Abstract—This paper generalises the notion of output strictly
negative imaginary systems and provides a complete character-isation both in frequency domain and time domain. The paper
also reveals the missing link between the negative imaginary
theory and dissipativity. A new time domain supply rate is
introduced to characterise the class of output strictly negative
imaginary systems that consists of input to the system, the
derivative of an auxiliary output of the system and a real
parameter \( \delta > 0 \). Further, in addition to the output strictly
negative imaginary systems, all stable negative imaginary sys-
tems are shown to be dissipative with respect to the same supply
rate with \( \delta = 0 \). An equivalence is also established between the
output strictly negative imaginary systems property and time
domain dissipativity of this class of systems with respect to the
proposed supply rate and a specific positive definite storage
function. Several numerical examples are studied to elucidate
the essence of the theoretical developments.

I. INTRODUCTION

Negative imaginary (NI) systems theory has drawn atten-
tion from both control theorists and practising engineers over
the last decade due to its wide applicability in problems
such as vibration control of lightly damped flexible structures
[1], cantilever beams [4], large space structures [10], robotic
manipulators [10], control of nano-positioning systems [11],
control of large vehicle platoons [5], observer-based control
of non-square plants [15], etc. Negative imaginary systems
theory was first introduced in [1] inspired by the ‘positive
position feedback control’ [6] of highly resonant mechanical
systems. The negative imaginary framework offers a simple
internal stability criteria \( \lambda_{\text{max}}[M(0)N(0)] < 1 \) for a positive
feedback interconnection of two NI systems \( M(s) \) and \( N(s) \)
of which one must be strictly negative imaginary (SNI) [1], [2]. Since the SNI system \( N(s) \) must satisfy the strict
frequency domain condition \( j[N(j\omega) - N(j\omega)^*] > 0 \) for all
\( \omega \in (0, \infty) \), the existing stability results of NI-SNI intercon-
nection fail to capture the cases where both the imaginary-
Hermitian parts \( [M(s) - M(s)^*] \) and \( [N(s) - N(s)^*] \) have
transmission/blocking zeros on the \( j\omega \) axis for \( \omega \in (0, \infty) \).
Furthermore, due to this strict frequency domain condition
defined on the punctured \( j\omega \) axis (i.e., excluding \( \omega = 0 \)), NI
theory faces significant technical difficulties in both analysis
and synthesis involving SNI systems. In order to circumvent
these issues, a new class of strict negative imaginary systems,
termed as the Output Strictly Negative Imaginary (OSNI)
systems class, has been introduced in [14] for which the
strictness property is defined in terms of Output Strictly
Passive (OSP) systems. An OSNI system is not required
to satisfy the same strict frequency domain condition of
SNI systems and hence, allows blocking/transmission zeros
on the \( j\omega \) axis for \( \omega \in (0, \infty) \). In this paper, the notion of
Output Strictly Negative Imaginary (OSNI) systems has
been generalised to capture an wider class of systems. In
contrast to [14], the present definition of OSNI systems
(say \( M(s) \)) does not impose the full normal rank constraint
on \( [M(s) - M(s)^*] \), which implies that the OSNI systems
defined in [14] is a subset of the OSNI class proposed in
this paper. It is found that the OSNI and SNI subclasses
intersect and the intersection contains the set of strongly
strict negative imaginary (SSNI [3]) systems as illustrated
via the Venn diagram (Fig. 1b). Apart from providing the
definition and characterisation for OSNI systems, this paper
also explores the connections between negative imaginary
theory and classical dissipativity [16].

