, via contradiction, we also keep the notation from the case analysis earlier.

Case 1: See Figure 3. Pick an innermost bigon $B$ between the pair $\gamma', \alpha$. We have $\partial B = a_{\gamma'} \cup a_\alpha$, where $a_{\gamma'} \subset \gamma'$ and $a_\alpha \subset \alpha$ are both arcs. We must have $a \subset a_{\gamma'} \subset \partial B$ otherwise $a_{\gamma'} \subset \gamma$ implying that $\gamma$ and $\alpha$ share a bigon. Observe that the cardinality of $a \cap b$ is equal to 1, and since $\gamma' \cap b$ do not share a bigon, we must have the other endpoint of $b \cap B$ in $a_\alpha \subset \partial B$. Let $\{p\} = b \cap B \cap c$, and $c_\gamma \subset \gamma - a$ be the arc with $p \in c_\gamma$. Now $\gamma$ and $\beta$ do not share a bigon, and $c_\gamma$ does not intersect the interior of $a$, thus $c_\gamma$ is contained within the disc $B$. We conclude that $c_\gamma$ and $\alpha$ cobound a bigon, contradicting $\gamma$ and $\alpha$ do not share a bigon.

Case 2: It suffices to show each connected component of $S - (\gamma' \cup \alpha)$ adjacent to at least one of the arcs $a_1, a_2$ is not a bigon. We start with the component containing $p_2$: if this is a bigon $B$, then the arc $b' \subset b - \gamma'$ with $p_2 \in b'$ satisfies $b' \subset B$ hence $b'$ cobounds a bigon with $\gamma'$. This contradicts $\gamma'$ and $\beta$ do not share a bigon. Now we argue that the component containing $p_1$ is not a bigon (and similarly $p_3$). Suppose this component was a bigon $B$, first suppose that $a_1 \subset \partial B$ but $a_2 \cap \partial B = \emptyset$, then one can follow a similar argument as in Case 1. If $a_1, a_2 \subset \partial B$, then see Figure 3 on the right. A similar argument again can be given as in Case 1.

Case 3: One can argue similarly to that of Case 1.

Finally, for each arc $b' \neq b$ with $b' \subset \beta - a$, we have $\gamma' \cap b' = \gamma \cap b'$ with opposite signs where applicable, and by definition of $\gamma'$ we have $|\gamma' \cap b| \leq 2$ with equality if and only if the intersection has opposite signs: $\gamma'$ is an $(\alpha, \beta)$-loop. □

Our Lemma 2.3 is the generalization of [22, Proposition 3.1.7].

**Lemma 2.3.** Suppose $\gamma_1, \gamma_2, \alpha, \beta$ are sensible and $\gamma_1$ misses $\gamma_2$. Then the $(\alpha, \beta)$-loops $\gamma'_1, \gamma'_2$ constructed by the surgery method above satisfy $i(\gamma'_1, \gamma'_2) \leq 4$. Furthermore, if $\gamma_1$ misses $\alpha$, or $\beta$, then $\gamma'_1$ misses $\alpha$, or $\beta$, respectively.

**Proof.** We may decompose $N = N(\beta) = S^1 \times [-1, 1]$. Call the subsets $\{z\} \times [-1, 1]$ vertical, and $S^1 \times \{t\}$ horizontal. We may assume $\beta$ is horizontal and since the collection is sensible, we may assume that $\alpha \cap N, \gamma_1 \cap N$ and $\gamma_2 \cap N$ are unions of
vertical subsets. We may assume that the surgery arcs of $\gamma_1', \gamma_2'$ are transverse to each vertical and horizontal subset.

Note that $\gamma_1'$ and $\gamma_2'$ are disjoint outside the surgery rectangles. If the pair have the same surgery rectangle then $i(\gamma_1', \gamma_2') \leq 4$. If they have different surgery rectangles, let $R$ be the surgery rectangle for $\gamma_1'$. There exists a vertical $\lambda$ which separates the arcs $\gamma_1' \cap R \subset R$ if there are two (see Figure 2). Note $\gamma_2' \cap R$ is a union of at most 2 vertical subsets. Since $\lambda$ separates $\gamma_1' \cap R \subset R$, each arc of $\gamma_2' \cap R$ intersects at most one arc of $\gamma_1' \cap R$. Thus there are only at most two intersections in $R$. Adding up both surgery rectangles, we have $i(\gamma_1', \gamma_2') \leq 4$.

Finally, the last statement follows by definition of the surgery. \hfill \qed

**Lemma 2.4.** Let $\alpha'$ be a component of a multicurve $\alpha$ on $S$ and let $\beta$ be a curve on $S$. Suppose $\alpha', \beta$ fill $S$. Then there exists a $(4,0)$-quasigeodesic $\alpha' = \gamma_0, \gamma_1, \ldots, \gamma_n = \beta$ with $\gamma_i$ a $(\alpha, \beta)$-loop for every $0 < i < n$.