The connections between negative imaginary theory and
classical dissipativity are not yet well explored. In the case of
passive systems, a complete characterisation already exists in
the literature using Willems’s dissipative framework [16] as
well as Hill-Moylan’s \((Q, S, R)\)-dissipative framework [17],
[18]. In [24], the authors introduced the notion of ‘mixed’
input-output passive and finite-gain system properties using a
frequency domain dissipative approach and being inspired by
[24], in [8], a frequency domain \((Q, S, R)\)-dissipative supply
rate was introduced to characterise the class of systems
with ‘mixed’ finite-gain and input-output (strictly) negative
imaginary properties. Later on, [9] and [12] have pursued
a similar approach alike [8] to establish internal stability
conditions for interconnected systems with ‘mixed’ NI, pas-
sive and finite-gain properties. Unlike [8], [9] and [12], this
paper theoretically proves that the OSNI systems with
reachable (from the origin) state-space are dissipative with
respect to a particular time domain supply rate \( w(u, \dot{y}) = 2\dot{y}^T u - \delta \dot{y}^T \dot{y} \) with \( \delta > 0 \) by showing the existence of a
positive semidefinite storage function \( V(x) \). Note that the
auxiliary output \( \dot{y} = y - Du \) is considered particularly to
capture bi-proper OSNI systems; in strictly-proper cases the
supply rate simplifies to \( 2\dot{y}^T u - \delta \dot{y}^T \dot{y} \). This supply rate
finds an interesting physical interpretation for electrical networks
having voltage and charge flow as the input-output variables
and for mechanical systems having force and displacement
as the input-output pair. For example, in case of a spring-
mass-damper system (Fig. 1a) being strictly-proper, the term

\[ y = \dot{x} \]

\[ \ddot{x} = -\kappa x - \gamma \dot{x} \]

where \( \kappa \) and \( \gamma \) are the spring and damping constants,
respectively. This system can be brought in the form of a
negative imaginary system by choosing the input \( u \) as the
velocity \( \dot{x} \) and the output \( y \) as the displacement \( x \). The
supply rate \( w(u, \dot{y}) \) in this case is given by

\[ w(u, \dot{y}) = -\gamma \dot{x}^2 \]

which is dissipative according to the theory presented in
this paper.

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$y^T u$ gives the mechanical power input (velocity ($\dot{y}$) × force ($u$)) while the term $\dot{y}^T \dot{y}$ represents the power dissipated in the damper ($d \dot{y}^2$ assuming $d = 1$). But, for more general systems, the supply rate provides an abstraction of the net power inflow to the system and often, it is not possible to find an exact physical interpretation. It is also shown in this paper that, in addition to the OSNI systems, all stable NI systems are dissipative with respect to the same time domain supply rate $w(u, \dot{y})$ when $\delta = 0$.

For simplicity of presentation, the dependence on time $t \in \mathbb{R}_{\geq 0}$ is omitted.

$x(T) = \Phi(T, 0, x(0), u)$ and $w(u, y)$ have been evaluated along any trajectory of (1).

Inequality (2) is known as the ‘dissipation inequality’ in the sense of Willems. If $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a differentiable storage function, then the dissipation inequality (2) can be expressed in the differential form as

$$w(u, y) \geq \dot{V}(x).$$

Note that for finite dimensional LTI systems with minimal state-space realisation, the storage function $V(x)$ can be characterized with a quadratic form $x^T P x$, without loss of generality, where $P = P^T > 0$ [16], [22]. Moreover, in the LTI setting, the storage function $V(x)$ can always be assumed to be a differentiable function of $x \in \mathbb{R}^n$ [17], [26].

For a dissipative system with a reachable (from the origin) state-space, the ‘required supply’ is defined as [21]

$$V_r(x_1) = \inf_{x_1 \mapsto x_2} \int_0^T w(u, y) \, dt$$

where $x^* \in \mathbb{R}^n$ represents the point of minimum storage. In general, the origin of a state-space is the point of minimum storage where $V(x^*) = V(0) = 0$. The ‘required supply’ is the least amount of energy required to excite a system to a desired state from the state of minimum energy level [19]. $V_r(x)$ is a possible storage function for any dissipative system with a reachable (from the origin) state-space.

**Definition 2:** ($Q, S, R$)-dissipativity in Hill-Moylan’s framework) [17] A dynamical system $M$, given in (1) with $x_0 = 0$, is said to be ($Q, S, R$)-dissipative if there exist $Q = Q^T \in \mathbb{R}^{p \times p}$, $S \in \mathbb{R}^{p \times m}$ and $R = R^T \in \mathbb{R}^{m \times m}$ such that

$$\int_0^T y^T Q y + 2 y^T S u + u^T R u \, dt \geq 0$$

for all $T \in (0, \infty)$ and all admissible $u(t) \in \mathbb{R}^m$ for all $t \in \mathbb{R}_{\geq 0}$.

If the supply rate function in Willems’s framework is specialized as $w(u, y) = y^T Q y + 2 y^T S u + u^T R u$ where $Q = Q^T \in \mathbb{R}^{p \times p}$, $S \in \mathbb{R}^{p \times m}$ and $R = R^T \in \mathbb{R}^{m \times m}$, then (3) takes the form

$$y^T Q y + 2 y^T S u + u^T R u \geq \dot{V}(x).$$

**B. Definitions for negative imaginary systems theory**

In this subsection, we recall the definitions of NI and SNI systems.

**Definition 3:** (NI System) [10], [13] Let $M(s)$ be the real, rational, proper transfer function matrix of a square and causal system without any poles on the open right-half plane. $M(s)$ is said to be Negative Imaginary (NI) if

- $j [M(j \omega) - M(-j \omega)]' \geq 0$ for all $\omega \in (0, \infty)$ except the values of $\omega$ where $j \omega$ is a pole of $M(s)$;

- For LTI systems, reachability (to and from the origin) is equivalent to complete state controllability [20].
• If \( s = j\omega \) with \( \omega \in (0, \infty) \) is a pole of \( M(s) \), then it is at most a simple pole and the residue matrix
\[
K_0 = \lim_{s \to j\omega} j(s - j\omega)M(s)
\]
is Hermitian and positive semidefinite;
• If \( s = 0 \) is a pole of \( M(s) \), then \( \lim_{s \to 0} s^k M(s) = 0 \) for all \( k \geq 3 \) and \( \lim_{s \to 0} s^2 M(s) \) is positive semidefinite.

**Definition 4:** (SNI System) [1] Let \( M(s) \) be the real, rational, proper transfer function matrix of a square and causal system. \( M(s) \) is said to be Strictly Negative Imaginary (SNI) if \( M(s) \) has no poles in \( \mathbb{R}[s] \geq 0 \) and \( j|M(\omega) - M(j\omega)^*| > 0 \) for all \( \omega \in (0, \infty) \).

**III. OUTPUT STRICTLY NEGATIVE IMAGINARY SYSTEMS**

In this section, we define output strictly negative imaginary (OSNI) systems\(^3\) in the frequency domain, discuss its properties and depict the set-theoretic relationship between OSNI and SNI subclasses. A state-space characterisation is also provided to test the OSNI property of an LTI system based on its minimal state-space realisation.

**Definition 5:** (OSNI systems) Let \( M(s) \in \mathbb{H}_\infty^{m \times m} \).

Then, \( M(s) \) is said to be Output Strictly Negative Imaginary (OSNI) if there exists a scalar \( \delta > 0 \) such that
\[
j\omega[M(j\omega) - M(j\omega)^*] - \delta \omega^2 M(j\omega)^* M(j\omega) \geq 0
\]
\( \forall \omega \in \mathbb{R} \cup \{\infty\} \) where \( M(j\omega) = M(\omega) - \infty \).

The parameter \( \delta > 0 \) is an index which quantifies the level of output strictness of a given OSNI system.

**Remark 1:** The index \( \delta \) corresponds to all stable NI systems because when \( \delta = 0 \), the inequality (7) reduces simply to the negative imaginary condition \( j\omega[M(j\omega) - M(j\omega)^*] \geq 0 \) for all \( \omega \in \mathbb{R} \cup \{\infty\} \). Therefore, \( M(j\omega) = M(\omega) - \infty \).

The parameter \( \delta > 0 \) is an index which quantifies the level of output strictness of a given OSNI system.

**Example 1:** Consider the SNI transfer function \( N(s) = \frac{s}{s^2 + 8s + 32} \).
In this case, \( j\omega[N(j\omega) - N(j\omega)^*] - \delta \omega^2 \geq 0 \) for all \( \omega \in \mathbb{R} \cup \{\infty\} \) only when \( \delta = 0 \). Therefore, \( N(s) \) is not an OSNI system.

**Example 2:** Let \( M(s) = \frac{1}{s + 1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \) which violates the full normal rank condition in Definition 5 to obtain \( j\omega[M(j\omega) - M(j\omega)^*] - \delta \omega^2 M(j\omega)^* M(j\omega)^* \geq 0 \) for all \( \omega \in \mathbb{R} \cup \{\infty\} \).