**Proof.** Step 1: Set $\gamma_0, \ldots, \gamma_m$ equal to a geodesic connecting $\alpha'$ to $\beta$.

Step 2: Using the surgery process on each $\gamma_i$ with $1 \leq i \leq m - 1$, we obtain a sequence $\gamma_1', \ldots, \gamma_m'$ of $(\alpha, \beta)$-loops. We have $i(\gamma_i', \gamma_{i+1}') \leq 4$ for each $i$ by Lemma 2.3, therefore $d_{C(S)}(\gamma_i', \gamma_{i+1}') \leq 4$ and $d_{C(S)}(\gamma_i', \gamma_j') \leq 4|j - i|$ for each $i, j$.

Step 3: If we cannot find $i > j$ such that $i - j > d_{C(S)}(\gamma_i', \gamma_j')$ then $\gamma_0', \ldots, \gamma_m'$ is the required $(4,0)$-quasigeodesic. However, if there exists such a pair $i > j$ then in the sequence we replace $\gamma_i', \ldots, \gamma_j'$ with a geodesic connecting $\gamma_i'$ to $\gamma_j'$. We set $\gamma_0, \ldots, \gamma_m$ equal to this new sequence and go to Step 2.

Each time the process goes from Step 2 to Step 3 to Step 2 the sequence is strictly shorter therefore it must terminate. \hfill \qed

We remark that if $S \neq S_{1,2}$ then in Lemma 2.4 we can take a $(3,0)$-quasigeodesic, and for all but finitely many surfaces we can take a $(2,0)$-quasigeodesic.

**Lemma 2.5.** Let $Y$ be a subsurface of $S$ and suppose $\partial Y$ and $\beta$ fill $S$. Let $\gamma$ be a $(\partial Y, \beta)$-loop that cuts $\partial Y$. Then $d_{\gamma}(\gamma, \beta) \leq 2$ if $Y$ is non-annular and $d_{\gamma}(\gamma, \beta) \leq 5$ otherwise.

**Proof.** If $Y$ is non-annular then any pair of arcs in the projection will intersect at most twice by Definition 2.1 so one can consider a closed regular neighbourhood of the arcs to prove the required bound on distance.

If $Y$ is annular then suppose for contradiction that $d_{\gamma}(\gamma, \beta) \geq 6$. Then there exist arcs $\delta^* \in \pi_Y(\gamma)$ and $\epsilon^* \in \pi_Y(\beta)$ with $|\delta^* \cap \epsilon^*| \geq 5$. Following a claim from [27] Section 10, if we isotope the triangles cobounded by $\partial Y, \beta, \gamma$ into $Y$ (this retains minimal position), we have that $|\delta^* \cap \epsilon^* \cap Y'| \geq 3$, where $Y'$ is the homeomorphic
lift of $Y$. Therefore there exists an arc of $\beta - \partial Y$ which intersects $\gamma$ at least two times with the same sign, contradicting $\gamma$ a $(\partial Y, \beta)$-loop.

\[ \square \]

3. The Proof

Let $\gamma$ be a curve and $P$ be a set of curves. We say $\gamma$ is $\epsilon$-close to $P$ if for some curve $\beta$ of $P$ we have $d_C(S)(\gamma, \beta) \leq \epsilon$. Throughout this section, $\delta$ is a constant such that $C(S)$ is $\delta$-hyperbolic, see Theorem 1.1.

**Lemma 3.1.** There exists $D = D(\delta)$ such that for any subsurface $Y$, component $\alpha \subset \partial Y$, and geodesic $\alpha = \gamma_0, \gamma_1, \ldots, \gamma_n = \beta$ with $n \geq 3$, we have $d_Y(\gamma_i, \beta) \leq D$ whenever $i \geq 2$.

**Proof.** Use Lemma 2.4 to construct a $(4,0)$-quasigeodesic $Q$ of $(\partial Y, \beta)$-loops from $\alpha$ to $\beta$. For each $i$, we have $\gamma_i$ is $D'$-close to $Q$ where $D' = D(\delta)$. For an explicit $D'$, we can take $D' = D'' + 2$, where $D''$ is the largest integer with $D'' \leq \delta[\log_2(26D'')]$. See for example [8, Chapter III.H]. Using Lemma 1.2, we can take $D = 2D' + B$, where $B$ is the bound provided in Lemma 2.5.

**Theorem 3.2.** Given a surface $S$ there exists $M = M(\delta)$ such that whenever $Y$ is a subsurface and $g = (\gamma_i)$ is a geodesic such that $\gamma_i$ cuts $Y$ for all $i$, then $d_Y(g) \leq M$.

**Proof.** Take $M = 4\delta + 2D + 4$, where $D$ is defined as in Lemma 3.1. Fix $i < j$. We shall show that $d_Y(\gamma_i, \gamma_j) \leq M$. Fix $\alpha$ a component of $\partial Y$. Let $I = N_{\delta+1}(\alpha) \cap g$. There exists $g' = (\gamma_i', \ldots, \gamma_{j'}')$ a geodesic of length at most $2\delta + 2$ such that $I \subset g' \subset g$.