**Example 3:** Consider the spring-mass-damper system shown in Fig. 2a. For the set of parameters \( m_1 = 1 \mathrm{kg}, \)
\( m_2 = 4 \mathrm{kg}, \) \( k_1 = 30 \mathrm{N/m}, \) \( k_2 = 20 \mathrm{N/m} \) and \( d_1 = 1 \mathrm{Ns/m}, \) the above system has the transfer function \( M(s) = \frac{1}{s^2 + 125} \).

\( U(s) = \frac{25}{s^2 + 5s + 25} \) is also an OSNI system with \( \delta \in [0, 0.4] \). However, this system is not SNI since \( j[M(\omega) - M(j\omega)^*] = 0 \) for all \( \omega \in \mathbb{R} \cup \{\infty\} \) and for all \( \delta \in [0, 2] \). This is revealed in Fig. 2b by the Nyquist plot of the system transfer function \( M(s) \) from \( U(s) \) to \( X_1(s) \).

Fig. 2: (a) A mechanical system realization of an OSNI system which is not SNI; (b) Nyquist plot of the system transfer function \( M(s) \) from \( U(s) \) to \( X_1(s) \).

**Lemma 1:** Let \( M(s) \in \mathbb{H}_\infty^{m \times m} \) and \( \tilde{M}(s) = M(s) - M(\infty) \). Then, the following statements hold:

a) \( M(s) \) is OSNI with \( \delta > 0 \) if and only if \( \tilde{M}(s) = s\tilde{M}(s) \) is OSP\(^4\) and \( M(\infty) = M(\infty)^T \);

b) \( M(s) \) is NI with \( \delta = 0 \) if and only if \( \tilde{M}(s) = s\tilde{M}(s) \) is passive and \( M(\infty) = M(\infty)^T \).

**Proof:** Since \( F(j\omega) = \frac{F(j\omega)^*}{\delta F(j\omega)^*} \), \( (j\omega M(j\omega) + (j\omega M(j\omega)^*) - \delta(j\omega M(j\omega))^*(j\omega M(j\omega)) = j\omega [M(j\omega) - M(j\omega)^*] - \delta \omega^2 M(j\omega)^* M(j\omega)^* \) for all \( \omega \in \mathbb{R} \cup \{\infty\} \) by following [14, MIMO OSP Lemma] and on exploiting the property \( M(\infty) = M(\infty)^T \), Part a) holds for \( \delta > 0 \), while Part b) holds for \( \delta = 0 \).

\(^3\)OSNI property is always defined for finite dimensional, causal and asymptotically stable NI systems.

\(^4\)A system \( F(s) \in \mathbb{H}_\infty^{m \times m} \) with \( F(s) + F(s)^* \) having full normal rank is Output Strictly Passive (OSP) if and only if there exists \( \delta_0 \) such that \( F(j\omega) + F(j\omega)^* \geq \delta_0 F(j\omega)^* F(j\omega) \forall \omega \in \mathbb{R} \cup \{\infty\} \) [14].
The following lemma derives a necessary and sufficient condition for a system to be OSNI and is a generalized version of [14, Lemma 6].

Lemma 2: (OSNI Lemma) Let $M(s) \in \mathbb{R}_{\infty}^{m \times m}$ have a minimal state-space realization $(A, B, C, D)$. Let $\delta > 0$ be a scalar. Then, $M(s)$ is OSNI with a level of output strictness $\delta$ if and only if $D = D^T$ and there exists a real matrix $Y = Y^T > 0$ such that

$$AY + Y A^T + \delta(CAY)^T CAY \leq 0 \text{ and } B + AY C^T = 0. \quad (8)$$

Proof: For the sake of convenience, we denote a shorthand

$$\Pi = \begin{bmatrix}
-P A - A^T P & -P B + A^T C^T \\
-\delta A^T C^T CA & -\delta A^T C^T CB \\
-P B + A^T C^T & C B + B^T C^T \\
-\delta A^T C^T CB & -\delta B^T C^T CB
\end{bmatrix}. \quad (9)$$

The proof proceeds through the following sequence of equivalent statements:

- $M(s)$ is OSNI with a level of output strictness $\delta$
- $\Leftrightarrow M(s) - D$ is OSNI with a level of output strictness $\delta$ and $D = D^T$
- $\Leftrightarrow F(s) = s[M(s) - D] = (A, B, C, A, C, B)$ is OSP with a level of output strictness $\delta$ and $D = D^T$ [via Lemma 1]
- $\Leftrightarrow D = D^T$ and there exists $P = P^T > 0$ such that $\Pi \geq 0$ [invoking [14, Lemma 2] with $\delta > 0$ and $\varepsilon = 0$]
- $\Leftrightarrow D = D^T$ and there exists $Y = Y^T > 0$ such that $AY + Y A^T + \delta(CAY)^T CAY \leq 0$ and $B + AY C^T = 0$

[On letting $Y = P^{-1}$ and following the same algebraic manipulation as in the proof of [14, Lemma 5]].

This completes the proof. \Box

Note that the matrix inequality in (8) is not in LMI form but can be readily converted into an LMI by applying the Schur Complement Lemma [25, Appendix A.61].

IV. CONNECTIONS BETWEEN OSNI SYSTEMS PROPERTY AND DISSIPATIVITY

This section is a main contribution of this paper. Subsection IV-A extends the classical notion of dissipativity to include supply rates that involve time derivative of the system’s output being inspired by [18], [21], [23] and introduces a time domain dissipative framework for characterising the OSNI as well as stable NI systems. While subsection IV-B shows the equivalence between the state-space characterisation and time domain dissipativity of the class of OSNI systems.

A. OSNI systems in time domain dissipative framework

The following theorem establishes that for an initially relaxed OSNI system $M$ with a reachable (from origin) state-space, there always exists a positive semidefinite storage function $V(x)$ such that the system satisfies the dissipation inequality (2) with a particular time domain supply rate $w(u, \dot{y}) = 2\dot{y}^T u - \delta \dot{y}^T \dot{y}$ with $\delta > 0$, where $\dot{y} = y - Du$ is defined as an auxiliary output of $M$. Note that in this section, the admissible inputs $u(t)$ are taken to be $\mathbb{R}^m$-valued locally square integrable functions of time $t \in \mathbb{R}_{\geq 0}$ along with sufficient smoothness properties such that $\dot{y}(t) = C \dot{x}(t) = CAx(t) + CBu(t)$ exists and belongs to $\mathbb{R}^m$ for all $t \in \mathbb{R}_{\geq 0}$, and also, $\dot{y}(t)$ remains locally square integrable.

Theorem 1: Let $M$ be a causal, square, LTI system given by the state-space equations $\dot{x} = Ax + Bu$ and $y = Cx + Du$ with zero initial condition and the state-space being reachable from the origin. Let the associated transfer function matrix be $M(s) \in \mathbb{R}_{\infty}^{m \times m}$, $\delta > 0$, and $\dot{y} = y - Du$. Then, $M$ is dissipative with respect to the supply rate $w(u, \dot{y}) = 2\dot{y}^T u - \delta \dot{y}^T \dot{y}$ if $M(s)$ is OSNI with the same $\delta$.

Proof: To show that the OSNI system $M$ with $\delta > 0$ is dissipative with respect to the supply rate $w(u, \dot{y}) = 2\dot{y}^T u - \delta \dot{y}^T \dot{y}$, we have to establish that there exists a storage function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ such that $M$ satisfies the dissipation inequality (2). Since the state-space is assumed to be reachable from the origin, there exists an admissible input $u(t)$ defined as

$$u(t) = \begin{cases} 0 & \text{when } t < t_{-1}; \\ \tilde{u}(t) & \text{when } t_{-1} \leq t \leq 0; \\ 0 & \text{when } t > 0, \end{cases}$$

which steers the system from $x(t_{-1}) = 0$ to any $x(t) \in \mathbb{R}^n$. In this proof, let $y(t)$ be the corresponding output; $Y(j\omega), \dot{Y}(j\omega)$ and $U(j\omega)$ denote respectively the Fourier Transform of the real-valued time domain signals $y(t), \dot{y}(t)$ and $u(t)$. $Y(j\omega) = Y(j\omega) - DU(j\omega) = M(j\omega)U(j\omega)$, where $M(j\omega) = M(j\omega) - D$. Now,

$$\int_{t_{-1}}^{0} w(u, \dot{y}) dt = \int_{t_{-1}}^{0} (2\dot{y}^T u - \delta \dot{y}^T \dot{y}) dt = \int_{-\infty}^{\infty} (2\dot{y}^T u - \delta \dot{y}^T \dot{y}) dt + \delta \int_{0}^{\infty} \dot{y}^T \dot{y} dt$$

[since $M$ is causal and time-invariant]

$$\geq \int_{-\infty}^{\infty} (2\dot{y}^T u - \delta \dot{y}^T \dot{y}) dt + \delta \int_{0}^{\infty} \dot{y}^T \dot{y} dt$$

[by Parseval’s identity [25]]

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ (j\omega \dot{Y}(j\omega))^* U(j\omega) + U(j\omega)^* (j\omega \dot{Y}(j\omega)) - \delta \omega^2 \dot{Y}(j\omega)^* \dot{Y}(j\omega) \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} U(j\omega)^* \left[ j\omega \{M(j\omega) - M(j\omega)^*\} - \delta \omega^2 \{M(j\omega)^* M(j\omega)\} \right] U(j\omega) d\omega$$

[since $M(\infty) = M(\infty)^T$ is implied by (7)]

$$\geq 0$$

[by Definition 5 and Remark 1].

Hence, for arbitrary $t_{-1} \leq 0$ and $x(t_{-1}) = 0$, we have $\int_{t_{-1}}^{0} w(u, \dot{y}) dt \geq 0$. We now construct the required supply function as $V_r(x) = \inf_{w(u, \dot{y})} \int_{t_{-1}}^{0} w(u, \dot{y}) dt \geq 0$ where $x$ is the point of minimum storage (i.e., $x^* = 0$). Thus, $V_r(x)$ can be considered as a storage function candidate associated with the OSNI system $M$.

It remains to be shown that $V_r(x)$ satisfies the dissipation inequality (2). Towards this end, note that in taking the
system from \( x = 0 \) at \( t = 0 \) to \( x_1 \in \mathbb{R}^n \) at \( t = t_1 \), we could first take it to \( x_0 \in \mathbb{R}^n \) at time \( t_0 \) while minimizing the energy, and then take it to \( x_1 \) at time \( t_1 \) along the path for which the dissipation inequality is to be evaluated. This is possible by virtue of \( M \) being a causal and time-invariant system. Since \( V_r(x_1) \) gives the infimum of the amount of energy required to reach \( x_1 \) at \( t = t_1 \) from \( x = 0 \) at \( t = 0 \), the energy required to reach the same destination \( x_1 \) from the same starting point \( x = 0 \) via any other path will be greater than or equal to \( V_r(x_1) \). Therefore, \( V_r(x_0) + \int_{x_0}^{x_1} w(u, \dot{y}) \, dt \geq V_r(x_1) \) follows. It can hence be concluded that the OSNI system \( M \) is dissipative with respect to the supply rate \( w(u, \dot{y}) = 2\dot{y}^T u - \delta \dot{y}^T \dot{y} \) for the same \( \delta > 0 \).

Following Theorem 1, a similar dissipative characterisation can be given for all stable NI (i.e., with \( \delta = 0 \)) systems.

**Lemma 3:** Let \( M(s) \in \mathbb{R}^{m \times m} \) be the transfer function matrix of an NI system \( M \) with \( \delta = 0 \) and it has a reachable state-space from the origin. Let \( M \) have time domain input \( u \), time domain output \( y \), and define \( \bar{y} = y - M(\infty)u \). Then, \( M \) is dissipative with respect to the supply rate \( w(u, \dot{y}) = 2\dot{y}^T u - \delta \dot{y}^T \dot{y} \) for the same \( \delta > 0 \).

**Proof:** The proof readily follows from Theorem 1 by setting \( \delta = 0 \) and applying the frequency domain criteria \( j\omega [M(j\omega) - M(j\omega)^*] \geq 0 \) for all \( \omega \in \mathbb{R} \cup \{\infty\} \leftrightarrow j[M(j\omega) - M(j\omega)^*] \geq 0 \) for all \( \omega \in (0, \infty) \), since the latter implies \( M(0) = M(0)^T \) and \( M(\infty) = M(\infty)^T \) for all stable NI systems via Remark 1.