Let $P$ be a geodesic from $\alpha$ to $\gamma_i$ and $Q$ be a geodesic from $\alpha$ to $\gamma_j$. Let $i'' = \max\{i, i'-1\}$ and $j'' = \min\{j, j'+1\}$. Since geodesic triangles are $\delta$-slim: Case 1, we have $\gamma_{i''}$ is $\delta$-close to $P$ and $\gamma_{j''}$ is $\delta$-close to $Q$, or, Case 2, without loss of generality $\gamma_{j''}$ is not $\delta$-close to $Q$ so it must be $\delta$-close to $P$, but $\gamma_j$ is $\delta$-close to $Q$, thus there exists adjacent vertices of $g - g'$ with one $\delta$-close to $P$ and the other $\delta$-close to $Q$. By lemmas 1.2 and 3.1, we have that $d_Y(\gamma_i, \gamma_j) \leq D + \delta + (2\delta + 4) + \delta + D = M$.

We remark that $M$ need not be optimal for each surface. For example, for $S_2$ it may be better to consider Leasure’s cycles, which give $(2,0)$-quasigeodesics, whereas a priori we are taking $(3,0)$-quasigeodesics in Lemma 2.4. Similar surgery arguments may produce better results for other surfaces. Also, for all but finitely many surfaces, in Lemma 2.4 we can take a $(2,0)$ quasigeodesic; this improves on the constant $D$, but only slightly.

4. Filling multiarcs

**Lemma 4.1.** Let $\alpha$ be a multicurve on $S$. Let $\beta$ be a curve that fills $S$ with $\alpha$. Suppose that $\gamma$ is an $(\alpha, \beta)$-loop that fills $S$ with $\alpha$. Then $d_{F.M.A(S,\alpha)}(\kappa_\alpha(\gamma), \kappa_\alpha(\beta)) \leq 3$.

**Proof.** Write $c = \kappa_\alpha(\gamma)$ and $b = \kappa_\alpha(\beta)$. We shall first construct a filling multiarc $m$ that misses $b$, with the property that each arc of $m$ intersects $c$ at most once.

To construct $m$ we shall describe each arc $m' \subset m$. Let $c' \subset c$ be an arc of $c$. If $c'$ misses $b$ then we include $c'$ as an arc of $m$.

If $c'$ cuts $b$ then we can decompose $c' = c_1 \cup c_2 \cup \ldots \cup c_n$ into subarcs, with $c' \cap b = \cup(c_i \cap c_{i+1})$, in the obvious ordered fashion. For $1 \leq i \leq n-1$ write $p_i$ for unique point $p_i \in c_i \cap c_{i+1}$, and $p_0$ and $p_n$ the endpoints of $c'$.
For $i$ such that $1 \leq i \leq n-1$, let $a_i \subset b$ be a choice of subarc with interior disjoint from $c$ and with one endpoint on the boundary (i.e. $\alpha$) and the other equal to $p_i$. The existence of a choice $a_i$ for each $i$ is due to each arc of $b$ intersecting $c$ at most twice because $\gamma$ is a $(\alpha, \beta)$-loop.

Now for each $i$ with $2 \leq i \leq n-1$, we let $m_i$ be the (not necessarily essential) arc $a_{i-1} \cup c_i \cup a_i$. We let $m_1 = a_1 \cup c_1$ and $m_n = a_{n-1} \cup c_n$, these are essential otherwise $m$ and $b$ are not in minimal position, so $m$ always has an essential arc. The $m_i$ can be isotoped disjoint, the $m_i$ miss $b$ and each $m_i$ intersects $c$ at most once. Also, for $c'$ and $c''$ different arcs of $c$, we have that these choices of $m_i(c')$ and $m_j(c'')$ miss.

We now let $m$ be the union of all the arcs $m_i(c')$ above, for each $c' \subset c$. To finish the proof, we now show that $m$ is filling. Suppose $\zeta$ is a curve that misses $m$. We shall show that $\zeta$ must miss $c'$, for each $c'$, contradicting that $c$ is filling. Indeed, fix $c'$ an arc of $c$. Then either $c'$ is already an arc of $m$, or the concatenation $m_0(c'), ..., m_n(c')$ is isotopic to $c'$. But $\zeta$ misses the $m_i(c')$, thus $\zeta$ misses $c'$.

Now we claim that $d_{F,M,A}(S,\alpha)(m,c) \leq 2$ to finish the proof. Using the above construction with $m$ in place of $b$ we can construct another filling multiarc $\tilde{m}$. Since each arc of $m$ intersects $c$ at most once, we can choose the $\tilde{\alpha}_i(c')$ such that the $\tilde{m}_i(c')$ miss $c$. Therefore $\tilde{m}$ misses $m$ and $c$.

**Theorem 4.2.** Suppose a multicurve $\alpha$ and a geodesic $g = (\gamma_i)$ satisfy $\gamma_i, \alpha$ fill $S$ for each $i$. Then $diam_{F,M,A}(S,\alpha)(k_\alpha(g)) \leq M$, where $M$ is a universal constant.

**Proof.** Following along the same lines of Theorem 3.2 we use Lemma 4.1 in place of Lemma 2.5.

**Acknowledgements.** The author would like to thank Saul Schleimer and the referee for comments and suggestions on the paper. We thank Brian Bowditch, Saul Schleimer and Robert Tang for interesting conversations.

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