The following example shows that an OSNI system satisfies the time domain dissipation inequality with the proposed supply rate \( w(u, \dot{y}) = 2\dot{y}^T u - \delta \dot{y}^T \dot{y} \), once an appropriate storage function is chosen.

**Example 4:** Consider the OSNI transfer function \( M(s) = \frac{s^2 + 5s + 25}{s^2 + 5s + 25} \) with a minimal state-space description \( x_1 = x_2, x_2 = -25x_1 - 5x_2 + 25u \) and \( y = x_1 \) with \( x(0) = 0 \). For this system, without loss of generality, we can choose a positive definite storage function \( V(x) = x_1^2 + \frac{1}{25} x_2^2 \) such that \( w(u, \dot{y}) = \dot{V}(x) = (2\dot{y}^T u - \delta \dot{y}^T \dot{y}) - \frac{\partial V(x)}{\partial x} \cdot \dot{x} = (0.4 - \delta)x_2^2 \geq 0 \) for all \( \delta \in [0, 0.4] \) and all admissible inputs \( u(t) \in \mathbb{R} \) for all \( t \in \mathbb{R}_{\geq 0} \). Therefore, \( M(s) \) is dissipative with respect to the time domain supply rate \( w(u, \dot{y}) \) with \( \delta \in [0, 0.4] \). Note, \( \delta = 0 \) indicates that \( M(s) \) is also a stable NI system (see Remark 1).

B. Equivalence between time domain dissipativity and state-space characterisation of OSNI systems

We have already established that OSNI systems (with \( \delta > 0 \)) are dissipative with respect to the time domain supply rate \( w(u, \dot{y}) = 2\dot{y}^T u - \delta \dot{y}^T \dot{y} \) where \( \bar{y} = y - Du \) is selected as an auxiliary output of the system. In this subsection, we will show that for a stable LTI system with a minimal state-space realisation, the OSNI Lemma conditions are equivalent to time domain dissipativity with respect to the proposed supply rate \( w(u, \dot{y}) \) and a specific storage function given by \( V(x) = x^T Px \) with \( P = P^T > 0 \) for all \( x \in \mathbb{R}^n \).

**Theorem 2:** Let \( M \) be a causal, square, LTI system described by the state-space equations \( \dot{x} = Ax + Bu \), \( x(0) = 0 \) and \( y = Cx + Du \), where \( A \) is Hurwitz, \( D = D^T \) and \((A, B, C, D)\) is minimal. Let the associated transfer function matrix be \( M(s) \) and define \( \bar{y} = y - Du \). Let a scalar \( \delta > 0 \). Then, the following statements are equivalent:

1. \( M(s) \) is OSNI with a level of output strictness \( \delta \);
2. there exists a real matrix \( Y = Y^T > 0 \) such that \( AY + YA^T + \delta(CAY)^T(CAY) \leq 0 \) and \( B = -AYC^T \);
3. \( M \) is dissipative with respect to the supply rate \( w(u, \dot{y}) = 2\dot{y}^T u - \delta \dot{y}^T \dot{y} \).

**Proof:**

1. \( \Rightarrow \) (ii) This equivalence is due to OSNI Lemma.

(ii) \( \Rightarrow \) (iii) There exists a real matrix \( Y = Y^T > 0 \) such that \( AY + YA^T + \delta(CAY)^T(CAY) \leq 0 \) and \( B = -AYC^T \).

\( \Rightarrow \) there exists \( P = P^T > 0 \) such that \( \Pi \geq 0 \) [on letting \( P = Y^{-1} \) and following the proof of Lemma 2. The shorthand \( \Pi \) has been introduced in (9)]]

\( \Rightarrow \) there exists \( P = P^T > 0 \) such that \( [x^T u^T] \Pi [x u] \geq 0 \) for all \( [x u] \in \mathbb{R}^{n+m} \)

\( \Rightarrow \) there exists \( P = P^T > 0 \) such that \( 2(CAx + CBu)^T u - \delta(CAx + CBu)^T(CAx + CBu) \geq x^T(PA + AT P)x + 2x^T P Bu \) for all \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \)

\( \Rightarrow \) there exists \( P = P^T > 0 \) such that the differentiable storage function \( V(x) \) satisfies \( 2\dot{y}^T u - \delta \dot{y}^T \dot{y} \geq V(x) \) evaluated along any trajectory of \( M \) subjected to any admissible input \( u(t) \in \mathbb{R}^m \) for all \( t \in \mathbb{R}_{\geq 0} \)

\( \Rightarrow \) there exists a differentiable storage function \( V(x) = x^T Px \) with \( P = P^T > 0 \) such that \( \int_0^T 2\dot{y}^T u - \delta \dot{y}^T \dot{y} \, dt \geq V(x(T)) - V(x(0)) \) for all \( T \in [0, \infty) \) and evaluated along any trajectory of \( M \) subjected to any admissible input \( u(t) \in \mathbb{R}^m \) for all \( t \in \mathbb{R}_{\geq 0} \)

\( \Rightarrow \) \( M \) is dissipative with respect to the supply rate \( w(u, \dot{y}) = 2\dot{y}^T u - \delta \dot{y}^T \dot{y} \) and a specific storage function \( V(x) = x^T Px \) with \( P = P^T > 0 \).

(iii) \( \Rightarrow \) (ii): This follows via the necessity part of the proof of [17, Theorem 1]. Suppose there exists a differentiable storage function \( V(x) = x^T Px \) with \( P = P^T > 0 \) such that \( 2\dot{y}^T u - \delta \dot{y}^T \dot{y} \geq \dot{V}(x) \) (10) along any trajectory of \( M \) for any admissible input \( u(t) \in \mathbb{R}^m \) for all \( t \in \mathbb{R}_{\geq 0} \). To turn this inequality into equality, let us introduce a function \( d : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) so that

\[ 2\dot{y}^T u - \delta \dot{y}^T \dot{y} = \dot{V}(x) + d(\cdot, \cdot). \]
Note that $x, y, \bar{y}$ and $u$ all are now variables of time and they take real vectored values at each $t \in \mathbb{R}_{\geq 0}$. As

\[(2\bar{y}^T u - \delta \hat{y}^T \hat{y}) - \hat{V}(x) = \begin{bmatrix} x^T & u^T \end{bmatrix} \Pi \begin{bmatrix} x \\ u \end{bmatrix}, \tag{12}\]

it is evident that $d(\cdot, \cdot)$ is a function of both $x$ and $u$, and $d(\cdot, \cdot)$ must be quadratic in $x$ and $u$. Moreover, $d(\cdot, \cdot) \geq 0$ is implied from (10). Based on these observations, $d(x, u)$ can be factored as

\[d(x, u) = (Lx + Wu)^T (Lx + Wu) \tag{13}\]

for some suitable choice of the matrices $L \in \mathbb{R}^{m \times n}$ and $W \in \mathbb{R}^{m \times m}$. Note, the choice of $L$ and $W$ may not be unique. Substituting (12) and (13) into (11), we have

\[
\begin{bmatrix} x^T & u^T \end{bmatrix} \Pi \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} L^T L & L^T W \\ W^T L & W^T W \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}. \tag{14}\]

Since $L$ and $W$ are constant matrices, (14) holds for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$, not necessarily related by the state equations. Equating the coefficients of the terms involving $x$ and $u$, we obtain the set of equality conditions

\[-PA - AT P - \delta A^T C^T CA = L^T L, \tag{15a}\]
\[-PB + AT C^T - \delta A^T C^T CB = L^T W, \tag{15b}\]
\[CB + B^T C^T - \delta B^T C^T CB = W^T W, \tag{15c}\]

which is equivalent to the set of conditions $AY + YA^T + \delta (CA^T)^T (CA^T) \leq 0$ and $B + AY^T C^T = 0$. This completes the proof.

V. CONCLUSIONS

This paper generalises the notion of output strictly negative imaginary systems (OSNI) introduced in [14] to widen the applicability of this theory to more general class of LTI systems. The present work has also underpinned the connections between negative imaginary (NI) theory and classical dissipativity. It is established that the class of OSNI systems is equivalent to a class of dissipative systems with respect to a particular supply rate comprised of the time domain input $(u)$, the time derivative of an auxiliary output of the system $(\bar{y})$ where $\bar{y}$ is defined as $y - Du$ and a real parameter $\delta > 0$. This dissipative characterisation may be extended in future to develop a unified framework for analysis and synthesis of NI and SNI systems.

REFERENCES


